Far East Journal of Mathematical Sciences (FJMS)

http://www.pphmj.com
http://dx.doi.org/10.17654/MS101112393
Volume 101, Number 11, 2017, Pages 2393-2403 **ISSN: 0972-0871**

THE ISOLATION FORM OF BRUNN-MINKOWSKI INEQUALITY AND MINKOWSKI INEQUALITY IN L_p SPACE

Fengfan Xie and Qian Yin*

College of Science
Wuhan University of Science and Technology
Wuhan, Hubei 430065, P. R. China

Abstract

This article is devoted to the study of inequality form of segregation. First, we establish the isolate forms of the Brunn-Minkowski inequality for the dual p-quermassintegrals of the dual Firey linear combination. Then we give the isolate forms of the new dual L_p-Brunn-Minkowski inequality for dual quermassintegrals of the L_p-radial Minkowski linear combination. Finally, we improve the Minkowski inequality for the dual mixed p-quermassintegrals and the Minkowski inequality of L_p-dual mixed quermassintegral.

Received: January 19, 2017; Revised: February 10, 2017; Accepted: March 14, 2017

2010 Mathematics Subject Classification: 52A40, 52A20.

Keywords and phrases: Brunn-Minkowski inequality, Minkowski inequality, dual quermassintegral, dual mixed quermassintegral.

This work is supported by youth fund projects province Department of Education (B20111101), by Hubei Province Key Laboratory of Systems Science in Metallurgical Process (Y201316), by School fund of Wuhan University of Science and Technology (2012XG011).

*Corresponding author

1. Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in the n-dimensional Euclidean space $\mathbb{R}^n$, $\mathcal{K}_o^n$ denote the set of convex bodies containing the origin in their interiors, $\mathcal{S}^n$ denote the set of star bodies in $\mathbb{R}^n$, and $\mathcal{S}_o^n$ denote the set of star bodies about the origin in $\mathbb{R}^n$.

The kernel of Brunn-Minkowski theory has been extended in several important ways (see [6-9]). In [1, 2], Lutwak established the L_p-Brunn-Minkowski theory: The core of Brunn-Minkowski theory is various Brunn-Minkowski inequalities and Minkowski inequalities (see [3-5]). Recently, Li and He [12] proved the extension of Brunn-Minkowski inequality for the dual p-quermassintegrals of the dual Firey linear combination as follows:

Theorem 1.1. *If* $p \geq 1$, $K, L \in \mathcal{S}_o^n$, $0 \leq i \leq n-1$, *then*

$$\widetilde{W}_i(K +_p L)^{\frac{p}{n-i}} \leq \widetilde{W}_i(K)^{\frac{p}{n-i}} + \widetilde{W}_i(L)^{\frac{p}{n-i}} \tag{1.1}$$

with equality if and only if K and L are dilates.

Wei and Wang [14] gave the new dual L_p-Brunn-Minkowski inequality for dual quermassintegrals of the L_p-radial Minkowski linear combination.

Theorem 1.2. *If* $K, L \in \mathcal{S}^n$, $p > 0$, $i \leq n - p$, *then*

$$\widetilde{W}_i(K \mathbin{\tilde{+}_p} L)^{\frac{p}{n-i}} \leq \widetilde{W}_i(K)^{\frac{p}{n-i}} + \widetilde{W}_i(L)^{\frac{p}{n-i}} \tag{1.2}$$

with equality if and only if K and L are dilates.

Li and He [12] defined the dual mixed p-quermassintegrals. Then they proved the extension of Minkowski inequality for the dual mixed p-quermassintegrals.

Theorem 1.3. *If* $p \geq 1$, $K, L \in \mathcal{S}_o^n$, $0 \leq i \leq n-1$, *then*

$$\widetilde{W}_{p,i}(K, L)^{n-i} \leq \widetilde{W}_i(K)^{n-i-p}\widetilde{W}_i(L)^p \tag{1.3}$$

with equality if and only if K and L are dilates.

Hu [11] gave a new definition of L_p-dual mixed quermassintegral, and established the Minkowski inequality of L_p-dual mixed quermassintegral.

Theorem 1.4. *If* $p > 0$, $K, L \in \mathcal{S}_o^n$, $i \neq n$, $i < n - p$, *then*

$$\widetilde{W}_{p,i}(K, L)^{n-i} \leq \widetilde{W}_i(K)^{n-i-p}\widetilde{W}_i(L)^p \tag{1.4}$$

with equality if and only if K and L are dilates. *For* $n - p < i < n$ *or* $i > n$, (1.4) *gets reversed*. *For* $i = n - p$, *there is an equality in* (1.4).

2. Preliminaries

If K is a compact star-shaped (about the origin) in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, +\infty)$, is defined by (see [10, 13])

$$\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in \mathcal{K}\}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

If ρ_K is positive and continuous, then K will be called a *star body* (about the origin). Two star bodies K and L are said to be *dilates* (*of one another*) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in \mathcal{S}^{n-1}$.

For $p \geq 1$, $K, L \in \mathcal{S}_o^n$ and $\alpha, \beta \geq 0$ (not both zero), the dual Firey linear combination $\alpha \circ K +_p \beta \circ L \in \mathcal{S}_o^n$ is defined by (see [12])

$$\rho(\alpha \circ K +_p \beta \circ L, \cdot)^p = \alpha\rho(K, \cdot)^p + \beta\rho(L, \cdot)^p. \tag{2.5}$$

Note that "$\circ$" rather than "$\circ_p$" is written for dual Firey scalar multiplication, which should create no confusion. Obviously, dual Firey and Minkowski scalar multiplications are related by $\alpha \circ K = \alpha^{\frac{1}{p}} K$.

For $K, L \in \mathcal{S}^n$, $p \neq 0$ and $\alpha, \beta \geq 0$ (not both 0), the L_p-radial Minkowski linear combination $\alpha \cdot K \tilde{+}_p \beta \cdot L$ is a star body defined by (see [14])

$$\rho(\alpha \cdot K \tilde{+}_p \beta \cdot L, u)^p = \alpha\rho(K, u)^p + \beta\rho(L, u)^p. \tag{2.6}$$

For $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and $\alpha, \beta \geq 0$ (not both 0), Lutwak defined that the radial sum combination $\alpha \circ K \tilde{+}_p \beta \circ L$ is a star body defined by (see [2])

$$\rho(\alpha \circ K \tilde{+}_p \beta \circ L, \cdot)^p = \alpha\rho(K, \cdot)^p + \beta\rho(L, \cdot)^p. \tag{2.7}$$

For $K \in \mathcal{S}_o^n$ and any real i, the dual quermassintegrals, $\tilde{W}_i(K)$ of K are defined by (see [10, 13])

$$\tilde{W}_i(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K, u)^{n-i} dS(u). \tag{2.8}$$

For $K, L \in \mathcal{S}^n$, $p \geq 1$, $0 \leq i \leq n-1$, the dual mixed p-quermassintegrals $\tilde{W}_{p,i}(K, L)$ has the following integral representation (see [11, 12]):

$$\tilde{W}_{p,i}(K, L) = \frac{1}{n}\int_{S^{n-1}} \rho(K, u)^{n-p-i} \rho(L, u)^p dS(u). \tag{2.9}$$

Obviously,

$$\tilde{W}_{p,i}(K, K) = \tilde{W}_i(K). \tag{2.10}$$

3. Main Results

In this paper, we continuously study the L_p-Brunn-Minkowski inequality. Here, we first give an isolate form of the Brunn-Minkowski inequality (1.1).

Lemma 3.1. *If* $p \geq 1$, $K, L \in \mathcal{S}_o^n$, $0 \leq i \leq n-1$ *and* $\lambda, \mu > 0$, *then*

$$\widetilde{W}_i(\lambda \circ K +_p \mu \circ L)^{\frac{p}{n-i}} \le \lambda \widetilde{W}_i(K)^{\frac{p}{n-i}} + \mu \widetilde{W}_i(L)^{\frac{p}{n-i}} \tag{3.11}$$

with equality if and only if K and L are dilates.

Proof. According to (2.5), (2.9), for $\forall Q \in \mathcal{S}_o^n$, we have

$$\widetilde{W}_{p,i}(Q, \lambda \circ K +_p \mu \circ L) = \frac{1}{n}\int_{S^{n-1}} \rho(Q, u)^{n-p-i} \rho(\lambda \circ K +_p \mu \circ L, u)^p \, dS(u)$$

$$= \lambda \widetilde{W}_{p,i}(Q, K) + \mu \widetilde{W}_{p,i}(Q, L).$$

Using (1.3), we get

$$\widetilde{W}_{p,i}(Q, \lambda \circ K +_p \mu \circ L) \le \widetilde{W}_i(Q)^{\frac{n-i-p}{n-i}} (\lambda \widetilde{W}_i(K)^{\frac{p}{n-i}} + \mu \widetilde{W}_i(L)^{\frac{p}{n-i}}).$$

Let $Q = \lambda \circ K +_p \mu \circ L$, together with (2.10). Then inequality (3.11) is proved.

Theorem 3.1. *If* $p \ge 1$, $K, L \in \mathcal{S}_o^n$, $0 \le i \le n-1$ *and* $\alpha \in [0, 1]$, *then*

$$\widetilde{W}_i(K +_p L)^{\frac{p}{n-i}} \le \widetilde{W}_i(\lambda \circ K +_p (1-\alpha) \circ L)^{\frac{p}{n-i}} + \widetilde{W}_i((1-\alpha) \circ K +_p \alpha \circ L)^{\frac{p}{n-i}}$$

$$\le \widetilde{W}_i(K)^{\frac{p}{n-i}} + \widetilde{W}_i(L)^{\frac{p}{n-i}} \tag{3.12}$$

with equality if and only if K and L are dilates.

Proof. Let

$$M = \alpha \circ K +_p (1-\alpha) \circ L, \quad N = (1-\alpha) \circ K +_p \alpha \circ L \tag{3.13}$$

for $\alpha \in [0, 1]$. Since $K, L \in \mathcal{S}_o^n$, $M, N \in \mathcal{S}_o^n$. Using (2.9), (2.5) and (3.13), for any $Q \in \mathcal{S}_o^n$, we have

$$\widetilde{W}_{p,i}(Q, K +_p L)$$

$$= \frac{1}{n}\int_{S^{n-1}} \rho(Q, u)^{n-p-i} \rho(K +_p L, u)^p \, dS(u)$$

$$= \frac{1}{n}\int_{S^{n-1}} \rho(Q, u)^{n-p-i}(\rho(K, u)^p + \rho(L, u)^p)dS(u)$$

$$= \frac{1}{n}\int_{S^{n-1}} \rho(Q, u)^{n-p-i}$$

$$\cdot (\alpha\rho(K, u)^p + (1-\alpha)\rho(L, u)^p + (1-\alpha)\rho(K, u)^p + \alpha\rho(L, u)^p)dS(u)$$

$$= \frac{1}{n}\int_{S^{n-1}} \rho(Q, u)^{n-p-i}(\rho(M, u)^p + \rho(N, u)^p)dS(u)$$

$$= \frac{1}{n}\int_{S^{n-1}} \rho(Q, u)^{n-p-i}\rho(M +_p N, u)^p dS(u)$$

$$= \widetilde{W}_{p,i}(Q, M +_p N). \tag{3.14}$$

Hence, let $Q = K +_p L$ in (3.14) and using (2.10) and (1.3), we get

$$\widetilde{W}_i(K +_p L) \le \widetilde{W}_i(M +_p N) \tag{3.15}$$

with equality if and only if $K +_p L$ and $M +_p N$ are dilates.

Using (1.1), we also have

$$\widetilde{W}_i(M +_p N)^{\frac{p}{n-1}} \le \widetilde{W}_i(M)^{\frac{p}{n-i}} + \widetilde{W}_i(N)^{\frac{p}{n-i}} \tag{3.16}$$

with equality if and only if M and N are dilates.

From inequalities (3.15), (3.16) and (3.13), we obtain the first inequality of inequality (3.12) in Theorem 3.1.

Because of $M \in \mathcal{S}_o^n$ and $N \in \mathcal{S}_o^n$ are dilates, we know $M +_p N \in \mathcal{S}_o^n$ and M (or N) are dilates, since $K +_p L \in \mathcal{S}_o^n$ and $M +_p N$ also are dilates, thus $K +_p L$ and M (or N) are dilates. Associated with (3.13) and (2.5), we see $K \in \mathcal{S}_o^n$ and $L \in \mathcal{S}_o^n$ are dilates. In turn, if $K \in \mathcal{S}_o^n$ and $L \in \mathcal{S}_o^n$ are dilates, then we easily know $M \in \mathcal{S}_o^n$ and $N \in \mathcal{S}_o^n$ are dilates and

$K +_p L \in \mathcal{S}_o^n$ and $M +_p N \in \mathcal{S}_o^n$ also are dilates. Hence, the equality holds in first inequality of inequality (3.12) if and only if *K* and *L* are dilates.

Otherwise, using inequalities (3.13) and (3.11), we have

$$\widetilde{W}_i(M)^{\frac{p}{n-i}} = \widetilde{W}_i(\alpha \circ K +_p (1-\alpha) \circ L)^{\frac{p}{n-i}}$$

$$\leq \alpha \widetilde{W}_i(K)^{\frac{p}{n-i}} + (1-\alpha)\widetilde{W}_i(L)^{\frac{p}{n-i}}. \tag{3.17}$$

Similarly,

$$\widetilde{W}_i(N)^{\frac{p}{n-i}} \leq (1-\alpha)\widetilde{W}_i(K)^{\frac{p}{n-i}} + \alpha \widetilde{W}_i(L)^{\frac{p}{n-i}}.$$

Therefore, the second inequality of inequality (3.12) in Theorem 3.1 is obtained.

Lemma 3.2. *If* $p \geq 1$, $K, L \in \mathcal{S}_o^n$, $0 \leq i \leq n-1$ *and* $\lambda, \mu > 0$, *then*

$$\widetilde{W}_i(\lambda \cdot K \,\tilde{+}_p\, \mu \cdot L)^{\frac{p}{n-i}} \leq \lambda \widetilde{W}_i(K)^{\frac{p}{n-i}} + \mu \widetilde{W}_i(L)^{\frac{p}{n-i}} \tag{3.18}$$

with equality if and only if K and L are dilates.

Proof. According to (2.6), (2.8), we have

$$\widetilde{W}_i(\lambda \cdot K \,\tilde{+}_p\, \mu \cdot L)$$

$$= \frac{1}{n}\int_{S^{n-1}} \rho(\lambda \cdot K \,\tilde{+}_p\, \mu \cdot L, u)^{n-p-i} \rho(\lambda \cdot K \,\tilde{+}_p\, \mu \cdot L, u)^p \, dS(u)$$

$$= \frac{1}{n}\int_{S^{n-1}} \rho(\lambda \cdot K \,\tilde{+}_p\, \mu \cdot L, u)^{n-p-i} (\lambda \rho(K, u)^p + \mu \rho(L, u)^p) \, dS(u)$$

$$= \lambda \widetilde{W}_{p,i}(\lambda \cdot K \,\tilde{+}_p\, \mu \cdot L, K) + \mu \widetilde{W}_{p,i}(\lambda \cdot K \,\tilde{+}_p\, \mu \cdot L, L).$$

Using (1.3), the inequality (3.18) is proved.

Theorem 3.2. *If* $K, L \in \mathcal{S}^n$, $p > 0$, $i \leq n-p$, $\alpha \in [0, 1]$, *then*

$$\widetilde{W}_i(K \,\tilde{+}_p\, L)^{\frac{p}{n-i}} \leq \widetilde{W}_i(\alpha \cdot K \,\tilde{+}_p\, (1-\alpha)\cdot L)^{\frac{p}{n-i}} + \widetilde{W}_i((1-\alpha)\cdot K \,\tilde{+}_p\, \alpha \cdot L)^{\frac{p}{n-i}}$$

$$\leq \widetilde{W}_i(K)^{\frac{p}{n-i}} + \widetilde{W}_i(L)^{\frac{p}{n-i}} \tag{3.19}$$

with equality if and only if K and L are dilates.

Proof. The Minkowski integral inequality, together with (2.6) and (2.8), gives

$$\widetilde{W}_i(K \,\tilde{+}_p\, L)^{\frac{p}{n-i}}$$

$$= \left(\frac{1}{n}\int_{S^{n-1}} \rho(K \,\tilde{+}_p\, L, u)^{n-i} dS(u)\right)^{\frac{p}{n-i}}$$

$$= \left(\frac{1}{n}\int_{S^{n-1}} (\rho(K, u)^p + \rho(L, u)^p)^{\frac{n-i}{p}} dS(u)\right)^{\frac{p}{n-i}}$$

$$= \left(\frac{1}{n}\int_{S^{n-1}} ((\alpha\rho(K,u)^p + (1-\alpha)\rho(L,u)^p) + ((1-\alpha)\rho(K,u)^p + \alpha\rho(L,u)^p))^{\frac{n-i}{p}} dS(u)\right)^{\frac{p}{n-i}}$$

$$\leq \left(\frac{1}{n}\int_{S^{n-1}} (\alpha\rho(K, u)^p + (1-\alpha)\rho(L, u)^p)^{\frac{n-i}{p}} dS(u)\right)^{\frac{p}{n-i}}$$

$$+ \left(\frac{1}{n}\int_{S^{n-1}} ((1-\alpha)\rho(K, u)^p + \alpha\rho(L, u)^p))^{\frac{n-i}{p}} dS(u)\right)^{\frac{p}{n-i}}$$

$$= \left(\frac{1}{n}\int_{S^{n-1}} (\rho(\alpha \cdot K \,\tilde{+}_p\, (1-\alpha)\cdot L, u)^p)^{\frac{n-i}{p}} dS(u)\right)^{\frac{p}{n-i}}$$

$$+ \left(\frac{1}{n}\int_{S^{n-1}} (\rho((1-\alpha)\cdot K \,\tilde{+}_p\, \alpha \cdot L, u)^p)^{\frac{n-i}{p}} dS(u)\right)^{\frac{p}{n-i}}$$

$$= \widetilde{W}_i(\alpha \cdot K \,\tilde{+}_p\, (1-\alpha)\cdot L)^{\frac{p}{n-i}} + \widetilde{W}_i((1-\alpha)\cdot K \,\tilde{+}_p\, \alpha \cdot L)^{\frac{p}{n-i}},$$

this is just first inequality of inequality (3.19) in Theorem 3.2.

Using Lemma 3.2, the second inequality of inequality (3.19) in Theorem 3.2 is obtained.

We establish an isolate form of the Minkowski inequalities (1.3) and (1.4).

Theorem 3.3. *If* $p \geq 1$, $K, L \in \mathcal{S}_o^n$, $0 \leq i \leq n-1$, *then*

$$\widetilde{W}_{p,i}(K, L)^{n-i} \leq \widetilde{W}_i(K)^{n-i-p}(\widetilde{W}_i(K +_p L)^{\frac{p}{n-i}} - \widetilde{W}_i(K)^{\frac{p}{n-i}})^{n-i}$$

$$\leq \widetilde{W}_i(K)^{n-i-p}\widetilde{W}_i(L)^p \tag{3.20}$$

with equality if and only if K and L are dilates.

Proof. Let $Q \in \mathcal{S}_o^n$. Then, using (2.5) and (2.9), we get

$$\widetilde{W}_{p,i}(Q, K +_p L) = \frac{1}{n}\int_{S^{n-1}} \rho(Q, u)^{n-p-i}\rho(K +_p L, u)^p\, dS(u)$$

$$= \frac{1}{n}\int_{S^{n-1}} \rho(Q, u)^{n-p-i}(\rho(K, u)^p + \rho(L, u)^p)\, dS(u)$$

$$= \widetilde{W}_{p,i}(Q, K) + \widetilde{W}_{p,i}(Q, L). \tag{3.21}$$

Let $Q = K$ in (3.21) and using (2.10), we get

$$\widetilde{W}_{p,i}(K, K +_p L) = \widetilde{W}_i(K) + \widetilde{W}_{p,i}(K, L). \tag{3.22}$$

Using (1.3), we have

$$\widetilde{W}_{p,i}(K, K +_p L) \leq \widetilde{W}_i(K)^{\frac{n-i-p}{n-i}}\widetilde{W}_i(K +_p L)^{\frac{p}{n-i}}. \tag{3.23}$$

From inequalities (3.22) and (3.23), we obtain the first inequality of inequality (3.20) in Theorem 3.3.

Using (1.1), the second inequality of inequality (3.20) in Theorem 3.3 is obtained.

Theorem 3.4. *If* $p \geq 1$, $K, L \in \mathcal{S}_o^n$, $0 \leq i \leq n-1$, *then*

$$\widetilde{W}_{p,i}(K, L)^{n-i} \le \widetilde{W}_i(K)^{n-i-p}(\widetilde{W}_i(K \tilde{+}_p L)^{\frac{p}{n-i}} - \widetilde{W}_i(K)^{\frac{p}{n-i}})^{n-i}$$

$$\le \widetilde{W}_i(K)^{n-i-p}\widetilde{W}_i(L)^p \tag{3.24}$$

with equality if and only if K and L are dilates. For $n - p < i < n$ *or* $i > n$, (3.24) *gets reversed. For* $i = n - p$, *there is an equality in* (3.24).

Proof. Using (2.7), (2.9) and (2.10), we get

$$\widetilde{W}_{p,i}(K, K \tilde{+}_p L) = \frac{1}{n}\int_{S^{n-1}} \rho(K, u)^{n-p-i}\rho(K \tilde{+}_p L, u)^p dS(u)$$

$$= \frac{1}{n}\int_{S^{n-1}} \rho(K, u)^{n-p-i}(\rho(K, u)^p + \rho(L, u)^p)dS(u)$$

$$= \widetilde{W}_i(K) + \widetilde{W}_{p,i}(K, L). \tag{3.25}$$

Using (1.4), we have

$$\widetilde{W}_{p,i}(K, K \tilde{+}_p L) \le \widetilde{W}_i(K)^{\frac{n-i-p}{n-i}}\widetilde{W}_i(K \tilde{+}_p L)^{\frac{p}{n-i}}. \tag{3.26}$$

From inequalities (3.25) and (3.26), this is just first inequality of inequality (3.24) in Theorem 3.4.

Using (1.2), the second inequality of inequality (3.24) in Theorem 3.4 is obtained.

Acknowledgement

The authors thank the anonymous referees for their valuable suggestions and constructive criticism which improved the presentation of the paper.

References

[1] E. Lutwak, The Brunn-Minkowski-Firey theory I: mixed volumes and the Minkowski problem, J. Differential Geom. 38 (1993), 131-150.

[2] E. Lutwak, The Brunn-Minkowski-Firey theory II: affine and geominimal surface areas, Adv Math. 118 (1996), 244-294.

[3] E. Lutwak, D. Yang and G. Zhang, Sharp affine L_p Sobolev inequalities, J. Differential Geom. 62 (2002), 17-38.

[4] E. Lutwak, D. Yang and G. Zhang, The Cramer-Rao inequality for star bodies, Duke Math. J. 112 (2002), 59-81.

[5] E. Lutwak and G. Zhang, Blaschke-Santalo inequalities, J. Differential Geom. 47 (1997), 1-16.

[6] E. Lutwak, D. Yang and G. Zhang, On the L_p-Minkowski problem, Trans. Amer. Math. Soc. 356 (2004), 4359-4370.

[7] G. Zhu, The logarithmic Minkowski problem for polytopes, Adv. Math. 262 (2014), 909-931.

[8] G. Zhu, The centro-affine Minkowski problem for polytopes, J. Differential Geom. 101 (2015), 159-174.

[9] G. Zhu, The L_p Minkowski problem for polytopes for $0 < p < 1$, J. Funct. Anal. 269 (2015), 1070-1094.

[10] R. J. Gardner, Geometric Tomography, Cambridge University Press, Cambridge, 2006.

[11] Y. Hu, On Several Problems of L_p-space Theory, Sciences College of Shanghai University, Shanghai, 2011.

[12] X. Li and B. He, The dual Brunn-Minkowski-Firey theorem, J. Math. 25 (2005), 545-548.

[13] R. Schneider, Convex Bodies: The Brunn-Minkowski Theory, Cambridge University Press, Cambridge, 1993.

[14] B. Wei and W. Wang, Dresher's type inequalities for dual quermassintegral, Chinese Quart. J. Math. 30 (2015), 199-205.