Advances and Applications in Statistics
© 2017 Pushpa Publishing House, Allahabad, India
http://www.pphmj.com
http://dx.doi.org/10.17654/AS050050423
Volume 50, Number 5, 2017, Pages 423-434 **ISSN: 0972-3617**

A NEW MODIFICATION OF THE CUADRAS COPULA

Rasha Ebaid* and Walid Elbadawy

Department of Mathematics, Insurance, and Applied Statistics
Faculty of Commerce
Helwan University
Cairo, Egypt

Abstract

Copulas were introduced as a method for modeling the dependence structure between random variables and for constructing a multivariate distribution function from its marginals. Many copulas were introduced in the literature to suit different models having different properties. Among which is the Cuadras copula introduced by Cuadras [3]. A drawback of the Cuadras copula is the limited range of dependence structure that it allows for modeling random variables. In this paper, we propose a new extension of the Cuadras copula that permits a wider range of dependence between the involved random variables, thus opening the way for an increased number of applications. Properties of the new extended copula with an example illustrating our findings will be given.

Introduction

Copulas are gaining more interest in high-dimensional statistical

Received: January 12, 2017; Revised: February 19, 2017; Accepted: February 23, 2017
2010 Mathematics Subject Classification: 97K50, 97K70.
Keywords and phrases: copula, dependence structure, Spearman's rho, Cuadras copula, exponential distribution.

*Corresponding author

applications for two main reasons; they allow one to model the dependence structure between random variables, and construct the multivariate distribution of random variables from its marginals. A copula function is a multivariate distribution function which joins or "couples" marginal distribution functions (cdfs), where the marginal distribution is uniform on the interval $[0, 1]$. Many copulas have recently been proposed in the literature to suit different models with different properties (see Hutchinson and Lai [6], Joe [7], Drouet-Mari and Kotz [5], Nelsen [10], and Balakrishnan and Lai [2]). Cuadras copula introduced by Cuadras [3] is one of the most important bivariate copulas introduced in the literature with one drawback that it does not allow the modeling of large dependence between the associated random variables. In [8], Klein and Christa proposed a new extension on the Cuadras copula with no improvement on the level of dependence structure. In [1], Abdelghaly et al. proposed another extension which admitted an increased level of dependence between the associated variables. In this paper, we will propose a new extension of the Cuadras copula. This new extension has admitted a wider range of dependence structure between the associated variables, and hence can be used in a wider range of applications. The rest of this paper is organized as follows. A brief introduction to copulas and their properties are given. The Cuadras copula is introduced. The new extended copula and its properties are then introduced with an example.

Background and Definitions

The concept of copulas dates back to Sklar [12]. Sklar's theorem, proves how copulas link joint distribution functions with their marginals.

Sklar's Theorem. *Suppose H is a joint bivariate distribution function that has marginal distribution functions F and G. Then, C is a copula for all those* $x, y \in (-\infty, \infty)$,

$$H(x, y) = C(F(x), G(y)).$$

If F and G are continuous, then the copula C is unique; but if F and G are not continuous, then C is uniquely specified on (*Range of F* $\times$ *Range of G*).

On the contrary, if C is a copula and F and G are the marginal cdfs of variables x and y, respectively, then H is a joint bivariate distribution function.

A two-dimensional copula is a bivariate distribution function whose marginals (F and G) are uniform on $[0, 1]$, then $H(x, y) = C(x, y)$, which indicates that the copula is in the form of a bivariate distribution function.

The joint density function $h(x, y)$ can be obtained by

$$h(x, y) = c(F(x), G(y)) f(x) g(y), \tag{1}$$

where

$$c(F(x), G(y)) = \frac{\partial^2 C(F(x), G(y))}{\partial F(x) \partial G(y)}$$

is the copula density function, and $f(x)$, $g(y)$ denote the marginal density functions.

Hence,

$$h(x, y) = \frac{\partial^2 H(x, y)}{\partial x \partial y} = \frac{\partial^2 C(F(x), G(y))}{\partial x \partial y}.$$

In general, copulas depend on one or more parameters - at least one - called association parameters related to the level of dependence between margins.

Copula properties

Let $C(F(x), G(y))$ denote a bivariate copula. Then, (Nelsen [9]):

- For every $F(x), G(y) \in [0, 1]$, $C(F(x), 0) = C(0, G(y)) = 0$,

$$C(F(x), 1) = F(x),\ C(1, G(y)) = G(y).$$

- For every $u_1, u_2, v_1, v_2 \in [0, 1]$, $u_1 \leq u_2$, and $v_1 \leq v_2$,

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0.$$

- C is non-decreasing in each variable.

Measures of dependence between two variables

Several dependence measures are "scale-invariant" and can be formulated using copulas. The most widely used measure of dependence between two random variables is Spearman's rho (ρ), which is the coefficient of correlation of the marginal distribution functions $F(X)$, and $G(Y)$. Spearman's rho can be expressed by using the copula function as

$$\rho = 12\int_0^1\int_0^1 C(F(x), G(y))dF(x)dG(y) - 3. \tag{2}$$

Cuadras copula

The single-parameter Cuadras copula, introduced by Cuadras [3], takes the form (Zhang et al. [13])

$$C(u, v) = uve^{a(1-u)(1-v)}, \quad -1 \leq a \leq 1$$

and the copula density function is given by

$$c(u, v) = e^{a(1-u)(1-v)}\{1 + a[uv - u(1-v) - v(1-u) + auv(1-u)(1-v)]\}.$$

The Spearman's rho (ρ) of Cuadras copula, which is limited to $[-0.29616, 0.380616]$, is given by:

$$\rho = 12\sum_{i=1}^{\infty}\frac{a^i}{i!}B(2, i+1)^2.$$

In [8], Klein, and Christa extended the Cuadras copula with an additional parameter on the copula defined by

$$C(u, v) = uve^{a(1-u^b)(1-v^b)}, \quad -1 \leq a \leq 1, \quad 0 < b \leq 1.$$

However, this extension did not improve the dependence structure.

In [1], Abdelghaly et al. proposed another extension on the Cuadras copula defined by

$$C(u, v) = uve^{a(1-u^b)(1-v^b)}, \quad -\frac{1}{b^2} \leq a \leq \frac{1}{b}, \ \ b > 0.$$

This extension has admitted an increased level of dependence, [–0.523848, 0.421596]. In this paper, we will propose a new modification on the Cuadras copula and determine the range of the parameters in the new modified Cuadras copula. We will investigate the corresponding level of dependence that it permits between the associated variables. Properties of the new modified Cuadras copula will also be provided.

The proposed copula

In this section, we introduce a new modification on the Cuadras copula defined as

$$C(u, v) = uve^{a(1-u)^b(1-v)^b}, \quad b \geq 1. \tag{3}$$

If $b = 1$, we then have the classical Cuadras copula. The new modified Cuadras copula density is given by

$$\begin{aligned} c(u, v) = e^{a(1-u)^b(1-v)^b}\{1 + ab[&abuv(1-u)^{2b-1}(1-v)^{2b-1} \\ &+ buv(1-u)^{b-1}(1-v)^{b-1} - u(1-u)^{b-1}(1-v)^b \\ &- v(1-u)^b(1-v)^{b-1}]\}, \end{aligned} \tag{4}$$

which is reduced to the classical Cuadras copula density when $b = 1$.

To determine the appropriate range of the parameter *a*, we determine the range of values of parameter *a* for which the copula density is non-negative,

$$c(u, v) \geq 0. \tag{5}$$

Using Maclaurin series, we have

$$e^{a(1-u)^b(1-v)^b} = \sum_{i=0}^{\infty} \frac{a^i}{i!}(1-u)^{ib}(1-v)^{ib}.$$

Accordingly, the new Cuadras copula can be expressed as follows

$$C(u, v) = \sum_{i=0}^{\infty} \frac{a^i}{i!} uv(1-u)^{ib}(1-v)^{ib},$$

and its density as

$$c(u,\ v) = \sum_{i=0}^{\infty} \frac{a^i}{i!} (1-u)^{ib-1} (1-v)^{ib-1} [1-(ib+1)u][1-(ib+1)v], \quad (6)$$

$$\frac{\partial c(u,\ v)}{\partial u} = \sum_{i=0}^{\infty} \frac{a^i}{i!} ib(1-u)^{ib-2} (1-v)^{ib-1} [(ib+1)u-2][1-(ib+1)v] = 0,$$

$$\frac{\partial c(u,\ v)}{\partial v} = \sum_{i=0}^{\infty} \frac{a^i}{i!} ib(1-u)^{ib-1} (1-v)^{ib-2} [1-(ib+1)u][(ib+1)v-2] = 0.$$

From which we get $u^* = \frac{2}{ib+1}$, and $v^* = \frac{2}{ib+1}$, as an extreme point of $c(u,\ v)$. The second partial derivatives are given by:

$$\frac{\partial^2 c(u,\ v)}{\partial u^2}$$

$$= \sum_{i=0}^{\infty} \frac{a^i}{i!} ib(ib-1)(1-u)^{ib-3} (1-v)^{ib-1} [3-(ib+1)u][1-(ib+1)v]$$

$$\frac{\partial^2 c(u,\ v)}{\partial v^2}$$

$$= \sum_{i=0}^{\infty} \frac{a^i}{i!} ib(ib-1)(1-u)^{ib-1} (1-v)^{ib-3} [1-(ib+1)u][3-(ib+1)v]$$

$$\frac{\partial^2 c(u,\ v)}{\partial u \partial v}$$

$$= \sum_{i=0}^{\infty} \frac{a^i}{i!} (ib)^2 (1-u)^{ib-2} (1-v)^{ib-2} [(ib+1)u-2][(ib+1)v-2].$$

Since

$$\frac{\partial^2 c(u^*,\ v^*)}{\partial u^2} = -\sum_{i=0}^{\infty} \frac{1}{i!} ib(ib-1) \left[-\left(\frac{b+1}{b-1}\right)^{2b-2} \right]^i \left(\frac{ib-1}{ib+1}\right)^{2ib-1}$$

$$\simeq 0 + b(b-1)\left(\frac{b+1}{b-1}\right)^2 > 0,$$

$$\frac{\partial^2 c(u^*, v^*)}{\partial v^2} \simeq 0 + b(b-1)\left(\frac{b+1}{b-1}\right)^2 > 0,$$

and

$$\frac{\partial^2 c(u^*, v^*)}{\partial u^2} \cdot \frac{\partial^2 c(u^*, v^*)}{\partial v^2} - \left[\frac{\partial^2 c(u^*, v^*)}{\partial u \partial v}\right]^2 = b^2(b-1)^2\left(\frac{b+1}{b-1}\right)^4 > 0.$$

$c(u, v)$ attains its minimum when $u^* = \dfrac{2}{ib+1}$, and $v^* = \dfrac{2}{ib+1}$. Hence,

$$c(u^*, v^*) = \sum_{i=0}^{\infty} \frac{a^i}{i!}\left(\frac{ib-1}{ib+1}\right)^{2ib-2} \geq 0.$$

One can easily check that this is a convergent series in a, and accordingly

$$1 + a\left(\frac{b-1}{b+1}\right)^{2b-2} \geq 0,$$

or

$$a \geq -\left(\frac{b+1}{b-1}\right)^{2b-2}.$$

Consider a univariate copula density function

$$\phi(u) = \sum_{i=0}^{\infty} \frac{a^i}{i!}(1-u)^{ib-1}[1-(ib+1)u], \tag{7}$$

$$\frac{\partial \phi(u)}{\partial u} = \sum_{i=0}^{\infty} \frac{a^i}{i!} ib(1-u)^{ib-2}[(ib+1)u-2] = 0,$$

from which we get $u^* = \dfrac{2}{ib+1}$ as an extreme point of $\phi(u)$,

$$\frac{\partial^2 \phi(u)}{\partial u^2} = \sum_{i=0}^{\infty} \frac{a^i}{i!} ib(ib-1)(1-u)^{ib-3}[3-(ib+1)u]$$

$$\frac{\partial^2 \phi(u^*)}{\partial u^2} = \sum_{i=0}^{\infty} \frac{1}{i!} ib(ib-1)\left(\frac{b+1}{b-1}\right)^{ib-i}\left(\frac{ib-1}{ib+1}\right)^{ib-3}$$

$$\simeq 0 + b(b-1)\left(\frac{b+1}{b-1}\right)^2 > 0.$$

Thus, u^* is a minimum point of ϕ, and accordingly,

$$\phi(u^*) = -\sum_{i=0}^{\infty} \frac{a^i}{i!}\left(\frac{ib-1}{ib+1}\right)^{ib-1} \geq 0$$

and the nonnegative condition becomes

$$1 - a\left(\frac{b-1}{b+1}\right)^{b-1} \geq 0$$

or equivalently,

$$a \leq \left(\frac{b+1}{b-1}\right)^{b-1}.$$

Therefore, the admissible range of a is

$$-\left(\frac{b+1}{b-1}\right)^{2b-2} \leq a \leq \left(\frac{b+1}{b-1}\right)^{b-1}.$$

Using the Maclaurin series, the Spearman's rho for X and Y is given by

$$\rho = 12\sum_{i=0}^{\infty} \frac{a^i}{i!} \int_0^1 v(1-v)^{ib} \int_0^1 u(1-u)^{ib}\, dudv - 3$$

$$= 12\sum_{i=0}^{\infty} \frac{a^i}{i!} [B(2, ib+1)]^2 - 3$$

$$= 12\sum_{i=1}^{\infty} \frac{a^i}{i!} [B(2, ib+1)]^2.$$

If $a = 0$, the Spearman's rho (ρ) will equal zero, and then X and Y are independent. If $b = 1$, we obtain the Spearman's rho for the classical Cuadras copula which is restricted to $[-0.29616, 0.380616]$. However, our new proposed copula has permitted a wider range of dependence structure between the associated variables as can be seen from Table 1, where the strongest positive correlation $\rho = 0.47826$ and the weakest negative correlation $\rho = -0.53857$ are attained.

Table 1. Spearman's rho (ρ) for the new extended copula

b	a		ρ	
	Lower bound	Upper bound	Lower bound	Upper bound
1	–1	1	–0.29616	0.380616
1.0001	–1.00198	1.000991	–0.29664	0.380976
1.001	–1.01532	1.00763	–0.29971	0.383256
1.01	–1.1119	1.054464	–0.32014	0.397404
1.1	–1.83842	1.355882	–0.42818	0.456216
1.2	–2.6095	1.615394	–0.49102	0.476388
1.25	–3	1.732051	–0.5103	0.47826
1.3	–3.39441	1.842392	–0.52369	0.476532
1.4	–4.19296	2.047673	–0.53704	0.465516
1.45	–4.59577	2.143775	–0.53856	0.457368
1.451	–4.60384	2.145657	–0.53857	0.457188
1.46	–4.67652	2.162527	–0.53854	0.45558
1.5	–5	2.236068	–0.53748	0.44802
1.7	–6.61869	2.572681	–0.51536	0.403944
2	–9	3	–0.45696	0.334092
3	–16	4	–0.26591	0.167124
5	–25.6289	5.0625	–0.09232	0.049452
10	–37.0427	6.086275	–0.01342	0.006156
50	–50.4268	7.101183	–4.8E-05	2.4E-05
100	–52.4643	7.243221	0	0
1000	–54.3803	7.374298	0	0
10000	–54.5763	7.387578	0	0

Example: Exponential distribution

Let X and Y be random variables that follow the exponential distribution with parameters λ_1 and λ_2, respectively, and extended Cuadras copula C given by (3). The cdfs for X and Y are

$$F(x) = 1 - e^{-\lambda_1 x}, \quad x \geq 0, \quad \lambda_1 > 0,$$

and

$$G(y) = 1 - e^{-\lambda_2 y}, \quad y \geq 0, \quad \lambda_2 > 0.$$

Then,

$$H(x, y) = C(F(x), G(y)) = C(1 - e^{-\lambda_1 x}, 1 - e^{-\lambda_2 y})$$

$$= (1 - e^{-\lambda_1 x})(1 - e^{-\lambda_2 y})e^{a[1-(1-e^{-\lambda_1 x})]^b [1-(1-e^{-\lambda_2 y})]^b}$$

$$= (1 - e^{-\lambda_1 x})(1 - e^{-\lambda_2 y})e^{ae^{-b(\lambda_1 x + \lambda_2 y)}}$$

and with density copula given by

$$c(F(x), G(y)) = e^{ae^{-b(\lambda_1 x+\lambda_2 y)}}\{1 + ab[ab(1 - e^{-\lambda_1 x})(1 - e^{-\lambda_2 y})$$

$$e^{-(2b-1)(\lambda_1 x+\lambda_2 y)} + b(1 - e^{-\lambda_1 x})(1 - e^{-\lambda_2 y})$$

$$e^{-(b-1)(\lambda_1 x+\lambda_2 y)} - (1 - e^{-\lambda_1 x})e^{-b(\lambda_1 x+\lambda_2 y)+\lambda_1 x}$$

$$- (1 - e^{-\lambda_2 y})e^{-b(\lambda_1 x+\lambda_2 y)+\lambda_2 y}]\}.$$

Hence, the joint density function is

$$h(x, y) = e^{ae^{-b(\lambda_1 x+\lambda_2 y)}}$$

$$\times \{1 + ab[ab(1 - e^{-\lambda_1 x})(1 - e^{-\lambda_2 y})e^{-(2b-1)(\lambda_1 x+\lambda_2 y)}$$

$$+ b(1 - e^{-\lambda_1 x})(1 - e^{-\lambda_2 y})e^{-(b-1)(\lambda_1 x+\lambda_2 y)} - (1 - e^{-\lambda_1 x})$$

$$e^{-b(\lambda_1 x+\lambda_2 y)+\lambda_1 x} - (1 - e^{-\lambda_2 y})e^{-b(\lambda_1 x+\lambda_2 y)+\lambda_2 y}]\}$$

$$(\lambda_1\lambda_2 e^{-(\lambda_1 x+\lambda_2 y)}).$$

Conclusion

In this paper, we have proposed a new extension of the Cuadras copula and determined the range of the parameters for the new extended copula. The new extension has permitted a wider range of dependence structure between the random variables involved. This opens the scope for an increased number of applications in various fields.

References

[1] A. Abdelghaly, E. Ahmed, R. Ebaid and W. Elbadawy, An extension of the Cuadras copula, Advances and Applications in Statistics 50(2) (2017), 97-106.

[2] N. Balakrishnan and C. Lai, Continuous Bivariate Distributions, 2nd ed., Springer, New York, 2009.

[3] C. Cuadras, Constructing copula functions with weighted geometric means, J. Statist. Plann. Inference 139 (2009), 3766-3772.

[4] A. Dolati and M. Úbeda-Flores, Constructing copulas by means of pairs of order statistics, Kybernetika 45 (2009), 992-1002.

[5] D. Drouet-Mari and S. Kotz, Correlation and Dependence, Imperial College Press, London, 2001.

[6] T. Hutchinson and C. Lai, The Engineering Statistician's Guide to Continuous Bivariate Distributions, Rumsby Scientific Publishing, Adelaide, 1991.

[7] H. Joe, Multivariate Models and Dependence Concepts, Chapman and Hall, London, 1997.

[8] I. Klein and F. Christa, Families of copulas closed under the construction of generalized linear means, IWQW Discussion Paper Series, No. 04/2011, 2011.

[9] R. Nelsen, Properties and applications of copulas: a brief survey, Proceedings of the First Brazilian Conference on Statistical Modeling in Insurance and Finance, J. Dhaene, N. Kolev and P. Morettin, eds., University Press USP: São Paulo, 2003, pp. 10-28.

[10] R. Nelsen, An Introduction to Copulas, 2nd ed., Springer, USA, 2006.

[11] B. Schweizer and A. Sklar, Probabilistic Metric Spaces, Elsevier North-Holland, New York, 1983.

[12] A. Sklar, Fonctions de répartition à n dimensions et leurs marges, Publications de L'Institut de Statistique de L'Université de Paris, 1959.

[13] K. Zhang, J. Lin and C. Huang, Some new results on weighted geometric mean for copulas, Inter. J. Uncertain. Fuzziness Knowledge-Based Systems 21(3) (2013), 277-288.