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ASYMPTOTIC PROPERTIES OF PERIODOGRAMS OF WEAKLY DEPENDENT FUNCTIONAL PROCESSES

Maryam Hashemi and Atefeh Zamani

Department of Statistics
Khansar Faculty of Mathematics and Computer Science
Khansar, Iran

Department of Statistics
Faculty of Science
Shiraz University
Shiraz, Iran

Abstract

Although in classical theory of time series analysis, it is customary to consider white noise processes as the error term, in functional time series analysis, this assumption can be put in abeyance. An approach to weaken this assumption is to consider the notion of weakly dependent functional processes. In this paper, we study the periodograms and their asymptotic properties in L^2-m-approximable processes that constitute a special class of weakly dependent functional processes.

1. Introduction

Functional time series (FTS) is an important branch in functional data analysis. Bosq [2] formed the basic theoretical foundation for FTS.

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Afterwards, many researchers worked on FTS such as Antoniadis and Sapatinas [1], Ferraty et al. [5, 6], Hyndman and Shang [11], Panaretos and Tavakoli [12], Horváth et al. [9, 10], Hormänn et al. [8] and so on. In classical time series analysis, it is customary to consider Gaussian white noise processes as the error term. In recent years, by increasing interest in functional data analysis, some researchers tried to extend the classical contexts into this field of study. In functional time series analysis, the error terms can be considered to be Gaussian functional white noise processes or follow some kind of weak dependence. The notion of weak dependence can be formalized in various ways, such as mixing conditions. In 2010, Hörmann and Kokoszka [7] introduced L^2-m-dependent processes as a class of weakly dependent functional processes. They showed that this kind of dependence is applicable to linear as well as nonlinear FTS. They studied the estimation of the functional principal component, the long-run covariance matrix, change point detection and the functional linear model. Recently, Cerovecki and Hörmann [4] in their working paper presented the central limit theorem for the discrete Fourier transform (DFT) of functional L^2-m-dependent time series.

In this paper, we study the asymptotic properties of the periodograms of L^2-m-dependent time series in detail. Consequently, this paper is organized as follows. The next section gives background materials based on Hörmann and Kokoszka [7]. Section 3 addresses the asymptotic behavior of periodograms such as unbiasedness, asymptotic distribution and consistency.

2. Preliminary Notations and Definitions

Let $H := L^2([0, 1])$ be the Hilbert space of square integrable functions defined on $[0, 1]$, which is endowed by the inner product $\langle f, q\rangle = \int_0^1 f(\tau)g(\tau)d\tau$ and the norm $\| f \| = \sqrt{\langle f, f\rangle}$. Moreover, let us denote the space of bounded linear operators on H by $\mathcal{L}(H)$. Nuclear operators set up for an important subspace of $\mathcal{L}(H)$, which will be demonstrated by $\mathcal{N}(H)$. This space will be equipped with the norm $\| A \|_{\mathcal{N}} = \sum_{k=1}^{\infty} \langle A\phi_k, \phi_k \rangle$, where $\{\phi_k\}$ is any orthonormal basis on H. Another subspace of $\mathcal{L}(H)$ is the class of Hilbert-Schmidt operators, $\mathcal{HS}(H)$, which form a Banach space endowed with the norm $\| A \|_{\mathcal{HS}} = \left\{ \sum_{k=1}^{\infty} \| Ae_i \|_H^2 \right\}^{1/2}$. For x and y in H, the tensorial product of x and y, $x \otimes y \in \mathcal{N}(H)$, is introduced as $(x \otimes y)z := \langle x, z \rangle_H y, \ y \in H$.

Furthermore, let $\mathcal{H} := L^2(\Omega, H, P)$ stand for the Hilbert space of all H-valued random variables X with finite second moment. We use $N_H(\mu, \Sigma)$ to denote a Gaussian element in H with mean μ and covariance operator $\sum$. In fact, $X \sim N_H(\mu, \Sigma)$ if and only if, for any $u \in H$, the projection $\langle X, u \rangle$ is normally distributed with mean $\langle \mu, u \rangle$ and variance $\langle \Sigma(u), u \rangle$. Although in the sequel, all observations are assumed to be real, in some situations, we will face some complex setting. In case of complex setting, we will assume that the Hilbert space $H = H_0 + iH_0$ is complex. Let $c = \mu = \mu_{Re} + i\mu_{Im}$. For $u \in H$, define

$$\Gamma(u) = \mathbb{E}[(X - \mu)\langle u, X - \mu \rangle] \quad \text{and} \quad C(u) = \mathbb{E}[(X - \mu)\langle X - \mu, u \rangle].$$

We say that X is *complex Gaussian* with mean μ, covariance Γ and relation operator C if

$$\begin{pmatrix} Re(X) \\ Im(X) \end{pmatrix} \sim N_{H_0 \times H_0}\left(\begin{pmatrix} \mu_{Re} \\ \mu_{Im} \end{pmatrix}, \frac{1}{2} \begin{pmatrix} Re(\Gamma + C) & -Im(\Gamma - C) \\ Im(\Gamma + C) & Re(\Gamma - C) \end{pmatrix} \right). \tag{2.1}$$

Henceforth, we will only need the circularly-symmetric case, i.e., when $\mu = 0$ and $C = 0$. Then we write $X \sim CN(0, \Gamma)$.

Definition 2.1. The H-valued random variable X is called *complex Gaussian*, $X \sim CN_H(0, \Gamma)$, if and only if, for any $u \in H$,

$$\begin{pmatrix} Re\langle X, u\rangle \\ Im\langle X, u\rangle \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} \langle \Gamma(u), u\rangle & 0 \\ 0 & \langle \Gamma(u), u\rangle \end{pmatrix}\right). \tag{2.2}$$

Let X be a noncentral Gaussian random variable with mean μ and covariance operator Γ, $X \sim N_H(\mu, \Gamma)$, the distribution of $X \otimes X$ will be called the *Wishart distribution* with one degree of freedom with parameters μ and Γ, denoted by $W_H(1, \mu, \Gamma)$. If X follows $X \sim CN_H(\mu, \Gamma)$, then the distribution of $X \otimes X$ will be called *complex Wishart distribution*, $CW_H(1, \mu, \Gamma)$.

A sequence of H-valued random variables, namely $\{X_t\}_{t\in\mathbb{Z}}$, is called *functional time series*. The mean function of X_t is defined in terms of Bochner integral and will be denoted by $\mu_t := \mathbb{E}(X_t) \in H$. For $X_t \in \mathcal{H}$, the autocovariance operator at lag s is defined by

$$\Gamma_s := \mathbb{E}[(X_{t+s} - \mu_{t+s}) \otimes (X_t - \mu_t)].$$

Despite the importance of white noise processes in the analysis of time series, an approach to weaken the assumption of independence is to consider the notion of weak dependence and, as a special case, L^2-m-approximable processes, which are introduced by Hörmann and Kokoszka [7].

Definition 2.2. A sequence $\{\varepsilon_t\} \in H$ is called L^2-m*-approximable* if each ε_t admits the representation

$$\varepsilon_t = f(\varepsilon_t, \varepsilon_{t-1}, \ldots), \tag{2.3}$$

where ε_i are i.i.d. elements taking values in a measurable space S, and f is a measurable function, $f : S^\infty \to H$. Moreover, it is assumed that if $\{\varepsilon_i'\}$ is an independent copy of $\{\varepsilon_i\}$ defined on the same probability space, then letting

$$\varepsilon_t^{(m)} = f(\varepsilon_t, \varepsilon_{t-1}, \ldots, \varepsilon_{t-m+1}, \varepsilon_{t-m}, \varepsilon_{t-m-1}, \ldots), \tag{2.4}$$

we have

$$\sum_{t=1}^{\infty}(E\| \varepsilon_t - \varepsilon_t^{(m)} \|^2)^{1/2} < \infty. \tag{2.5}$$

Let us denote the discrete Fourier transform of $X_1, X_2, ..., X_n$ by

$$S_n(\theta) := \sum_{t=1}^{n} X_t e^{-it\theta}, \quad \theta \in [-\pi, \pi]. \tag{2.6}$$

It is well known that if the autocovariance operators satisfy $\sum_{h\in\mathbb{Z}} \| \Gamma_h \|_{\mathcal{N}} < \infty$, then the spectral density operator, $\mathcal{F}_\theta$, is defined as

$$\mathcal{F}_\theta := \frac{1}{2\pi}\sum_{h\in\mathbb{Z}} \Gamma_h e^{-ih\theta}. \tag{2.7}$$

The following theorem, which is proved by Cerovecki and Hörmann [4] demonstrates the asymptotic distribution of DFT of L^2-m-approximable time series.

Theorem 2.1 (Cerovecki and Hörmann [4]). *Suppose that* (X_t) *is* L^2-m*-approximable*. *Then*, *for all* $\theta \in [-\pi, \pi]$,

$$\frac{1}{\sqrt{n}} S_n(\theta) \to CN_H(0, \pi\mathcal{F}_\theta). \tag{2.8}$$

Moreover, $\sum_{h\in\mathbb{Z}} \| \Gamma_h \|_{\mathcal{HS}} < \infty$ *and therefore* $\mathcal{F}_\theta = \frac{1}{2\pi}\sum_{h\in\mathbb{Z}} \Gamma_h e^{-i\theta h}$. *Additionally*, *we have*

- $n^{-1}E[S_n(\theta) \otimes S_n(\theta)] \to 2\pi\mathcal{F}_\theta$,
- $n^{-1}E\| S_n(\theta) \|_H^2 = n^{-1}\| E[S_n(\theta) \otimes S_n(\theta)] \|_{\mathcal{N}} \to 2\pi\| \mathcal{F}_\theta \|_{\mathcal{N}} < \infty$,
- $\Gamma_h = \int_{-\pi}^{\pi} \mathcal{F}_\theta e^{ih\theta} d\theta$,
- *the components of* $n^{-1}(S_n(\theta), S_n(\theta'))$ *are asymptotically independent if* $\theta \neq \theta'$.

The following section is devoted to the asymptotic distribution of the periodograms and some asymptotic properties.

3. Asymptotic Distribution of Periodograms and their Properties

Periodograms play an important role in the analysis of time series data. These operators, which are defined based on DFT of time series, are considered as spectral density operator estimators. In case of H-valued random processes, periodograms are nuclear operators and, for Fourier frequencies $\theta_k = 2\pi k/n$, $k = 0, 1, \ldots, n$, are defined as

$$\mathcal{I}_{\theta_k} := \frac{1}{n}(S_n(\theta_k) \otimes S_n(\theta_k)).$$

Following Fuller [13], for any arbitrary $\theta \in [-\pi, \pi]$, the periodogram is defined as a piecewise constant function, which coincides with $\mathcal{I}_{\theta_k}$ at the Fourier frequencies:

$$\mathcal{I}_\theta = \begin{cases} \mathcal{I}_{\theta_k}, & \text{if } \theta_k - \frac{\pi}{n} < \theta \leq \theta_k + \frac{\pi}{n} \text{ and } 0 \leq \theta \leq \pi, \\ \mathcal{I}_{-\theta_k}, & \text{if } -\pi \leq \theta < 0, \end{cases} \tag{3.1}$$

Brockwell and Davis [3]. Based on Theorem 2.1, it can be easily seen that $(2\pi)^{-1}\mathcal{I}_\theta$ is an asymptotically unbiased estimator of $\mathcal{F}_\theta$. The following theorem establishes the asymptotic distribution of the periodogram of L^2-m-approximable time series.

Theorem 3.1. *Let $X_1, \ldots, X_n$ be a sequence of L^2-m-approximable time series. Then*

(i) $\mathcal{I}_\theta \sim CW_H(1, 0, \pi\mathcal{F})$,

(ii) *for $\theta \neq \theta'$, $\mathcal{I}_\theta$ and $\mathcal{I}'_\theta$ are asymptotically independent.*

Proof. The proof of parts (i) and (ii) are easy consequences of Theorem 2.1 and definition of complex Wishart distribution.

In functional time series analysis, norm of operators can be applied in various situations. The next theorem demonstrates the asymptotic distribution of $\| \mathcal{I}_\theta \|_{\mathcal{N}}$, which is a multiple of chi-square distribution.

Theorem 3.2. *Let* $X_1, \ldots, X_n$ *be a sequence of* L^2*-*m*-approximable time series*. *Then*

$$\| \mathcal{I}_\theta \|_{\mathcal{N}} \to c\chi^2_{(f)}, \tag{3.2}$$

where $c = \pi \| \mathcal{F}_\theta^{1/2} \|^2_{\mathcal{HS}} / (2 \| \mathcal{F}_\theta \|_{\mathcal{N}})$ *and* $f = 2 \| \mathcal{F}_\theta \|^2_{\mathcal{N}} / (\| \mathcal{F}_\theta^{1/2} \|^2_{\mathcal{HS}})$.

Proof. It is well known that

$$\begin{aligned} \| \mathcal{I}_\theta \|_{\mathcal{N}} &= n^{-1} \| S_n(\theta) \otimes S_n(\theta) \|_{\mathcal{N}} \\ &= n^{-1} \| S_n(\theta) \|^2 \\ &= n^{-1} \sum_{j=1}^{\infty} | \langle S_n(\theta), e_j \rangle |^2 \\ &= n^{-1} \left\{ \sum_{j=1}^{\infty} | Re\langle S_n(\theta), e_j \rangle |^2 + | Im\langle S_n(\theta), e_j \rangle |^2 \right\}. \end{aligned}$$

Based on Definition 2.1 and Theorem 2.1, $n^{-1/2} Re\langle S_n(\theta), e_j \rangle$ and $n^{-1/2} Im\langle S_n(\theta), e_j \rangle$ are asymptotically independent Gaussian random variables with mean 0 and variance $\frac{\pi}{2} \langle \mathcal{F}(e_j), e_j \rangle$. Consequently,

$$\| \mathcal{I}_\theta \|_{\mathcal{N}} \to \sum_{j=1}^{\infty} \frac{\pi}{2} \langle \mathcal{F}_\theta(e_j), e_j \rangle \chi^2_{(2)}. \tag{3.3}$$

Moreover, Satterthwaite's approximation [14] states that if $\sum_{k=1}^{\infty} r_k < \infty$, then $c\chi^2_{(f)}$ can be used to approximate $\sum_{k=1}^{\infty} r_k \chi^{(2)}_{(m)}$, where c and

f are determined, respectively, by $c = \left(\sum_{k=1}^{\infty} r_k^2\right) / \sum_{k=1}^{\infty} r_k$ and $f = m\left(\left(\sum_{k=1}^{\infty} r_k\right) / \sum_{k=1}^{\infty} r_k^2\right)$ and $[x]$ denotes the closest integer to x. Based on this approximation, the distribution of $\| \mathcal{I}_\theta \|_{\mathcal{N}}$ can be approximated by $\| \mathcal{I}_\theta \|_{\mathcal{N}} \to c\chi^2_{(f)}$, where $c = \pi \| \mathcal{F}_\theta^{1/2} \|^2_{\mathcal{HS}} / (2 \| \mathcal{F}_\theta \|_{\mathcal{N}})$ and $f = 2 \| \mathcal{F}_\theta \|^2_{\mathcal{N}} / (\| \mathcal{F}_\theta^{1/2} \|^2_{\mathcal{HS}})$.

Various testing procedures are defined in multivariate time series analysis for detecting hidden periodicities based on norm of periodograms. Theorem 3.2 can be applied to extend these theorems to functional time series analysis.

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