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FUNDAMENTAL SOLUTIONS TO TWO TYPES OF 2D BOUNDARY VALUE PROBLEMS OF ANISOTROPIC MATERIALS

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Abstract

We discuss the derivation of the fundamental solutions to two types of 2D boundary value problems of anisotropic materials. Once the fundamental solutions have been obtained, we derive the boundary integral equations associated with the boundary value problems. Then we implement the boundary element method by discretizing the boundary of the domain. Some examples of boundary value problems are solved using the boundary element method to see the validity of the mathematical analysis in deriving the fundamental solutions.

1. Introduction

In a discussion of boundary integral equation method or sometimes called as *boundary element method* for finding solutions to boundary value problems governed by partial differential equations, the fundamental solutions to that problems play an important role. The fundamental solutions

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are always involved as the integrand function in the integral equations derived from the governing differential equations. Therefore, availability of the fundamental solution is very important for the implementation of the boundary integral equation method.

Anisotropic materials are materials having specific characteristics of which their values are varying with geometrical directions in the materials. Wood, for example, may have elastic characteristics that differ between in horizontal and vertical directions due to its grain. That is why wood can be treated as an anisotropic material. Isotropic materials, on the other hand, have properties of which their values are equal in all directions. Most of metals can be treated as isotropic materials.

In general, fundamental solutions for isotropic materials are now available widely for most of the prototype governing equations such as Laplace, Helmholtz, diffusion-convection equations and others. However, this is not true for anisotropic materials. Recently, some works have been done on deriving the fundamental solutions for anisotropic materials. Manolis et al. [6] derived time-harmonic fundamental solutions for the general case of anisotropic, inhomogeneous continua under in-plane and anti-plane conditions using the Radon transform. Yaslan [8] studied the fundamental solution of the time-dependent differential equations of anisotropic elasticity in 3D quasicrystals using Fourier transform. Iovane et al. [5] obtained an explicit representation of the elastodynamic Green's function for the antiplane problem of concentrated point force moving with constant velocity and oscillating with constant frequency in unbounded homogeneous anisotropic elastic medium using Fourier integral transform techniques. Daros [4] derived a fundamental solution for SH waves in a class of inhomogeneous anisotropic media as well as the fundamental solution for inhomogeneous media with linear velocity variation by employing what they as "transmutation" formula. Marczak and Denda [7] studied the fundamental solution for three-dimensional heat transfer problems in the general anisotropic media using algebraic manipulation as well as through Fourier and Radon transforms.

In this paper, a technique for deriving fundamental solutions for anisotropic media by using the associated ones for isotropic media is developed. The technique uses a transformation of the reference geometrical coordinate system to a new one such that the governing equation of the anisotropic media is transformed to a relevant governing equation for isotropic media.

2. The Boundary Value Problems

Referred to the Cartesian frame Ox_1x_2, we will consider the boundary value problems governed by the following two types of governing equations for anisotropic materials:

$$\lambda_{ij} \frac{\partial^2 \phi(\mathbf{x})}{\partial x_i \partial x_j} + k\phi(\mathbf{x}) = 0, \tag{1}$$

$$\lambda_{ij} \frac{\partial^2 \phi(\mathbf{x})}{\partial x_i \partial x_j} - v_i \frac{\partial \phi(\mathbf{x})}{\partial x_i} = 0, \tag{2}$$

where $\mathbf{x} = (x_1, x_2)$, λ_{ij}, k and v_i are constant coefficients, i, $j = 1, 2$. The coefficient $[\lambda_{ij}]$ is a real definite positive symmetrical matrix. Also, in (1) and (2), the summation convention for repeated indices apply, so that (1) and (2) can be written explicitly as

$$\lambda_{11} \frac{\partial^2 \phi}{\partial x_1^2} + 2\lambda_{12} \frac{\partial^2 \phi}{\partial x_1 \partial x_2} + \lambda_{22} \frac{\partial^2 \phi}{\partial x_2^2} + k\phi = 0,$$

$$\lambda_{11} \frac{\partial^2 \phi}{\partial x_1^2} + 2\lambda_{12} \frac{\partial^2 \phi}{\partial x_1 \partial x_2} + \lambda_{22} \frac{\partial^2 \phi}{\partial x_2^2} - v_1 \frac{\partial \phi}{\partial x_1} - v_2 \frac{\partial \phi}{\partial x_2} = 0.$$

Equations (1) and (2) are relevant for anisotropic media, but also cover the case of isotropic media as a special case that occurs when $\lambda_{11} = \lambda_{22}$ and $\lambda_{12} = 0$. For the case of isotropic media, the corresponding governing equations are, respectively,

$$\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \kappa\phi = 0, \tag{3}$$

$$D\left(\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2}\right) - v_1 \frac{\partial\phi}{\partial x_1} - v_2 \frac{\partial\phi}{\partial x_2} = 0, \tag{4}$$

where $\kappa = k/\lambda_{11}$ and $D = \lambda_{11}$.

Solutions ϕ to (1) and (2) are sought which are valid in a region Ω in R^2 with boundary Γ which consists of a finite number of piecewise smooth curves. On Γ_1, the dependent variable $\phi(\mathbf{x})$ is specified, and $P = \lambda_{ij}(\partial\phi/\partial x_i)n_j$ is specified on Γ_2, where $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\mathbf{n} = (n_1, n_2)$ denotes the outward pointing normal to Γ.

3. The Boundary Integral Equation

Multiplying both sides of (1) and (2) by function ϕ^* and then integrating it over the domain Ω, respectively, yields

$$\int_\Omega \left(\lambda_{ij} \frac{\partial^2\phi}{\partial x_i \partial x_j} + k\phi\right)\phi^* d\mathbf{x} = 0, \tag{5}$$

$$\int_\Omega \left(\lambda_{ij} \frac{\partial^2\phi}{\partial x_i \partial x_j} - v_i \frac{\partial\phi}{\partial x_i}\right)\phi^* d\mathbf{x} = 0. \tag{6}$$

Using Gauss divergence theorem in (5) and (6), we obtain

$$\int_\Gamma \lambda_{ij} \frac{\partial\phi}{\partial x_i} n_j \phi^* d\mathbf{x} - \int_\Omega \left(\lambda_{ij} \frac{\partial\phi}{\partial x_i}\frac{\partial\phi^*}{\partial x_j} - k\phi\phi^*\right) d\mathbf{x} = 0, \tag{7}$$

$$\int_\Gamma \left(\lambda_{ij} \frac{\partial\phi}{\partial x_i} n_j - \phi v_i n_i\right)\phi^* d\mathbf{x} - \int_\Omega \left(\lambda_{ij} \frac{\partial\phi}{\partial x_i}\frac{\partial\phi^*}{\partial x_j} - \phi v_i \frac{\partial\phi^*}{\partial x_i}\right) d\mathbf{x} = 0. \tag{8}$$

Use of Gauss divergence theorem once again for the integrand function $\lambda_{ij} \frac{\partial\phi}{\partial x_i}\frac{\partial\phi^*}{\partial x_j}$ in the domain integral in (7) and (8), respectively, yields

$$\int_\Gamma \lambda_{ij} \frac{\partial \phi}{\partial x_i} n_j \phi^* d\mathbf{x} - \int_\Gamma \phi \lambda_{ij} \frac{\partial \phi^*}{\partial x_i} n_j \, d\mathbf{x}$$

$$+ \int_\Omega \left(\phi \lambda_{ij} \frac{\partial^2 \phi^*}{\partial x_i \, \partial x_j} + k \phi \phi^* \right) d\mathbf{x} = 0,$$

$$\int_\Gamma \left(\lambda_{ij} \frac{\partial \phi}{\partial x_i} n_j - \phi v_i n_i \right) \phi^* d\mathbf{x} - \int_\Gamma \phi \lambda_{ij} \frac{\partial \phi^*}{\partial x_i} n_j d\mathbf{x}$$

$$+ \int_\Omega \left(\phi \lambda_{ij} \frac{\partial^2 \phi^*}{\partial x_i \, \partial x_j} + \phi v_i \frac{\partial \phi^*}{\partial x_i} \right) d\mathbf{x} = 0$$

or

$$\int_\Omega \left(\lambda_{ij} \frac{\partial^2 \phi^*}{\partial x_i \partial x_j} + k \phi^* \right) \phi d\mathbf{x} = \int_\Gamma (P^* \phi - P \phi^*) d\mathbf{x}, \tag{9}$$

$$\int_\Omega \left(\lambda_{ij} \frac{\partial^2 \phi^*}{\partial x_i \partial x_j} + v_i \frac{\partial \phi^*}{\partial x_i} \right) \phi d\mathbf{x} = -\int_\Gamma [P \phi^* - (P_v \phi^* + P^*) \phi] d\mathbf{x}, \tag{10}$$

where

$$P_v(\mathbf{x}) = v_i n_i(\mathbf{x}) \quad \text{and} \quad P^* = \lambda_{ij} \frac{\partial \phi^*}{\partial x_i} n_j.$$

If the function ϕ^* for (1) and (2) is, respectively, taken such that

$$\lambda_{ij} \frac{\partial^2 \phi^*}{\partial x_i \partial x_j} + k \phi^* = \delta(\mathbf{x} - \boldsymbol{\xi}), \tag{11}$$

$$\lambda_{ij} \frac{\partial^2 \phi^*}{\partial x_i \partial x_j} + v_i \frac{\partial \phi^*}{\partial x_i} = -\delta(\mathbf{x} - \boldsymbol{\xi}), \tag{12}$$

where δ is the Dirac delta function and $\boldsymbol{\xi} = (\xi_1, \xi_2)$, then (9) and (10) may be written, respectively, as

$$\int_\Omega \phi(\mathbf{x}) \delta(\mathbf{x} - \xi) d\mathbf{x} = \int_\Gamma [P^*(\mathbf{x}, \xi) \phi(\mathbf{x}) - P(\mathbf{x}) \phi^*(\mathbf{x}, \xi)] d\mathbf{x}, \tag{13}$$

$$\int_{\Omega} \phi(\mathbf{x})\delta(\mathbf{x}-\xi)\,d\mathbf{x} = \int_{\Gamma} \{P(\mathbf{x})\phi^{*}(\mathbf{x}, \xi)$$

$$- [P_{v}(\mathbf{x})\phi^{*}(\mathbf{x}, \xi) + P^{*}(\mathbf{x}, \xi)]\phi(\mathbf{x})\}\,d\mathbf{x}. \quad (14)$$

As one of the Dirac delta function's properties, the following equation holds:

$$\int_{\Omega} \phi(\mathbf{x})\delta(\mathbf{x}-\xi)\,d\mathbf{x} = \eta(\xi)\phi(\xi) \quad (15)$$

with $\eta = \frac{1}{2}$ if ξ lies on the boundary Γ, $\eta = 1$ if ξ is inside of the domain Ω, $\eta = 0$ if ξ is outside of the domain Ω. By substituting (15) into (13) and (14), we obtain the boundary integral equations

$$\eta(\xi)\phi(\xi) = \int_{\Gamma} [P^{*}(\mathbf{x}, \xi)\phi(\mathbf{x}) - P(\mathbf{x})\phi^{*}(\mathbf{x}, \xi)]\,d\mathbf{x}, \quad (16)$$

$$\eta(\xi)\phi(\xi) = \int_{\Gamma} [P(\mathbf{x})\phi^{*}(\mathbf{x}, \xi) - (P_{v}(\mathbf{x})\phi^{*}(\mathbf{x}, \xi) + P^{*}(\mathbf{x}, \xi))\phi(\mathbf{x})]\,d\mathbf{x}. \quad (17)$$

4. Fundamental Solutions for Anisotropic Case

The fundamental solutions $\Phi(\mathbf{x}, \mathbf{x}_0)$ for the isotropic cases (3) and (4) are defined by

$$\frac{\partial^2 \Phi}{\partial x_1^2} + \frac{\partial^2 \Phi}{\partial x_2^2} + \kappa\Phi = \delta(\mathbf{x}-\xi), \quad (18)$$

$$D\left(\frac{\partial^2 \Phi}{\partial x_1^2} + \frac{\partial^2 \Phi}{\partial x_2^2}\right) + v_1 \frac{\partial \Phi}{\partial x_1} + v_2 \frac{\partial \Phi}{\partial x_2} = -\delta(\mathbf{x}-\xi). \quad (19)$$

These fundamental solutions $\Phi(\mathbf{x}, \xi)$ are, respectively,

$$\Phi(\mathbf{x}, \xi) = \begin{cases} \frac{1}{2\pi} \ln R & \text{if } \kappa = 0 \\ \frac{\imath}{4} H_0^{(2)}(\sqrt{|\kappa|}R) & \text{if } \kappa > 0 \\ \frac{-1}{2\pi} K_0(\sqrt{|\kappa|}R) & \text{if } \kappa < 0, \end{cases} \quad (20)$$

$$\Phi(\mathbf{x}, \xi) = \frac{1}{2\pi D} \exp\left(-\frac{\mathbf{v}.\mathbf{R}}{2D}\right) K_0\left(\frac{vR}{2D}\right), \tag{21}$$

where vector $\mathbf{v} = (v_1, v_2)$, vector $\mathbf{R} = \mathbf{x} - \xi$, v is the magnitude of the vector $\mathbf{v}$ that is $v = \sqrt{v_1^2 + v_2^2}$, R is the magnitude of the vector $\mathbf{R}$, $\imath = \sqrt{-1}$, $H_0^{(2)}$ is the second kind Hankel function, and K_0 is the modified Bessel function.

Similar to the fundamental solutions Φ in (18) and (19) for the governing equations of isotropic cases (3) and (4), respectively, we define the fundamental solutions ϕ^* for the governing equations of anisotropic cases (1) and (2), respectively, as follows:

$$\lambda_{ij} \frac{\partial^2 \phi^*}{\partial x_i \partial x_j} + k\phi^* = \delta(\mathbf{x} - \xi), \tag{22}$$

$$\lambda_{ij} \frac{\partial^2 \phi^*}{\partial x_i \partial x_j} + v_i \frac{\partial \phi^*}{\partial x_i} = -\delta(\mathbf{x} - \xi). \tag{23}$$

Let

$$\begin{aligned} z &= x_1 + \tau x_2, \\ \bar{z} &= x_1 + \bar{\tau} x_2, \end{aligned} \tag{24}$$

where τ is the complex root with positive imaginary part of the quadratic equation

$$\lambda_{11} + 2\lambda_{12}\tau + \lambda_{22}\tau^2 = 0$$

and the symbol bar $(\bar{\cdot})$ denotes complex conjugate. Then

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial z}\frac{\partial z}{\partial x_1} + \frac{\partial}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x_1} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}},$$

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial z}\frac{\partial z}{\partial x_2} + \frac{\partial}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x_2} = \tau\frac{\partial}{\partial z} + \bar{\tau}\frac{\partial}{\partial \bar{z}},$$

$$\frac{\partial^2}{\partial x_1^2} = \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right)\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right) = \frac{\partial^2}{\partial z^2} + 2\frac{\partial^2}{\partial z \partial \bar{z}} + \frac{\partial^2}{\partial \bar{z}^2},$$

$$\frac{\partial^2}{\partial x_1 \partial x_2} = \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right)\left(\tau\frac{\partial}{\partial z} + \bar{\tau}\frac{\partial}{\partial \bar{z}}\right) = \tau\frac{\partial^2}{\partial z^2} + (\tau + \bar{\tau})\frac{\partial^2}{\partial z \partial \bar{z}} + \bar{\tau}\frac{\partial^2}{\partial \bar{z}^2},$$

$$\frac{\partial^2}{\partial x_2^2} = \left(\tau\frac{\partial}{\partial z} + \bar{\tau}\frac{\partial}{\partial \bar{z}}\right)\left(\tau\frac{\partial}{\partial z} + \bar{\tau}\frac{\partial}{\partial \bar{z}}\right) = \tau^2\frac{\partial^2}{\partial z^2} + 2\tau\bar{\tau}\frac{\partial^2}{\partial z \partial \bar{z}} + \bar{\tau}^2\frac{\partial^2}{\partial \bar{z}^2}.$$

Therefore,

$$\lambda_{ij}\frac{\partial^2 \phi^*}{\partial x_i \partial x_j} = 2[\lambda_{11} + \lambda_{12}(\tau + \bar{\tau}) + \lambda_{22}\tau\bar{\tau}]\frac{\partial^2 \phi^*}{\partial z \partial \bar{z}},$$

$$v_i\frac{\partial \phi^*}{\partial x_i} = (v_1 + \tau v_2)\frac{\partial \phi^*}{\partial z} + (v_1 + \bar{\tau} v_2)\frac{\partial \phi^*}{\partial \bar{z}}. \tag{25}$$

Furthermore, let

$$z = \dot{x}_1 + \imath \dot{x}_2, \tag{26}$$

$$\bar{z} = \dot{x}_1 - \imath \dot{x}_2, \tag{27}$$

$$\tau = \dot{\tau} + \imath \ddot{\tau}.$$

Then, from (26) and (27), we obtain

$$\dot{x}_1 = \frac{1}{2}(z + \bar{z}),$$

$$\dot{x}_2 = \frac{1}{2\imath}(z - \bar{z}) \tag{28}$$

so that

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \dot{x}_1}\frac{\partial \dot{x}_1}{\partial z} + \frac{\partial}{\partial \dot{x}_2}\frac{\partial \dot{x}_2}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial \dot{x}_1} - \imath\frac{\partial}{\partial \dot{x}_2}\right),$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial \dot{x}_1}\frac{\partial \dot{x}_1}{\partial \bar{z}} + \frac{\partial}{\partial \dot{x}_2}\frac{\partial \dot{x}_2}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial \dot{x}_1} + \imath\frac{\partial}{\partial \dot{x}_2}\right),$$

$$\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial \dot{x}_1^2} + \frac{\partial^2}{\partial \dot{x}_2^2}\right). \tag{29}$$

Also, from (24) and (28), it can be obtained

$$x_1 = \dot{x}_1 - \dot{x}_2\dot{\tau}/\ddot{\tau},$$

$$x_2 = \dot{x}_2/\ddot{\tau}. \tag{30}$$

Substitution of (29) into (25) and then into (22) and (23), respectively, yields

$$\frac{\partial^2 \phi^*}{\partial \dot{x}_1^2} + \frac{\partial^2 \phi^*}{\partial \dot{x}_2^2} + \frac{k}{C}\phi^* = \frac{1}{C}\delta(\mathbf{x} - \boldsymbol{\xi}), \tag{31}$$

$$C\left(\frac{\partial^2 \phi^*}{\partial \dot{x}_1^2} + \frac{\partial^2 \phi^*}{\partial \dot{x}_2^2}\right) + (v_1 + \dot{\tau}v_2)\frac{\partial \phi^*}{\partial \dot{x}_1} + \ddot{\tau}v_2\frac{\partial \phi^*}{\partial \dot{x}_2} = -\delta(\mathbf{x} - \boldsymbol{\xi}), \tag{32}$$

where

$$C = [\lambda_{11} + \lambda_{12}(\tau + \bar{\tau}) + \lambda_{22}\tau\bar{\tau}]/2.$$

Now, the Dirac delta function $\delta(\mathbf{x} - \boldsymbol{\xi})$ on the right hand side of (31) and (32) also needs to be transformed into the new coordinate system $O\dot{x}_1\dot{x}_2$. We know from (9), (11), (10) and (12) that the integral of the Dirac delta function is evaluated. This suggests that the Jacobian J of the transformation (30) which is given by

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial \dot{x}_1} & \frac{\partial x_1}{\partial \dot{x}_2} \\ \frac{\partial x_2}{\partial \dot{x}_1} & \frac{\partial x_2}{\partial \dot{x}_2} \end{vmatrix} = \begin{vmatrix} 1 & -\frac{\dot{\tau}}{\ddot{\tau}} \\ 0 & \frac{1}{\ddot{\tau}} \end{vmatrix} = \frac{1}{\ddot{\tau}}$$

must be evaluated. Specifically,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(\mathbf{x} - \boldsymbol{\xi})\,dx_1 dx_2 = 1,$$

that is,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(\dot{\mathbf{x}}-\dot{\boldsymbol{\xi}})\,|\,J\,|\,d\dot{x}_1 d\dot{x}_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(\dot{\mathbf{x}}-\dot{\boldsymbol{\xi}})\frac{1}{\ddot{\tau}}\,d\dot{x}_1 d\dot{x}_2 = 1.$$

Therefore,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(\dot{\mathbf{x}}-\dot{\boldsymbol{\xi}})\,d\dot{x}_1 d\dot{x}_2 = \ddot{\tau}.$$

This means that the quantity $\ddot{\tau}$ has to be a multiplier of right hand side of (31) and (32) so that (31) and (32) become

$$\frac{\partial^2\phi^*}{\partial\dot{x}_1^2}+\frac{\partial^2\phi^*}{\partial\dot{x}_2^2}+\frac{k}{C}\phi^* = \frac{\ddot{\tau}}{C}\delta(\dot{\mathbf{x}}-\dot{\boldsymbol{\xi}}), \tag{33}$$

$$C\left(\frac{\partial^2\phi^*}{\partial\dot{x}_1^2}+\frac{\partial^2\phi^*}{\partial\dot{x}_2^2}\right)+\dot{v}_1\frac{\partial\phi^*}{\partial\dot{x}_1}+\dot{v}_2\frac{\partial\phi^*}{\partial\dot{x}_2} = -\ddot{\tau}\delta(\dot{\mathbf{x}}-\dot{\boldsymbol{\xi}}), \tag{34}$$

where

$$\dot{v}_1 = v_1 + \dot{\tau}v_2, \quad \dot{v}_2 = \ddot{\tau}v_2,$$

$$\dot{x}_1 = x_1 + \dot{\tau}x_2, \quad \dot{x}_2 = \ddot{\tau}x_2.$$

Equations (33) and (34) are relevant for isotropic case in the new coordinate system $O\dot{x}_1\dot{x}_2$.

Finally, by comparing (33) and (18), we obtain fundamental solution ϕ^* for (1) from the fundamental solution Φ in (20):

$$\phi^*(\mathbf{x},\boldsymbol{\xi}) = \begin{cases} \dfrac{K}{2\pi}\ln\dot{R} & \text{if } k = 0 \\ \dfrac{\imath K}{4}H_0^{(2)}(\omega\dot{R}) & \text{if } k/C > 0 \\ \dfrac{-K}{2\pi}K_0(\omega\dot{R}) & \text{if } k/C < 0. \end{cases} \tag{35}$$

Similarly, with a comparison between (34) and (19), the fundamental solution ϕ^* for (2) can be obtained from fundamental solution Φ in (21):

$$\phi^*(\mathbf{x}, \xi) = \frac{K}{2\pi} \exp\left(-\frac{\dot{\mathbf{v}} \cdot \dot{\mathbf{R}}}{2C}\right) K_0\left(\frac{\dot{v}\dot{R}}{2C}\right), \tag{36}$$

where $K = \ddot{\tau}/C$, $\omega = \sqrt{|k/C|}$, $\dot{\mathbf{R}} = \dot{\mathbf{x}} - \dot{\xi}$, $\dot{\mathbf{x}} = (\dot{x}_1, \dot{x}_2)$, $\dot{\xi} = (\dot{a}, \dot{b})$, $\dot{\mathbf{v}} = (\dot{v}_1, \dot{v}_2)$, $\dot{a} = a + \dot{\tau}b$, $\dot{b} = \ddot{\tau}b$, $\xi = (a, b)$, $\dot{R}$ is the magnitude of vector $\dot{\mathbf{R}}$, and $\dot{v}$ is the length of vector $\dot{\mathbf{v}}$.

The derived fundamental solutions for anisotropic cases (35) and (36) have been used extensively by Azis and Clements in [2, 3], and Azis et al. in [1].

5. Numerical Examples

In this section, some particular boundary value problems governed by (1) and (2) are solved numerically by employing the boundary integral equations (16) and (17). Two problems with analytical solutions are considered to show the validity of the analysis used to derive the fundamental solutions (35) and (36). Standard boundary element procedures are employed to obtain the numerical results.

5.1. Problem 1

Consider the boundary value problem governed by (1) for three cases $k < 0$, $k > 0$ and $k = 0$ and with coefficients

$$[\lambda_{ij}] = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}.$$

The analytical solutions to (1) are taken to be

$$\begin{aligned} &\text{for } k > 0, \quad \phi = \sin(\mu x_2) && \mu = \sqrt{k/\lambda_{22}} && k = 0.5 \\ &\text{for } k < 0, \quad \phi = \exp(\mu x_2) && \mu = \sqrt{-k/\lambda_{22}} && k = -0.5 \\ &\text{for } k = 0, \quad \phi = \lambda_{11}x_1 + \lambda_{22}x_2 \end{aligned}$$

and the domain Ω is chosen to be a unit square with corner points $A = (0, 0)$, $B = (1, 0)$, $C = (1, 1)$ and $D = (0, 1)$. The boundary conditions are that P is given on AB, BC, CD and ϕ is given on AD.

Tables 1, 2 and 3 show a comparison between the boundary element method (BEM) solution and the analytical solution. The results show that the BEM solution converges to the analytical solution as the number of elements used increases from 80, 160 to 320. The results are as expected.

Table 1. BEM and analytical solutions of Problem 1 for the case $k = 0.5$

(x_1, x_2)	ϕ	$\partial\phi/\partial x_1$	$\partial\phi/\partial x_2$	ϕ	$\partial\phi/\partial x_1$	$\partial\phi/\partial x_2$
	BEM 80 elements			Analytical		
(.1,.5)	.4785	-.0001	.8795	.4794	.0000	.8776
(.3,.5)	.4787	.0013	.8769	.4794	.0000	.8776
(.5,.5)	.4790	.0017	.8760	.4794	.0000	.8776
(.7,.5)	.4793	.0016	.8754	.4794	.0000	.8776
(.9,.5)	.4796	.0006	.8766	.4794	.0000	.8776
	BEM 160 elements			Analytical		
(.1,.5)	.4789	.0002	.8777	.4794	.0000	.8776
(.3,.5)	.4790	.0004	.8774	.4794	.0000	.8776
(.5,.5)	.4791	.0005	.8771	.4794	.0000	.8776
(.7,.5)	.4792	.0005	.8769	.4794	.0000	.8776
(.9,.5)	.4793	.0005	.8767	.4794	.0000	.8776
	BEM 320 elements			Analytical		
(.1,.5)	.4792	.0000	.8777	.4794	.0000	.8776
(.3,.5)	.4792	.0001	.8776	.4794	.0000	.8776
(.5,.5)	.4792	.0002	.8774	.4794	.0000	.8776
(.7,.5)	.4793	.0002	.8773	.4794	.0000	.8776
(.9,.5)	.4793	.0002	.8773	.4794	.0000	.8776

Table 2. BEM and analytical solutions of Problem 1 for the case $k = -0.5$

(x_1, x_2)	ϕ	$\partial\phi/\partial x_1$	$\partial\phi/\partial x_2$	ϕ	$\partial\phi/\partial x_1$	$\partial\phi/\partial x_2$
	BEM 80 elements			Analytical		
(.1,.5)	1.6468	-.0032	1.6496	1.6487	.0000	1.6487
(.3,.5)	1.6465	-.0009	1.6454	1.6487	.0000	1.6487
(.5,.5)	1.6465	.0004	1.6439	1.6487	.0000	1.6487
(.7,.5)	1.6466	.0011	1.6436	1.6487	.0000	1.6487
(.9,.5)	1.6469	-.0002	1.6471	1.6487	.0000	1.6487
	BEM 160 elements			Analytical		
(.1,.5)	1.6478	-.0007	1.6477	1.6487	.0000	1.6487
(.3,.5)	1.6476	-.0005	1.6473	1.6487	.0000	1.6487
(.5,.5)	1.6476	-.0001	1.6468	1.6487	.0000	1.6487
(.7,.5)	1.6476	.0003	1.6466	1.6487	.0000	1.6487
(.9,.5)	1.6477	.0004	1.6467	1.6487	.0000	1.6487
	BEM 320 elements			Analytical		
(.1,.5)	1.6482	-.0003	1.6482	1.6487	.0000	1.6487
(.3,.5)	1.6482	-.0002	1.6481	1.6487	.0000	1.6487
(.5,.5)	1.6481	-.0001	1.6479	1.6487	.0000	1.6487
(.7,.5)	1.6481	.0000	1.6478	1.6487	.0000	1.6487
(.9,.5)	1.6482	.0001	1.6478	1.6487	.0000	1.6487

Table 3. BEM and analytical solutions of Problem 1 for the case $k = 0$

(x_1, x_2)	ϕ	$\partial\phi/\partial x_1$	$\partial\phi/\partial x_2$	ϕ	$\partial\phi/\partial x_1$	$\partial\phi/\partial x_2$
	BEM 80 elements			Analytical		
(.1,.5)	.3470	.9973	.5011	.3500	1.0000	.5000
(.3,.5)	.5465	.9975	.5003	.5500	1.0000	.5000
(.5,.5)	.7460	.9976	.5004	.7500	1.0000	.5000
(.7,.5)	.9456	.9978	.5007	.9500	1.0000	.5000
(.9,.5)	1.1452	.9974	.5023	1.1500	1.0000	.5000
	BEM 160 elements			Analytical		
(.1,.5)	.3485	.9989	.5001	.3500	1.0000	.5000
(.3,.5)	.5483	.9988	.5002	.5500	1.0000	.5000
(.5,.5)	.7480	.9988	.5014	.7500	1.0000	.5000
(.7,.5)	.9478	.9989	.5005	.9500	1.0000	.5000
(.9,.5)	1.1476	.9989	.5007	1.1500	1.0000	.5000
	BEM 320 elements			Analytical		
(.1,.5)	.3493	.9995	.5000	.3500	1.0000	.5000
(.3,.5)	.5491	.9994	.5001	.5500	1.0000	.5000
(.5,.5)	.7490	.9994	.5002	.7500	1.0000	.5000
(.7,.5)	.9489	.9994	.5003	.9500	1.0000	.5000
(.9,.5)	1.1488	.9994	.5004	1.1500	1.0000	.5000

5.2. Problem 2

The analytical solutions to (2) are taken to be

$$\phi = \exp(\alpha_1 x_1 + \alpha_2 x_2),$$

where α_1 and α_2 satisfy

$$\lambda_{22}\alpha_2^2 + (2\lambda_{12}\alpha_1 - v_2)\alpha_2 + (\lambda_{11}\alpha_1^2 - v_1\alpha_1) = 0. \tag{37}$$

The domain Ω is also chosen to be a unit square with corner points $A = (0, 0)$, $B = (1, 0)$, $C = (1, 1)$ and $D = (0, 1)$, and the boundary conditions are again that P is given on *AB*, *BC*, *CD* and ϕ is given on *AD*. The coefficients λ_{ij} and v_i are

$$\lambda_{11} = 1, \quad \lambda_{12} = 1, \quad \lambda_{22} = 2,$$

$$v_1 = 1, \quad v_2 = 1.$$

Also, we take $\alpha_1 = 1$ and α_2 is evaluated using (37). Table 4 shows the results. Again, the results indicate a convergence of the BEM solution to the analytical solution as the number of elements used increases from 80, 160 to 320.

Table 4. BEM and analytical solutions of Problem 2

(x_1, x_2)	ϕ	$\partial\phi/\partial x_1$	$\partial\phi/\partial x_2$	ϕ	$\partial\phi/\partial x_1$	$\partial\phi/\partial x_2$
		BEM 80 elements			Analytical	
(.1,.5)	0.8605	0.8580	-0.4298	0.8607	0.8607	-0.4304
(.3,.5)	1.0503	1.0463	-0.5230	1.0513	1.0513	-0.5256
(.5,.5)	1.2820	1.2788	-0.6385	1.2840	1.2840	-0.6420
(.7,.5)	1.5652	1.5622	-0.7801	1.5683	1.5683	-0.7842
(.9,.5)	1.9112	1.9095	-0.9524	1.9155	1.9155	-0.9578
		BEM 160 elements			Analytical	
(.1,.5)	0.8606	0.8596	-0.4301	0.8607	0.8607	-0.4304
(.3,.5)	1.0508	1.0492	-0.5245	1.0513	1.0513	-0.5256
(.5,.5)	1.2832	1.2818	-0.6405	1.2840	1.2840	-0.6420
(.7,.5)	1.5670	1.5658	-0.7824	1.5683	1.5683	-0.7842
(.9,.5)	1.9137	1.9129	-0.9554	1.9155	1.9155	-0.9578
		BEM 320 elements			Analytical	
(.1,.5)	0.8607	0.8602	-0.4302	0.8607	0.8607	-0.4304
(.3,.5)	1.0511	1.0503	-0.5251	1.0513	1.0513	-0.5256
(.5,.5)	1.2836	1.2830	-0.6413	1.2840	1.2840	-0.6420
(.7,.5)	1.5677	1.5672	-0.7834	1.5683	1.5683	-0.7842
(.9,.5)	1.9147	1.9144	-0.9567	1.9155	1.9155	-0.9578

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