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SEMIDETACHED *B*-ALGEBRAS BASED ON FUZZY POINTS

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Abstract

The concepts of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subalgebra, $(\overline{q}, \overline{\in} \vee \overline{q})$-fuzzy subalgebra and $(\overline{\in} \vee \overline{q}, \overline{\in} \vee \overline{q})$-fuzzy subalgebra are introduced, and relative relations and properties are discussed. Several conditions for a semidetached structure in *B*-algebras to be a semidetached *B*-algebra are provided.

1. Introduction and Preliminaries

The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [11], played a vital role to generate some different types of fuzzy subgroups, called (α, β)-*fuzzy subgroups*, introduced by Bhakat and Das [1]. In particular, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. In BCK/BCI-algebras and

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B-algebras, the concept of (α, β)-fuzzy *B*-algebras, which is studied in the papers [2-6, 12] and [13], is also important and useful generalization of the well-known concepts, called *fuzzy subalgebras*. In [10], Jun and Song introduced the notion of semidetached *B*-algebras, and investigated their properties. We provided several conditions for a semidetached structure to be a semidetached *B*-algebra. We considered characterization of a semidetached *B*-algebra.

In this paper, we introduce the concepts of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy *B*-algebra, $(\overline{q}, \overline{\in} \vee \overline{q})$-fuzzy *B*-algebra and $(\overline{\in} \vee \overline{q}, \overline{\in} \vee \overline{q})$-fuzzy *B*-algebra, and investigate relative relations and properties. We provide several conditions for a semidetached structure in *B*-algebras to be a semidetached *B*-algebra.

2. Preliminaries

A *B*-algebra is a set *X* with a constant 0 and a binary operation '$*$' satisfying the axioms:

(a1) $x * x = 0$,

(a2) $x * 0 = x$,

(a3) $(x * y) * z = x * (z * (0 * y))$

for all $x, y, z \in X$.

A nonempty subset *A* of a *B*-algebra *X* is called a *subalgebra* of *X* if $x * y \in A$ for all $x, y \in A$.

A fuzzy set λ in a *B*-algebra *X* is called a *fuzzy B-algebra* of *X* (see [9]) if it satisfies:

$$(\forall x, y \in X)(\lambda(x * y) \geq \min\{\lambda(x), \lambda(y)\}). \tag{2.1}$$

For any fuzzy set λ in a set *X* and any $t \in [0, 1]$, the set

$$\lambda_t = \{x \in X \mid \lambda(x) \geq t\}$$

is called a *level subset* of λ.

Note that a fuzzy set λ in X is a fuzzy B-algebra of X if and only if λ_t is a subalgebra of X for all $t \in (0, 1]$.

A fuzzy set λ in a set X of the form

$$\lambda(y) := \begin{cases} t \in (0, 1] & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases} \tag{2.2}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t.

For a fuzzy set λ in a set X, a fuzzy point x_t is said to be

- *contained* in λ, denoted by $x_t \in \lambda$ (see [11]), if $\lambda(x) \geq t$,
- *quasi-coincident* with λ, denoted by $x_t q\lambda$ (see [11]), if $\lambda(x) + t > 1$,
- $x_t \in \vee q\lambda$ if $x_t \in \lambda$ or $x_t q\lambda$.

3. Semidetached B-algebras

In what follows, let X denote a B-algebra unless otherwise specified.

Definition 3.1 [7]**.** A fuzzy set λ in X is called an $(\in, \in\vee q)$-*fuzzy* *B-algebra* of X if it satisfies:

$$x_t \in \lambda,\ y_r \in \lambda \Rightarrow (x * y)_{\min\{t, r\}} \in \vee q\lambda \tag{3.1}$$

for all $x, y \in X$ and $t, r \in (0, 1]$.

Given a set X and a subinterval Ω of [0, 1], a *semidetached structure* over Ω is defined to be a pair (X, f), where $f : \Omega \to \mathcal{P}(X)$ is a mapping (see [10]).

Definition 3.2 [8]**.** A semidetached structure (X, f) is called a *semidetached* *B-algebra* over Ω with respect to $t \in \Omega$ (briefly, t-semidetached B-algebra over Ω) if $f(t)$ is a B-subalgebra of X where $\mathcal{P}(X)$ is the power set of X.

We say that (X, f) is a *semidetached B-algebra* over Ω if it is a t-semidetached B-algebra over Ω with respect to all $t \in \Omega$.

Given a fuzzy set λ in X, consider the following mappings:

$$\mathcal{A}_U^{\lambda} : \Omega \to \mathcal{P}(X),\ t \mapsto \lambda_t, \tag{3.2}$$

$$\mathcal{A}_Q^{\lambda} : \Omega \to \mathcal{P}(X),\ t \mapsto Q(\lambda;\ t), \tag{3.3}$$

$$\mathcal{A}_{\mathcal{E}}^{\lambda} : \Omega \to \mathcal{P}(X),\ t \mapsto \mathcal{E}(\lambda;\ t), \tag{3.4}$$

where

$$Q(\lambda;\ t) := \{x \in X \mid x_t q \lambda\} \quad \text{and} \quad \mathcal{E}(\lambda;\ t) := \{x \in X \mid x_t \in \vee q\lambda\}$$

which are called the *q-set* and $\in \vee\, q$ *-set* with respect to t (briefly, t-q-set and t-$\in \vee\, q$ -set), respectively, of λ.

Note that, for any $t, r \in (0, 1],$ if $t \geq r,$ then every r-q-set is contained in the t-q-set, that is, $Q(\lambda;\ r) \subseteq Q(\lambda;\ t)$. Obviously, $\mathcal{E}(\lambda;\ t) = \lambda_t \cup Q(\lambda;\ t)$.

For $\alpha \in \{\in, q\}$ and $t \in (0, 1],$ we say that $x_t \overline{\alpha} \lambda$ if $x_t \alpha \lambda$ does not hold.

Definition 3.3. A fuzzy set λ in X is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$*-fuzzy B-algebra* of X if it satisfies:

$$(\forall x, y \in X)(\forall t, r \in (0, 1])((x * y)_{\min\{t, r\}}\ \overline{\in}\ \lambda \Rightarrow x_t\ \overline{\in} \vee \overline{q}\, \lambda$$
$$\text{or } y_r\ \overline{\in} \vee \overline{q}\, \lambda). \tag{3.5}$$

We provide a characterization of an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy B-algebra.

Theorem 3.4. *A fuzzy set λ in X is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy B-algebra of X if and only if the following inequality is valid*:

$$(\forall x, y \in X)(\max\{\lambda(x * y), 0.5\} \geq \min\{\lambda(x), \lambda(y)\}). \tag{3.6}$$

Proof. Let λ be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy B-algebra of X. Assume that (3.6) is not valid. Then there exist $a, b \in X$ such that

$$\max\{\lambda(a * b),\, 0.5\} < \min\{\lambda(a),\, \lambda(b)\} \overset{\Delta}{=} t.$$

Then $0.5 < t \leq 1$, $a_t \in \lambda$, $b_t \in \lambda$ and $(a * b)_t \,\overline{\in}\, \lambda$. It follows from (3.5) that $a_t \overline{q} \lambda$ or $b_t \overline{q} \lambda$. Hence

$$\lambda(a) \geq t \quad \text{and} \quad \lambda(a) + t \leq 1$$

or

$$\lambda(b) \geq t \quad \text{and} \quad \lambda(b) + t \leq 1.$$

In either case, we have $t \leq 0.5$ which is a contradiction. Therefore

$$\max\{\lambda(x * y),\, 0.5\} \geq \min\{\lambda(x),\, \lambda(y)\}$$

for all $x, y \in X$.

Conversely, suppose that (3.6) is valid. Let $(x * y)_{\min\{t,\, r\}} \,\overline{\in}\, \lambda$ for $x, y \in X$ and $t, r \in (0, 1]$. Then $\lambda(x * y) < \min\{t, r\}$. If $\max\{\lambda(x * y), 0.5\} = \lambda(x * y)$, then $\min\{t,\, r\} > \lambda(x * y) \geq \min\{\lambda(x),\, \lambda(y)\}$ and so $\lambda(x) < t$ or $\lambda(y) < r$. Thus, $x_t \,\overline{\in}\, \lambda$ or $y_r \,\overline{\in}\, \lambda$, which implies that $x_t \,\overline{\in \vee q}\, \lambda$ or $y_r \,\overline{\in \vee q}\, \lambda$. If $\max\{\lambda(x * y), 0.5\} = 0.5$, then $\min\{\lambda(x), \lambda(y)\} \leq 0.5$. Suppose $x_t \in \lambda$ or $y_r \in \lambda$. Then $t \leq \lambda(x) \leq 0.5$ or $r \leq \lambda(y) \leq 0.5$, and so

$$\lambda(x) + t \leq 0.5 + 0.5 = 1$$

or

$$\lambda(y) + r \leq 0.5 + 0.5 = 1.$$

Hence $x_t \overline{q} \lambda$ or $y_r \overline{q} \lambda$. Therefore, $x_t \,\overline{\in \vee q}\, \lambda$ or $y_r \,\overline{\in \vee q}\, \lambda$. This shows that λ is an $(\overline{\in}, \overline{\in \vee q})$-fuzzy B-algebra of X. □

Theorem 3.5. *A fuzzy set λ in X is an $(\overline{\in}, \overline{\in \vee q})$-fuzzy B-algebra of X if and only if $(X, \mathcal{A}_U^{\lambda})$ is a semidetached B-algebra over $\Omega = (0.5, 1]$.*

Proof. Assume that λ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy B-algebra of X. Let $x, y \in \mathcal{A}_U^{\lambda}(t)$ for $t \in \Omega = (0.5, 1]$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq t$. It follows from (3.6) that

$$\max\{\lambda(x * y), 0.5\} \geq \min\{\lambda(x), \lambda(y)\} \geq t.$$

Since $t > 0.5$, it follows that $\lambda(x * y) \geq t$ and so that $x * y \in \mathcal{A}_U^{\lambda}(t)$. Thus, $\mathcal{A}_U^{\lambda}(t)$ is a subalgebra of X, and $(X, \mathcal{A}_U^{\lambda})$ is a semidetached B-algebra over $\Omega = (0.5, 1]$.

Conversely, suppose that $(X, \mathcal{A}_U^{\lambda})$ is a semidetached B-algebra over $\Omega = (0.5, 1]$. If (3.6) is not valid, then there exist $a, b \in X$ such that

$$\max\{\lambda(a * b), 0.5\} < \min\{\lambda(a), \lambda(b)\} \stackrel{\Delta}{=} t.$$

Then $t \in (0.5, 1]$, $a, b \in \mathcal{A}_U^{\lambda}(t)$ and $ab \notin \mathcal{A}_U^{\lambda}(t)$. This is a contradiction, and so (3.6) is valid. Using Theorem 3.4, we know that λ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy B-algebra of X. $\square$

Theorem 3.6. *A fuzzy set* λ *in X is an* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-fuzzy B-algebra of X if and only if* $(X, \mathcal{A}_Q^{\lambda})$ *is a semidetached B-algebra over* $\Omega = (0, 0.5]$.

Proof. Assume that $(X, \mathcal{A}_Q^{\lambda})$ is a semidetached B-algebra over $\Omega = (0, 0.5]$. If (3.6) is not valid, then there exist $a, b \in X$, $t \in \Omega$ and $k \in [0, 1)$ such that

$$\max\{\lambda(a * b), 0.5\} + t \leq 1 < \min\{\lambda(a), \lambda(b)\} + t.$$

It follows that $a_t q \lambda$ and $b_t q \lambda$, that is, $a, b \in \mathcal{A}_Q^{\lambda}(t)$, but $(a * b)_t \overline{q} \lambda$, i.e., $ab \notin \mathcal{A}_Q^{\lambda}(t)$. This is a contradiction, and so (3.6) is valid. Using Theorem 3.4, we know that λ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy B-algebra of X.

Conversely, suppose that λ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy B-algebra of X. Let $x, y \in \mathcal{A}_Q^{\lambda}(t)$ for $t \in \Omega = (0, 0.5]$. Then $x_t q\lambda$ and $y_t q\lambda$, that is, $\lambda(x) + t > 1$ and $\lambda(y) + t > 1$. It follows from (3.6) that

$$\max\{\lambda(x * y), 0.5\} \geq \min\{\lambda(x), \lambda(y)\} > 1 - t \geq 0.5$$

and so that $\lambda(x * y) + t > 1$, that is, $x * y \in \mathcal{A}_Q^{\lambda}(t)$. Therefore, $\mathcal{A}_Q^{\lambda}(t)$ is a subalgebra of X, and $(X, \mathcal{A}_Q^{\lambda})$ is a semidetached B-algebra over $\Omega = (0, 0.5]$. □

Definition 3.7. A fuzzy set λ in X is called an $(\overline{\in} \vee \overline{q}, \overline{\in} \vee \overline{q})$-*fuzzy B-algebra* of X if for all $x, y \in X$ and $t, r \in (0, 1]$,

$$(x * y)_{\min\{t, r\}} \overline{\in} \vee \overline{q}\lambda \Rightarrow x_t \overline{\in} \vee \overline{q}\lambda \text{ or } y_r \overline{\in} \vee \overline{q}\lambda. \tag{3.7}$$

Theorem 3.8. *Every* $(\overline{\in} \vee \overline{q}, \overline{\in} \vee \overline{q})$-*fuzzy B-algebra is an* $(\overline{\in}, \overline{\in} \vee \overline{q})$-*fuzzy B-algebra*.

Proof. Let $x, y \in X$ and $t, r \in (0, 1]$ be such that $(x * y)_{\min\{t, r\}} \overline{\in} \lambda$. Then $(x * y)_{\min\{t, r\}} \overline{\in} \vee \overline{q}\lambda$, and so $x_t \overline{\in} \vee \overline{q}\lambda$ or $y_r \overline{\in} \vee \overline{q}\lambda$ by (3.7). Therefore, λ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy B-algebra of X. □

Definition 3.9. A fuzzy set λ in X is called a $(\overline{q}, \overline{\in} \vee \overline{q})$-*fuzzy B-algebra* of X if for all $x, y \in X$ and $t, r \in (0, 1]$,

$$(x * y)_{\min\{t, r\}} \overline{q}\lambda \Rightarrow x_t \overline{\in} \vee \overline{q}\lambda \quad \text{or} \quad y_r \overline{\in} \vee \overline{q}\lambda. \tag{3.8}$$

Theorem 3.10. *Assume that* $\min\{t, r\} \leq 0.5$ *for any* $t, r \in (0, 1]$. *Then every* $(\overline{q}, \overline{\in} \vee \overline{q})$-*fuzzy B-algebra is an* $(\overline{\in}, \overline{\in} \vee \overline{q})$-*fuzzy B-algebra*.

Proof. Let λ be a $(\overline{q}, \overline{\in} \vee \overline{q})$-fuzzy B-algebra of X. Assume that $(x * y)_{\min\{t, r\}} \overline{\in} \lambda$ for $x, y \in X$ and $t, r \in (0, 1]$ with $\min\{t, r\} \leq 0.5$. Then $\lambda(x * y) < \min\{t, r\} \leq 0.5$, and so

$$\lambda(x * y) + \min\{t, r\} < 0.5 + 0.5 = 1,$$

that is, $(x * y)_{\min\{t, r\}} \overline{q} \lambda$. It follows from (3.8) that $x_t \,\overline{\in} \vee \overline{q}\, \lambda$ or $y_r \,\overline{\in}\, \overline{q}\, \lambda$. Therefore, λ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy *B*-algebra of *X*. □

Theorem 3.11. *Assume that* $\min\{t, r\} > 0.5$ *for any* $t, r \in (0, 1]$. *Then every* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-fuzzy B-algebra is a* $(\overline{q}, \overline{\in} \vee \overline{q})$*-fuzzy B-algebra.*

Proof. Let λ be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy *B*-algebra of *X*. Assume that $(x * y)_{\min\{t, r\}} \overline{q} \lambda$ for $x, y \in X$ and $t, r \in (0, 1]$ with $\min\{t, r\} > 0.5$. If $(x * y)_{\min\{t, r\}} \in \lambda$, then $\lambda(x * y) \geq \min\{t, r\}$ and so

$$\lambda(x * y) + \min\{t, r\} > 0.5 + 0.5 = 1.$$

Hence $(x * y)_{\min\{t, r\}} q \lambda$, a contradiction. Thus $(x * y)_{\min\{t, r\}} \,\overline{\in}\, \lambda$, which implies from (3.5) that $x_t \,\overline{\in} \vee \overline{q}\, \lambda$ or $y_r \,\overline{\in} \vee \overline{q}\, \lambda$. Therefore, λ is a $(\overline{q}, \overline{\in} \vee \overline{q})$-fuzzy *B*-algebra of *X*. □

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