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METHOD OF POINT ESTIMATION OF THE PARETO SET IN LINEAR MULTICRITERIA PROBLEM

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Abstract

We discuss multicriteria linear programming problem.

In the classical theory of decision-making, multicriteria linear programming problem is formally stated as:

$$Cx \to \max_{x \in X}, \tag{1}$$

$$X = \{x \in R^n \mid Ax \le b,\ x \ge 0\}. \tag{2}$$

Unlike conventional problem of linear programming (LP), C is a matrix of dimension $l \times n$, not a vector. Thus, multicriteria problem (1), (2) involves maximization on the polyhedron X.

Note that from the normal form of the problem (1), (2), we can easily go to its canonical form. The restriction $x \ge 0$ can be ignored.

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As a rule, the traditional solution of the problem (1), (2) does not exist. However, there is no point $x \in X$ such that $Cx \geq Cy$ for all $y \in X$, $y \neq x$. In case a person making decisions (PMD) has no a priori information on the relative importance of criteria for the solution of the problem (1), (2), we have the so-called Pareto set. We denote it by $N \subset X$. The solution $x \in N$ is called *Pareto* (*non-dominated* if it cannot be improved on any one criterion not worsen, formally

$$x \in N \Leftrightarrow (\forall y \in X,\ y \neq x) \neg ((Cy \geq Cx) \wedge (\exists i\ C^i y > Cx)),$$

where C^i is *i*th row (*i*th criterion) of matrix *C*.

The problem of Pareto set in the problem of MLP has an extensive literature. However, one of the best (if not the best) publications on the subject is, apparently, the article of American mathematicians Yu and Zeleny [1]. Here are shown theoretically well-founded methods of constructing the set of Pareto vertices $N^{ex} \in N$ and the entire Pareto set.

The construction of the set N^{ex} is represented by so-called multicriteria simplex method based on the following two fundamental theorems:

Theorem 1 [1]**.** *The set* N^{ex} *is connected*.

Theorem 2 [1]**.** $x^0 \in N \Leftrightarrow \omega = 0.$ $x^0 \in D \Leftrightarrow \omega > 0.$

Here $D = X/N,$ and ω-solution of LP problem:

$$\omega = \max \sum_{i=1}^{l} e_i, \tag{3}$$

$$\tilde{X} = \{(x,\ l) \in R^{n+l} \mid x \in X,\ Cx - e \geq Cx^0,\ e \geq 0\}. \tag{4}$$

The essence of constructing the algorithm of the set N^{ex} described in [1] is as follows. First, we find the first Pareto vertex x^1. Then, it is enough to solve the problem of LP with objective function

$$\sum_{i=1}^{l} \lambda_i C^i x \to \max_{x \in X}, \quad \lambda > 0.$$

After that while solving the problem (3), (4), we check all adjacents to x^1 point. Those will be Pareto included in N^{ex}.

It should be noted that [1] shows (see, e.g., Theorem 3.1 in [1]) a series of simple sufficient conditions for some arbitrary point $y \in X$ of the set *D*, which greatly facilitates the search.

Multicriteria simplex method [1] shows that for any point $x \in N^{ex}$, there is a set of numbers $\lambda_i \in (0, 1)$, $i = \overline{1, l}$ such that

$$x = \arg\max_{y \in X} \sum_{i=1}^{l} \lambda_i C^i y. \tag{5}$$

This means that the set can be constructed by sorting nodes of l-dimensional Σ-network on the set

$$\Lambda = \{\lambda \in R^l \mid \lambda_i \in (0, 1),\ i = \overline{1, l}\}$$

and solving for each node LP problem (5).

Further, [1] shows how on the basis of N^{ex}, the set *N* is built. In this case, *N* will be a union of Pareto convex combinations (faces of polyhedron *X*) points from N^{ex}.

Note that with the entire Pareto set *N* , it is hard to work because it contains an infinite number of possible ‘equal’ decisions. LPR as a rule, requires to implement a single solution. At the same time, all objective information are already used as in constructions of sets N^{ex} and *N*.

It would seem that the way out of this situation could be the solution of the LP problem (5) with equal coefficients:

$$\lambda_i = \frac{1}{l}.$$

This, however, is not so because of two reasons. Firstly, this method assumes the same importance for LPR of criterion, which narrows the formulation of the original problem. Secondly, such a decision ‘withdraw’ some points from N^{ex} (that is, to the top) ignoring essentially the set N/N^{ex}.

The method allows a decision maker to allocate only one point from *N* and does not require additional considerations of a subjective nature [2].

First of all, note that every Pareto solution $x \in N$ is equal in relation to other Pareto solutions (not better, but not worse even). Consequently, with allocation of a single point from *N* (i.e., at the point characterization *N*), the whole set *N* should be taken into account (even implicitly) for the implementation.

Note further that this characterization denoted by $\tilde{x}$ should reflect configuration of the set *N*, to a large extent defined by the set N^{ex}.

Based on these two reasons, the idea of searching the solution $\tilde{x} \in N$ is as follows. Count points in N^{ex} equal in weight and find their convex combination x^*:

$$x^* = \frac{1}{p} \sum_{x \in N^{ex}} x, \tag{6}$$

where *p* is the number of elements of the set N^{ex}.

It is clear that in the general case, point x^* is not Pareto. It is therefore natural to highlight the point (previously denoted by $\tilde{x}$), in the set *N* to the maximum extent ‘improving’ x^* in all criteria.

For this, we use Theorem 2 and solve the problem (3) on the set

$$X^* = \{(x, l) \in R^{n+l} \mid Cx - e \geq Cx^*, \, e \geq 0\}. \tag{7}$$

The resulting solution is the required characterization $\tilde{x}$ of the set *N*.

Example. In order to maintain consistency of presentation, we take the example from [1]:

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 & 2 & 1 & 2 \\ -2 & -1 & 0 & 1 & 2 & 0 & 1 \\ -1 & 0 & 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & 2 & -1 & 1 & -2 & -1 \end{pmatrix};$$

$$b = \begin{pmatrix} 16 \\ 16 \\ 16 \\ 16 \end{pmatrix};$$

$$C = \begin{pmatrix} 1 & 2 & -1 & 3 & 2 & 0 & 1 \\ 0 & 1 & 1 & 2 & 3 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 & -1 & -1 \end{pmatrix}.$$

To solve this problem with the help of multicriteria simplex method, the set N^{ex} amounts following six points:

$$x^1 = (0, 0, 0, 0, 8, 0, 0),$$

$$x^2 = (0, 0, 0, 16, 0, 0, 0),$$

$$x^3 = (16, 0, 0, 0, 0, 0, 0),$$

$$x^4 = (8, 0, 8, 0, 0, 0, 0),$$

$$x^5 = \left(0, 0, \frac{32}{3}, \frac{16}{3}, 0, 0, 0\right),$$

$$x^6 = \left(0, 0, \frac{16}{3}, 0, \frac{16}{3}, 0, 0\right).$$

The image of the set N^{ex} in the criterion space consists of points:

$$Cx^1 = (16, 24, 0)',$$

$$Cx^2 = (48, 32, -16)',$$

$$Cx^3 = (16,\ 0,\ 16)',$$

$$Cx^4 = (0,\ 8,\ 16)',$$

$$Cx^5 = \left(\frac{16}{3},\ \frac{64}{3},\ \frac{16}{3}\right)',$$

$$Cx^6 = \left(\frac{16}{3},\ \frac{64}{3},\ \frac{16}{3}\right)'.$$

According to the formula (6), we find a convex combination of points from N^{ex} with equal weights:

$$x^* = (4,\ 0,\ 4,\ 3.55,\ 2.2,\ 0,\ 0).$$

Further, by solving the problem (3) on the set (7), we find a point characterization $\tilde{x}$ of the Pareto set N as follows:

$$\tilde{x} = (4.075,\ 0,\ 0.375,\ 0,\ 5.775,\ 0,\ 0).$$

The distance from the point Cx^* to the Pareto set in the criterion space is equal to 0.4 $\left(\omega = \sum_{i=1}^{3} e_i = 0.4\right)$.

The final solution of the problem is the point

$$C\tilde{x} = (15.25,\ 17.7,\ 4.45)'.$$

Note that if we solve the problem (5) with equal weights, then the decision will be the point $x^2 \left(\sum_{i=1}^{3} C^i x_i^2 = 64\right)$.

Other aspect of this problem is considered in [3-7].

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