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# FORMULAS FOR SAIGO FRACTIONAL INTEGRAL OPERATORS WITH ${}_2F_1$ GENERALIZED k-STRUVE FUNCTIONS

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**Abstract**

We aim to present certain interesting, new and potentially useful formulas for the Saigo fractional integral operators involving the Gauss hypergeometric function ${}_2F_1$ which have the generalized k-Struve function as one of the kernel factors. The main results presented here are also shown to reduce to yield the corresponding identities regarding the Riemann-Liouville fractional integral operators and the Erdélyi-Kober fractional integral operators.

## 1. Introduction and Preliminaries

The Fox-Wright function ${}_p\Psi_q$ is due to Fox [4] and Wright [17-19] who studied the asymptotic expansion of the generalized (Wright) hypergeometric function defined by (see [13, p. 21])

$$ {}_p\Psi_q\left[\begin{matrix}(\alpha_1, A_1), \ldots, (\alpha_p, A_p); \\ (\beta_1, B_1), \ldots, (\beta_q, B_q);\end{matrix}\ z\right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j k)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j k)} \frac{z^k}{k!}, \tag{1.1} $$

where the coefficients $A_1, \ldots, A_p \in \mathbb{R}^+$ and $B_1, \ldots, B_q \in \mathbb{R}^+$ such that

$$ 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geqq 0. $$

A special case of (1.1) is

$$ {}_p\Psi_q\left[\begin{matrix}(\alpha_1, 1), \ldots, (\alpha_p, 1); \\ (\beta_1, 1), \ldots, (\beta_q, 1);\end{matrix}\ z\right] = \frac{\prod_{j=1}^{p} \Gamma(\alpha_j)}{\prod_{j=1}^{q} \Gamma(\beta_j)}\ {}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q;\end{matrix}\ z\right], $$

where ${}_pF_q(p, q \in \mathbb{N}_0)$ is the generalized hypergeometric function defined by (see [9, p. 73]; see also [12, Section 1.5]):

$$ {}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q;\end{matrix} \; z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} $$

$$ = {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z), $$

where $(\lambda)_\nu$ denotes the Pochhammer symbol defined (for $\lambda, \nu \in \mathbb{C}$) by

$$ (\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1, & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}), \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1), & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases} \tag{1.2} $$

where $\Gamma$ is the familiar Gamma function, among several useful equivalent forms, whose Euler's integral is recalled (see, e.g., [12, Section 1.1]):

$$ \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \quad (\Re(z) > 0). $$

Here and in the following, let $\mathbb{C}, \mathbb{R}, \mathbb{R}^+, \mathbb{Z}_0^-$, and $\mathbb{N}$ be the sets of complex numbers, real numbers, positive real numbers, non-positive integers, and positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Díaz and Pariguan [2] introduced the k -Pochhammer symbol as follows:

$$ (\gamma)_{n, \mathrm{k}} := \begin{cases} \dfrac{\Gamma_{\mathrm{k}}(\gamma + n\mathrm{k})}{\Gamma_{\mathrm{k}}(\gamma)}, & (n \in \mathbb{N}; \mathrm{k} \in \mathbb{R}^+; \gamma \in \mathbb{C} \setminus \{0\}), \\ \gamma(\gamma + \mathrm{k}) \cdots (\gamma + (n - 1)\mathrm{k}), & (n \in \mathbb{N}; \gamma \in \mathbb{C}), \end{cases} \tag{1.3} $$

where $\Gamma_{\mathrm{k}}$ is the k -gamma function defined by

$$ \Gamma_{\mathrm{k}}(z) = \int_0^\infty e^{-\frac{t^{\mathrm{k}}}{\mathrm{k}}} t^{z-1} dt \quad (\Re(z) > 0; \mathrm{k} \in \mathbb{R}^+), \tag{1.4} $$

which has the following relationships:

$$ \Gamma_{\mathrm{k}}(z + \mathrm{k}) = z\Gamma_{\mathrm{k}}(z) \quad \text{and} \quad \Gamma_{\mathrm{k}}(\gamma) = \mathrm{k}^{\frac{\gamma}{\mathrm{k}} - 1} \Gamma\left(\frac{\gamma}{\mathrm{k}}\right). $$

Saigo [10] introduced the following left- and right-sided generalized fractional integral operators involving the Gauss hypergeometric function ${}_2F_1$ (see, e.g., [14-16, 20, 21]), respectively, defined (for $x \in \mathbb{R}^+$) as follows:

$$(I_{0+}^{\alpha,\beta,\eta} f)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \, {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) f(t)dt \quad (1.5)$$

and

$$(I_{-}^{\alpha,\beta,\eta} f)(x)$$

$$= \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} \, {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}\right) f(t)dt, \quad (1.6)$$

where $\alpha, \beta, \eta \in \mathbb{C}$ and $\Re(\alpha) > 0$.

The particular case of (1.5) and (1.6) when $\beta = -\alpha$ leads to the classical Riemann-Liouville left- and right-sided fractional integrals of order $\alpha$ $(\Re(\alpha) > 0)$, respectively, as follows (see, e.g., [7, 11]): For $x \in \mathbb{R}^+$,

$$(I_{0+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt \quad (1.7)$$

and

$$(I_{-}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)dt. \quad (1.8)$$

When $\beta = 0$, (1.5) and (1.6) reduce to give the well-known Erdélyi-Kober fractional integral operators, respectively, as follows (see, e.g., [7, 11]). For $x \in \mathbb{R}^+$,

$$(I_{0+}^{\alpha,0,\eta} f)(x) = (K_{\eta,\alpha}^{+} f)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{\eta} f(t)dt \quad (1.9)$$

and

$$(I_{-}^{\alpha, 0, \eta} f)(x) = (K_{\eta, \alpha}^{-} f)(x)$$

$$= \frac{x^{\eta}}{\Gamma(\alpha)} \int_{x}^{\infty} (t - x)^{\alpha - 1} t^{-\alpha - \eta} f(t) dt, \tag{1.10}$$

where $\alpha, \eta \in \mathbb{C}$ with $\Re(\alpha) > 0$.

The generalized k -Struve function is defined by (see [8])

$$S_{v, c}^{\mathrm{k}}(z) = \sum_{n=0}^{\infty} \frac{(-c)^{n}}{\Gamma_{\mathrm{k}}\left(n\mathrm{k} + v + \frac{3\mathrm{k}}{2}\right) \Gamma\left(n + \frac{3}{2}\right) n!} \left(\frac{z}{2}\right)^{2n + \frac{v}{\mathrm{k}} + 1}$$

$$(\mathrm{k} \in \mathbb{R}^{+};\ c \in \mathbb{R};\ v > -1), \tag{1.11}$$

where $\Gamma_{\mathrm{k}}$ is the k -gamma function in (1.4). Setting $\mathrm{k} \to 1$ and $c = 1$ (1.11) reduces to yield the well-known Struve function of order $v$ defined by (see [1])

$$H_{v}(z) = \sum_{n=0}^{\infty} \frac{(-1)^{n}}{\Gamma\left(n + v + \frac{3}{2}\right) \Gamma\left(n + \frac{3}{2}\right) n!} \left(\frac{z}{2}\right)^{2n + v + 1}.$$

We recall some known formulas for the fractional integral operators (1.5), (1.6), (1.9) and (1.10) as in the following lemma (see [6]).

**Lemma 1.** *Let* $\alpha, \beta, \eta, \lambda \in \mathbb{C}$ *with* $\Re(\alpha) > 0$ *and* $\Re(\lambda) > \max\{0, \Re(\beta - \eta)\}$. *Also, let* $x \in \mathbb{R}^{+}$. *Then*

$$(I_{0+}^{\alpha, \beta, \eta} t^{\lambda - 1})(x) = \frac{\Gamma(\lambda) \Gamma(\lambda + \eta - \beta)}{\Gamma(\lambda - \beta) \Gamma(\lambda + \alpha + \eta)} x^{\lambda - \beta - 1}$$

*and*

$$(I_{-}^{\alpha, \beta, \eta} t^{\lambda - 1})(x) = \frac{\Gamma(\eta - \lambda + 1) \Gamma(\beta - \lambda + 1)}{\Gamma(1 - \lambda) \Gamma(\alpha + \beta + \eta - \lambda + 1)} x^{\lambda - \beta - 1}.$$

*We also have*

$$(K^{+}_{\eta,\alpha}t^{\lambda-1})(x) = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda+\alpha+\eta)}\,x^{\lambda-1}$$

$$(\Re(\alpha) > 0;\ \Re(\lambda) > -\Re(\eta);\ x \in \mathbb{R}^{+})$$

*and*

$$(K^{-}_{\eta,\alpha}t^{\lambda-1})(x) = \frac{\Gamma(\eta-\lambda+1)}{\Gamma(\alpha+\eta-\lambda+1)}\,x^{\lambda-1}$$

$$(\Re(\alpha) > 0;\ \Re(\lambda) < 1 + \Re(\eta);\ x \in \mathbb{R}^{+}).$$

In this paper, we present certain interesting, new and potentially useful formulas for the Saigo fractional integral operators involving the Gauss hypergeometric function ${}_2F_1$ in (1.5) and (1.6) which have the generalized *k*-Struve function (1.11) as one of the kernel factors. The main results presented here are also shown to reduce to yield the corresponding identities regarding the Riemann-Liouville fractional integral operators (1.7) and (1.8) and the Erdélyi-Kober fractional integral operators (1.9) and (1.10).

## 2. Formulas for the Saigo Fractional Integrals Associated with the Generalized $\mathrm{k}$-Struve Function

Here, we present formulas for the Saigo fractional integrals (1.5) and (1.6) associated with the generalized $\mathrm{k}$-Struve function (1.11), which are expressed in terms of the Fox-Wright function ${}_p\Psi_q$ in (1.1).

**Theorem 1.** *Let* $\alpha, \beta, \eta, \lambda \in \mathbb{C}$ *and* $v \in \mathbb{R}$, $\mathrm{k} \in \mathbb{R}^{+}$ *such that* $\Re(\alpha) > 0$ *and* $v + \mathrm{k} > 0$

$$\Re(\lambda) + v > \mathrm{k}\max\{0,\ \Re(\beta-\eta)\}.$$

*Then, for* $x \in \mathbb{R}^+$,

$$(I_{0+}^{\alpha,\beta,\eta} t^{\frac{\lambda}{\mathrm{k}}-1} S_{\nu,c}^{\mathrm{k}}(t))(x)$$

$$= \frac{x^{\frac{\lambda}{\mathrm{k}}+\frac{\nu}{\mathrm{k}}-\beta}}{2^{\frac{\nu}{\mathrm{k}}+1}\mathrm{k}^{\frac{\nu}{\mathrm{k}}+\frac{1}{2}}}$$

$$\times {}_2\Psi_4\left[\begin{matrix}\left(\frac{\nu}{\mathrm{k}}+\frac{\lambda}{\mathrm{k}}+1,\,2\right),\left(1+\frac{\nu}{\mathrm{k}}+\frac{\lambda}{\mathrm{k}}+\eta-\beta,\,2\right);\\ \left(1+\frac{\nu}{\mathrm{k}}+\frac{\lambda}{\mathrm{k}}-\beta,\,2\right),\left(1+\frac{\nu}{\mathrm{k}}+\frac{\lambda}{\mathrm{k}}+\alpha+\eta,\,2\right),\left(\frac{\nu}{\mathrm{k}}+\frac{3}{2},\,1\right),\left(\frac{3}{2},\,1\right);\end{matrix}\ -\frac{cx^2}{4\mathrm{k}}\right]. \tag{2.1}$$

**Proof.** Let $\mathcal{L}$ be the left-hand side of (2.1). Applying (1.5) to (1.11) and interchanging the order of integral and summation, which is valid under the given conditions in this theorem, we have

$$\mathcal{L} = \sum_{n=0}^{\infty} \frac{(-c)^n\left(\frac{1}{2}\right)^{\frac{\nu}{\mathrm{k}}+2n+1}}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+\nu+\frac{3\mathrm{k}}{2}\right)\Gamma\left(n+\frac{3}{2}\right)n!}(I_{0+}^{\alpha,\beta,\eta} t^{\frac{\lambda+\nu}{\mathrm{k}}+2n})(x).$$

Using (1.12), we obtain

$$\mathcal{L} = \frac{x^{\frac{\lambda+\nu}{\mathrm{k}}-\beta}}{(2\mathrm{k})^{\frac{\nu}{\mathrm{k}}+1}} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{\nu}{\mathrm{k}}+\frac{\lambda}{\mathrm{k}}+2n+1\right)\Gamma\left(\frac{\nu}{\mathrm{k}}+\frac{\lambda}{\mathrm{k}}+\eta-\beta+2n+1\right)}{\Gamma\left(\frac{\nu}{\mathrm{k}}+\frac{\lambda}{\mathrm{k}}-\beta+2n+1\right)\Gamma\left(\frac{\nu}{\mathrm{k}}+\frac{\lambda}{\mathrm{k}}+\alpha+\eta+2n+1\right)}$$

$$\times \frac{1}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(\frac{\nu}{\mathrm{k}}+\frac{3}{2}+n\right)} \frac{(-cx^2)^n}{(4\mathrm{k})^n n!},$$

which, upon expressing in terms of the Fox-Wright function ${}_p\Psi_q$ in (1.1), leads to the right-hand side of (2.1). This completes the proof. □

**Theorem 2.** *Let* $\alpha, \beta, \eta, \lambda \in \mathbb{C}$ *and* $v \in \mathbb{R}$, $\mathrm{k} \in \mathbb{R}^{+}$ *such that* $\Re(\alpha) > 0$ *and* $v + \mathrm{k} > 0$

$$\Re(\lambda) < v + \mathrm{k} + \mathrm{k}\min\{\Re(\beta), \Re(\eta)\}.$$

*Then*, *for* $x \in \mathbb{R}^{+}$,

$$\left(I_{0-}^{\alpha,\beta,\eta} t^{\frac{\lambda}{\mathrm{k}}-1} S_{v,c}^{\mathrm{k}}(1/t)\right)(x)$$

$$= \frac{x^{\frac{\lambda - v}{\mathrm{k}} - \beta}}{2^{\frac{v}{\mathrm{k}}+1}\mathrm{k}^{\frac{v}{\mathrm{k}}+\frac{1}{2}}}$$

$$\times {}_2\Psi_4\left[\begin{matrix}\left(\beta - \frac{\lambda}{\mathrm{k}} + \frac{v}{\mathrm{k}} + 2, 2\right), \left(\eta - \frac{\lambda}{\mathrm{k}} + \frac{v}{\mathrm{k}} + 2, 2\right); \\ \left(2 - \frac{\lambda}{\mathrm{k}} + \frac{v}{\mathrm{k}}, 2\right), \left(\alpha + \beta + \eta - \frac{\lambda}{\mathrm{k}} + \frac{v}{\mathrm{k}} + 2, 2\right), \left(\frac{v}{\mathrm{k}} + \frac{3}{2}, 1\right), \left(\frac{3}{2}, n\right);\end{matrix} \; -\frac{c}{4\mathrm{k}x^{2}}\right].$$

**Proof.** Here, applying (1.6) instead of (1.5), the proof would run parallel to that of Theorem 1. We omit the details. □

Setting $\beta = -\alpha$ and $\beta = 0$ in the results presented in Theorems 1 and 2, we obtain four formulas for the Riemann-Liouville fractional integrals and the Erdélyi-Kober fractional integrals associated with the generalized $\mathrm{k}$-Struve function, which are recorded in the following corollary.

**Corollary 1.** *Each of the following formulas holds*:

(i) *Let* $\alpha, \eta, \lambda \in \mathbb{C}$ *and* $v \in \mathbb{R}$, $\mathrm{k} \in \mathbb{R}^{+}$ *such that* $\Re(\alpha) > 0$ *and* $v + \mathrm{k} > 0$

$$\Re(\lambda) + v > \mathrm{k}\max\{0, -\Re(\alpha + \eta)\}.$$

*Then, for* $x \in \mathbb{R}^{+}$,

$$\left(I_{0+}^{\alpha} t^{\frac{\lambda}{\mathrm{k}}-1} S_{\nu, c}^{\mathrm{k}}(t)\right)(x)$$

$$= \frac{x^{\frac{\nu}{\mathrm{k}}+\frac{\lambda}{\mathrm{k}}+\alpha}}{2^{\frac{\nu}{\mathrm{k}}+1} \mathrm{k}^{\frac{\nu}{\mathrm{k}}+\frac{1}{2}}} {}_{1}\Psi_{3}\left[\begin{matrix} \left(\frac{\nu}{\mathrm{k}}+\frac{\lambda}{\mathrm{k}}+1,\ 2\right); \\ \left(1+\frac{\nu}{\mathrm{k}}+\frac{\lambda}{\mathrm{k}}+\alpha,\ 2\right), \left(\frac{\nu}{\mathrm{k}}+\frac{3}{2}, 1\right), \left(\frac{3}{2}, 1\right); \end{matrix} -\frac{cx^{2}}{4\mathrm{k}}\right]. \tag{2.2}$$

(ii) *Let* $\alpha, \eta, \lambda \in \mathbb{C}$ *and* $\nu \in \mathbb{R}, \mathrm{k} \in \mathbb{R}^{+}$ *such that* $\Re(\alpha) > 0$ *and* $\nu + \mathrm{k} > 0$

$$\Re(\lambda) + \nu > \mathrm{k} \max\{0, -\Re(\eta)\}.$$

*Then, for* $x \in \mathbb{R}^{+}$,

$$\left(K_{\alpha, \eta}^{+} t^{\frac{\lambda}{\mathrm{k}}-1} S_{\nu, c}^{\mathrm{k}}(t)\right)(x)$$

$$= \frac{x^{\frac{\nu}{\mathrm{k}}+\frac{\lambda}{\mathrm{k}}}}{2^{\frac{\nu}{\mathrm{k}}+1} \mathrm{k}^{\frac{\nu}{\mathrm{k}}+\frac{1}{2}}} {}_{1}\Psi_{3}\left[\begin{matrix} \left(1+\frac{\nu}{\mathrm{k}}+\frac{\lambda}{\mathrm{k}}+\eta,\ 2\right); \\ \left(1+\frac{\nu}{\mathrm{k}}+\frac{\lambda}{\mathrm{k}}+\alpha+\eta,\ 2\right), \left(\frac{\nu}{\mathrm{k}}+\frac{3}{2}, 1\right), \left(\frac{3}{2}, 1\right); \end{matrix} -\frac{cx^{2}}{4\mathrm{k}}\right]. \tag{2.3}$$

(iii) *Let* $\alpha, \eta, \lambda \in \mathbb{C}$ *and* $\nu \in \mathbb{R}, \mathrm{k} \in \mathbb{R}^{+}$ *such that* $\Re(\alpha) > 0$ *and* $\nu + \mathrm{k} > 0$

$$\Re(\lambda) < \nu + \mathrm{k} + \mathrm{k} \min\{-\Re(\alpha), \Re(\eta)\}.$$

*Then, for* $x \in \mathbb{R}^{+}$,

$$\left(I_{-}^{\alpha} t^{\frac{\lambda}{\mathrm{k}}-1} S_{\nu, c}^{\mathrm{k}}(1/t)\right)(x)$$

$$= \frac{x^{\frac{\lambda}{\mathrm{k}}-\frac{\nu}{\mathrm{k}}+\alpha}}{(2)^{\frac{\nu}{\mathrm{k}}+1} \mathrm{k}^{\frac{\nu}{\mathrm{k}}+\frac{1}{2}}} {}_{1}\Psi_{3}\left[\begin{matrix} \left(2-\alpha-\frac{\lambda}{\mathrm{k}}+\frac{\nu}{\mathrm{k}},\ 2\right); \\ \left(2-\frac{\lambda}{\mathrm{k}}+\frac{\nu}{\mathrm{k}},\ 2\right), \left(\frac{\nu}{\mathrm{k}}+\frac{3}{2}, 1\right), \left(\frac{3}{2}, 1\right); \end{matrix} -\frac{c}{4\mathrm{k}x^{2}}\right]. \tag{2.4}$$

(iv) *Let* $\alpha, \eta, \lambda \in \mathbb{C}$ *and* $v \in \mathbb{R}, \mathrm{k} \in \mathbb{R}^{+}$ *such that* $\Re(\alpha) > 0$ *and* $v + \mathrm{k} > 0$

$$\Re(\lambda) < v + \mathrm{k} + \mathrm{k}\min\{0, \Re(\eta)\}.$$

*Then, for* $x \in \mathbb{R}^{+}$,

$$\left(K_{\alpha,\eta}^{-} t^{\frac{\lambda}{\mathrm{k}}-1} S_{v,c}^{\mathrm{k}}(1/t)\right)(x)$$

$$= \frac{x^{\frac{\lambda}{\mathrm{k}}-\frac{v}{\mathrm{k}}}}{2^{\frac{v}{\mathrm{k}}+1}\mathrm{k}^{\frac{v}{\mathrm{k}}+\frac{1}{2}}}\, {}_1\Psi_3\left[\begin{matrix}\left(2-\frac{\lambda}{\mathrm{k}}+\frac{v}{\mathrm{k}}+\eta,\, 2\right); \\ \left(2-\frac{\lambda}{\mathrm{k}}+\frac{v}{\mathrm{k}}+\alpha+\eta,\, 2\right), \left(\frac{v}{\mathrm{k}}+\frac{3}{2}, 1\right), \left(\frac{3}{2}, 1\right);\end{matrix}\; -\frac{c}{4\mathrm{k}x^{2}}\right]. \tag{2.5}$$

We can express the results in Theorems 1 and 2 together with Corollary 1 in terms of the generalized hypergeometric function ${}_pF_q$, by using the following duplication formula for the gamma function (see, e.g., [12, Section 1.1]):

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma(z+1/2) \tag{2.6}$$

together with (1.2). Here, we choose the result in Theorem 1 to demonstrate it as in the following corollary.

**Corollary 2.** *Let* $\alpha, \beta, \eta, \lambda \in \mathbb{C}$ *and* $v \in \mathbb{R}, \mathrm{k} \in \mathbb{R}^{+}$ *such that* $\Re(\alpha) > 0$ *and* $v + \mathrm{k} > 0$

$$\Re(\lambda) + v > \mathrm{k}\max\{0, \Re(\beta - \eta)\}.$$

*Then, for* $x \in \mathbb{R}^{+}$,

$$\left(I_{0+}^{\alpha,\beta,\eta} t^{\frac{\lambda}{\mathrm{k}}-1} S_{v,c}^{\mathrm{k}}(t)\right)(x)$$

$$= \frac{x^{\frac{\lambda}{\mathrm{k}}+\frac{v}{\mathrm{k}}-\beta}}{2^{\frac{v}{\mathrm{k}}+1}\mathrm{k}^{\frac{v}{\mathrm{k}}+\frac{1}{2}}} \frac{\Gamma\left(\frac{\lambda}{\mathrm{k}}+\frac{v}{\mathrm{k}}+1\right)\Gamma\left(\frac{\lambda}{\mathrm{k}}+\frac{v}{\mathrm{k}}+\eta-\beta+1\right)}{\Gamma\left(\frac{\lambda}{\mathrm{k}}+\frac{v}{\mathrm{k}}-\beta+1\right)\Gamma\left(\frac{\lambda}{\mathrm{k}}+\frac{v}{\mathrm{k}}+\alpha+\eta+1\right)\Gamma\left(\frac{v}{\mathrm{k}}+\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}$$

$$\times\, {}_4F_6\left[\begin{matrix}\frac{\lambda}{2k}+\frac{v}{2k}+1,\ \frac{\lambda}{2k}+\frac{v}{2k}+\frac{1}{2},\ \frac{\lambda}{2k}+\frac{v}{2k}+\frac{\eta-\beta}{2}+1,\\ \frac{v}{k}+\frac{3}{2},\ \frac{3}{2},\ \frac{\lambda}{2k}+\frac{v}{2k}-\frac{\beta}{2}+1,\ \frac{\lambda}{2k}+\frac{v}{2k}-\frac{\beta+1}{2},\ \frac{\lambda}{k}+\frac{v}{k}+\frac{\alpha+\eta}{2}+1,\end{matrix}\right.$$

$$\left.\begin{matrix}\frac{\lambda}{2k}+\frac{v}{2k}+\frac{\eta-\beta+1}{2};\\ \frac{\lambda}{2k}+\frac{v}{2k}+\frac{\alpha+\eta+1}{2};\end{matrix}\ -\frac{cx^2}{4k}\right].$$

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