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ON q-HYPERGEOMETRIC FUNCTIONS

K. A. Challab[1], **M. Darus**[1,*] **and F. Ghanim**[2]

[1]School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600, Selangor D. Ehsan
Malaysia

[2]Department of Mathematics
College of Sciences
University of Sharjah
Sharjah, United Arab Emirates

Abstract

In this article, we study some results on meromorphic functions defined by q-hypergeometric functions. In addition, certain sufficient conditions for these meromorphic functions to satisfy a subordination property are also pointed out. In fact, these results extend known results of starlikeness, convexity, and close to convexity.

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[*]Corresponding author

1. Introduction

In the present paper, we initiate the study of functions which are meromorphic in the punctured disk $\mathbf{U}^* = \{z : 0 < |z| < 1\}$ with a Laurent expansion about the origin, see [8].

Let A be the class of analytic functions $h(z)$ with $h(0) = 1$, which are convex and univalent in the open unit disk $\mathbf{U} = \mathbf{U}^* \bigcup \{0\}$ and for which

$$\Re\{h(z)\} > 0 \quad (z \in \mathbf{U}^*). \tag{1.1}$$

For functions f and g analytic in $\mathbf{U}$, we say that f is *subordinate to* g and write

$$f \prec g \text{ in } U \text{ or } f(z) \prec g(z) \quad (z \in \mathbf{U}) \tag{1.2}$$

if there exists an analytic function $\omega(z)$ in $\mathbf{U}$ such that

$$|\omega(z)| \le |z|, \quad f(z) = g(\omega(z)) \quad (z \in \mathbf{U}). \tag{1.3}$$

Furthermore, if the function g is univalent in $\mathbf{U}$, then

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0),$$

$$f(\mathbf{U}) \subseteq g(\mathbf{U}) \quad (z \in \mathbf{U}). \tag{1.4}$$

2. Preliminaries

Let Σ denote the class of meromorphic functions $f(z)$ normalized by

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} d_n z^n, \tag{2.1}$$

which are analytic in the punctured unit disk $\mathbf{U}^*$. For $0 \le \beta$, we denote by $S^*(\beta)$ and $K(\beta)$ the subclasses of Σ consisting of all meromorphic functions which are, respectively, starlike of order β and convex of order β in $\mathbf{U}^*$.

For functions $f_j(z)(j = 1, 2)$ defined by

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} d_{n,j} z^n, \tag{2.2}$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2) = \frac{1}{z} + \sum_{n=1}^{\infty} d_{n,1} d_{n,2} z^n. \tag{2.3}$$

Cho et al. [3] and Ghanim and Darus [6] studied the following function:

$$q_{\lambda,\mu}(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{\lambda}{n+1+\lambda}\right)^{\mu} z^n \quad (\lambda > 0, \mu \geq 0). \tag{2.4}$$

Corresponding to the function $q_{\lambda,\mu}(z)$ and using the Hadamard product for $f(z) \in \Sigma,$ Ghanim and Darus [7] defined a linear operator $L(\lambda, \mu)$ on Σ by

$$L_{\lambda,\mu} f(z) = (f(z) * q_{\lambda,\mu}(z)) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{\lambda}{n+1+\lambda}\right)^{\mu} |d_n| z^n. \tag{2.5}$$

As for the result of this paper on applications involving generalized hypergeometric functions, we need to utilize the well-known q-hypergeometric function.

For complex parameters $a_1, ..., a_l$ and $b_1, ..., b_m$ $(b_j \neq 0, -1, ...; j = 1, 2, ..., m),$ the q-hypergeometric function ${}_l\Psi_m(z)$ is defined by

$$ {}_l\Psi_m(a_1, ..., a_l; b_1, ..., b_m; q, z) := \sum_{n=0}^{\infty} \frac{(a_1, q)_n \ldots (a_l, q)_n}{(q, q)_n (b_1, q)_n \ldots (b_m, q)_n} \times \left[(-1)^n q^{\binom{n}{2}}\right]^{1+m-l} z^n, \tag{2.6}$$

with $\binom{n}{2} = n(n-1)/2$, where $q \neq 0$ when $l > m+1$ $(l, m \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\};\ z \in \mathbf{U})$.

The q-shifted factorial is defined for $a, q \in \mathbb{C}$ as a product of n factors by

$$(a : q)_n = \begin{cases} (1-a)(1-aq)\cdots(1-aq^{n-1}) & (n \in \mathbb{N}) \\ 1 & (n = 0) \end{cases} \tag{2.7}$$

and in terms of basic analogue of the gamma function

$$(q^a; q)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}, \quad n > 0. \tag{2.8}$$

It is of interest to note that

$$\lim_{q \to -1}((q^a; q)_n/(1-q)^n) = (a)_n = a(a+1)\cdots(a+n-1)$$

is the familiar Pochhammer symbol and

$${}_l\Psi_m(a_1, \ldots, a_l; b_1, \ldots, b_m; z) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_l)_n}{(b_1)_n \cdots (b_m)_n} \frac{z^n}{n!}. \tag{2.9}$$

Now for $z \in \mathbf{U}$, $0 < |q| < 1$ and $l = m+1$, the basic hypergeometric function defined in (2.9) takes the form

$${}_l\Psi_m(a_1, \ldots, a_l; b_1, \ldots, b_m; q, z) := \sum_{n=0}^{\infty} \frac{(a_1, q)_n \cdots (a_l, q)_n}{(q, q)_n (b_1, q)_n \cdots (b_m, q)_n} z^n, \tag{2.10}$$

which converges absolutely in the open unit disk $\mathbf{U}$.

Corresponding to the function ${}_l\Psi_m(a_1, \ldots, a_l; b_1, \ldots, b_m; q, z)$ recently for meromorphic functions $f \in \Sigma$ consisting functions of the form (2.1), Aldweby and Darus [1] introduced q-analogue of Liu-Srivastava operator as below:

$$
\begin{aligned}
& {}_l\Upsilon_m(a_1, \ldots, a_l; b_1, \ldots, b_m; q, z) * f(z) \\
&= \frac{1}{z}\, {}_l\Psi_m(a_1, \ldots, a_l; b_1, \ldots, b_m; q, z) * f(z) \\
&= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{l} (a_i, q)_{n+1}}{(q, q)_{n+1} \prod_{i=1}^{m} (b_i, q)_{n+1}} d_n z^n \quad (z \in \mathbf{U}^*),
\end{aligned} \tag{2.11}
$$

where $\prod_{k=1}^{s} (a_k, q)_{n+1} = (a_1, q)_{n+1} \ldots (a_s, q)_{n+1}$, and

$$
\begin{aligned}
{}_l\Upsilon_m(a_1, \ldots, a_l; b_1, \ldots, b_m; q, z) &= \frac{1}{z}\, {}_l\Psi_m(a_1, \ldots, a_l; b_1, \ldots, b_m; q, z) \\
&= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{l} (a_i, q)_{n+1}}{(q, q)_{n+1} \prod_{i=1}^{m} (b_i, q)_{n+1}} z^n.
\end{aligned}
$$

Corresponding to the functions ${}_l\Upsilon_m(a_1, \ldots, a_l; b_1, \ldots, b_m; q, z)$ and $q_{\lambda, \mu}(z)$ given in (2.4) and using the Hadamard product for $f(z) \in \Sigma$, we will define a new linear operator as $L_\mu^\lambda(a_1, a_2, \ldots, a_l; b_1, b_2, \ldots, b_m; q)$ on Σ by

$$
\begin{aligned}
& L_\mu^\lambda(a_1, a_2, \ldots, a_l; b_1, b_2, \ldots, b_m; q) f(z) \\
&= (f(z) * {}_l\Upsilon_m(a_1, \ldots, a_l; b_1, \ldots, b_m; q, z)) * q_{\lambda, \mu}(z) \\
&= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{l} (a_i, q)_{n+1}}{(q, q)_{n+1} \prod_{i=1}^{m} (b_i, q)_{n+1}} \left(\frac{\lambda}{n+1+\lambda}\right)^{\mu} | d_n | z^n,
\end{aligned} \tag{2.12}
$$

and for convenience, we shall henceforth denote

$$
L_\mu^\lambda(a_1, a_2, \ldots, a_l; b_1, b_2, \ldots, b_m; q) f(z) = L_\mu^\lambda(a_l, b_m, q) f(z). \tag{2.13}
$$

Remark 2.1. (i) For $\mu = 0$, $a_i = q^{a_i}$, $b_j = q^{b_j}$, $a_i > 0$, $b_j > 0$ $(i = 1, \ldots, l,\ j = 1, \ldots, m,\ l = m+1)$, $q \to 1$, the operator $L_{\mu}^{\lambda}(a_l, b_m, q) f(z) = H_m^l[a_l] f(z)$ which was investigated by Liu and Srivastava [11].

(ii) For $\mu = 0$, $l = 2$, $m = 1$, $a_2 = q$, $q \to 1$, the operator $L_{\mu}^{\lambda}(a_l, q, b_m, q) f(z) = L(a_l, b_m) f(z)$ was introduced and studied by Liu and Srivastava [10].

(iii) For $\mu = 0$, $l = 1$, $m = 0$, $a_i = \gamma + 1$, $q \to 1$, the operator $L_{\mu}^{\lambda}(\gamma + 1, b_m, q) f(z) = D^{\gamma} f(z) = \dfrac{1}{z(1-z)^{\gamma+1}} * f(z) (\gamma > -1)$, where D^{γ} is the differential operator which was introduced by Ganigi and Uralegaddi [4] and then it was generalized by Yang [13].

For a function $f \in L_{\mu}^{\lambda}(a_l, b_m, q) f(z)$, we define

$$I^0(L_{\mu}^{\lambda}(a_l, b_m, q) f(z)) = L_{\mu}^{\lambda}(a_l, b_m, q) f(z),$$

and for $k = 1, 2, 3, \ldots,$

$$I^k(L_{\mu}^{\lambda}(a_l, b_m, q) f(z)) = z(I^{k-1} L_{\mu}^{\lambda}(a_l, b_m, q) f(z))' + \frac{2}{z},$$

$$I^k(L_{\mu}^{\lambda}(a_l, b_m, q) f(z))$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} n^k \frac{\prod_{i=1}^{l} (a_i, q)_{n+1}}{(q, q)_{n+1} \prod_{i=1}^{m} (b_i, q)_{n+1}} \left(\frac{\lambda}{n+1+\lambda}\right)^{\mu} |d_n| z^n. \tag{2.14}$$

We note that I^k in (2.14) was studied by Ghanim and Darus [5], and Challab and Darus [2, 14]. Also, it follows from (2.12) that

$$z(L_{\mu}^{\lambda}(a_l, b_m, q) f(z))' = n L_{\mu}^{\lambda}(a_l, b_m, q) f(z) - \frac{n+1}{z}, \tag{2.15}$$

also, from (2.15), we get

$$z\left(I^k L_\mu^\lambda(a_l, b_m, q) f(z)\right)' = nI^k L_\mu^\lambda(a_l, b_m, q) f(z) - \frac{n+1}{z}. \tag{2.16}$$

We obtain certain sufficient conditions for a function $f \in \Sigma$ to satisfy either of the following subordinations:

$$\frac{I^k L_\mu^\lambda(a_l+1, b_m, q) f(z)}{I^k L_\mu^\lambda(a_l, b_m, q) f(z)} \prec \frac{\gamma(1-z)}{\gamma - z},$$

$$\frac{I^k L_\mu^\lambda(a_l, b_m, q) f(z)}{z} \prec \frac{1+Az}{1-z},$$

$$\frac{I^k L_\mu^\lambda(a_l, b_m, q) f(z)}{z} \prec \frac{\gamma(1-z)}{\gamma - z}.$$

To prove our main results, we need the following:

Lemma 2.1 (cf. Miller and Mocanu [12, Theorem 3.4h, p. 132]). *Let* $q(z)$ *be univalent in the unit disk* $\mathbf{U}$ *and let* ϑ *and* φ *be analytic in a domain* $q(\mathbf{U}) \subset D$, *with* $\varphi(\omega) \neq 0$ *when* $q(\mathbf{U}) \in \omega$. *Set*

$$Q(z) := zq'(z)\varphi(q(z)), \quad h(z) := \vartheta(q(z)) + Q(z).$$

Suppose that

(1) $Q(z)$ is starlike univalent in $\mathbf{U}$, and

(2) $\Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$ for $z \in \mathbf{U}$.

If $p(z)$ is analytic in $\mathbf{U}$ with $p(0) = q(0)$, $p(\mathbf{U}) \subset D$ and

$$\vartheta(p(z)) + zp'(z)\varphi(p(z)) \prec \vartheta(q(z)) + zq'(z)\varphi(q(z)), \tag{2.17}$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

3. Main Results

Theorem 3.1. *Let* $a \in \mathbb{R}$ *satisfy* $-1 \le a \le 1$ *and* $\gamma > 1$. *If* $f \in \Sigma$ *satisfies* $z(I^k L_\mu^\lambda(a_l, b_m, q) f(z)) \ne 0$ *in* $\mathbf{U}^*$ *and*

$$\left(\frac{I^k L_\mu^\lambda(a_l+1, b_m, q) f(z)}{I^k L_\mu^\lambda(a_l, b_m, q) f(z)}\right)^a \left(\frac{n+1}{z(I^k L_\mu^\lambda(a_l+1, b_m, q) f(z))}\right) \prec h(z), \quad (3.1)$$

where

$$h(z) = \left(\frac{\gamma(1-z)}{\gamma - z}\right)^{a+1}\left(\frac{n+1}{z(I^k L_\mu^\lambda(a_l+1, b_m, q) f(z))} - \frac{(\gamma-1)z}{\gamma(1-z)^2}\right),$$

then

$$\frac{I^k L_\mu^\lambda(a_l+1, b_m, q) f(z)}{I^k L_\mu^\lambda(a_l, b_m, q) f(z)} \prec \frac{\gamma(1-z)}{\gamma - z}.$$

Proof. The condition (3.1) and $z(I^k L_\mu^\lambda(a_l, b_m, q) f(z)) \ne 0$ in $\mathbf{U}^*$ imply that $z(I^k L_\mu^\lambda(a_l+1, b_m, q) f(z)) \ne 0$ in $\mathbf{U}^*$. Define the function $p(z)$ by

$$p(z) := \frac{I^k L_\mu^\lambda(a_l+1, b_m, q) f(z)}{I^k L_\mu^\lambda(a_l, b_m, q) f(z)}.$$

Clearly, $p(z)$ is analytic in $\mathbf{U}^*$. A computation shows that

$$\frac{zp'(z)}{p(z)} = \frac{z[I^k L_\mu^\lambda(a_l+1, b_m, q) f(z)]'}{I^k L_\mu^\lambda(a_l+1, b_m, q) f(z)} - \frac{z[I^k L_\mu^\lambda(a_l, b_m, q) f(z)]'}{I^k L_\mu^\lambda(a_l, b_m, q) f(z)}. \quad (3.2)$$

By using the identities (2.16) and (3.2), we get

$$\frac{n+1}{z(I^k L_\mu^\lambda(a_l+1, b_m, q) f(z))} = \frac{(n+1)p(z)}{z(I^k L_\mu^\lambda(a_l+1, b_m, q) f(z))} - \frac{zp'(z)}{p(z)}. \quad (3.3)$$

Using (3.3) in (3.1), we get

$$\frac{(n+1)(p(z))^{a+1}}{z(I^k L_\mu^\lambda(a_l+1, b_m, q) f(z))} - zp'(z)(p(z))^{a-1} \prec h(z). \tag{3.4}$$

Let $q(z)$ be the function defined by

$$q(z) := \frac{\gamma(1-z)}{\gamma - z}.$$

It is clear that q is convex univalent in $\mathbf{U}^*$. Since

$$h(z) = \frac{(n+1)(q(z))^{a+1}}{z(I^k L_\mu^\lambda(a_l+1, b_m, q) f(z))} - zq'(z)(q(z))^{a-1},$$

we see that (3.4) can be written as (2.17) when ϑ and φ are given by

$$\vartheta(\omega) = \frac{(n+1)}{z(I^k L_\mu^\lambda(a_l+1, b_m, q) f(z))}\,\omega^{a+1} \quad \text{and} \quad \varphi(\omega) = \omega^{a-1}.$$

Clearly, φ and ϑ are analytic in $\mathbb{C}\backslash 0$. Now

$$Q(z) := zq'(z)\varphi(q(z)) = zq'(z)(q(z))^{a-1} = \frac{(1-\gamma)z\gamma^a(1-z)^{a-1}}{(\gamma-z)^{1+a}},$$

$$\begin{aligned} h(z) &:= \vartheta(q(z)) + Q(z) \\ &= \left(\frac{\gamma(1-z)}{\gamma-z}\right)^{1+a}\left(\frac{(n+1)}{z(I^k L_\mu^\lambda(a_l+1, b_m, q) f(z))} - \frac{(\gamma-1)z}{\gamma(1-z)^2}\right). \end{aligned}$$

By our assumptions on the parameters a and γ, we see that

$$\begin{aligned} \Re\frac{zQ'(z)}{Q(z)} &= \Re\left(1 + \frac{z(1-a)}{1-z} + (1+a)\frac{z}{\gamma-z}\right) \\ &> 1 - \frac{1}{2}(1-a) - \frac{(1+a)\gamma}{1+\gamma} \\ &= \frac{(1+a)(\gamma-1)}{2(1+\gamma)} > 0, \end{aligned}$$

and therefore $Q(z)$ is starlike. Also, we have

$$\Re\frac{zh'(z)}{Q(z)}$$

$$= (n+1)\Re\frac{\gamma(z-1)[z(\gamma-z)(1+a)-(1-z)((\gamma-z)(n+1)-z(1+a))]}{z^2(\gamma-z)(1-\gamma)(I^kL_\mu^\lambda(a_l+1, b_m, q)f(z))}$$

$$+\Re\frac{zQ'(z)}{Q(z)} \geq 0.$$

By an application of Lemma 2.1, we have $p(z) \prec q(z)$ or

$$\frac{I^kL_\mu^\lambda(a_l+1, b_m, q)f(z)}{I^kL_\mu^\lambda(a_l, b_m, q)f(z)} \prec \frac{\gamma(1-z)}{\gamma - z}.$$

This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let* $-1 < a < 0$ *and* $-1 < A < 1$. *If* $f \in \Sigma$ *satisfies the condition* $\frac{I^kL_\mu^\lambda(a_l, b_m, q)f(z)}{z} \neq 0$ *in* $\mathbf{U}^*$ *and*

$$\left(\frac{I^kL_\mu^\lambda(a_l, b_m, q)f(z)}{z}\right)^a\left(\frac{n+1}{z^2}\right) \prec h(z), \tag{3.5}$$

where

$$h(z) = \left(\frac{1+Az}{1-z}\right)^{a+1}\left((n-1)+\frac{(1+A)z}{(1-z)(1+Az)}\right),$$

then

$$\frac{I^kL_\mu^\lambda(a_l, b_m, q)f(z)}{z} \prec \frac{1+Az}{1-z}.$$

Proof. Define the function $p(z)$ by

$$p(z) := \frac{I^kL_\mu^\lambda(a_l, b_m, q)f(z)}{z}. \tag{3.6}$$

It is clear that p is analytic in $\mathbf{U}^*$. By using the identity (2.16), we get from (3.6),

$$\frac{n+1}{z^2} = (n-1)\,p(z) - zp'(z). \tag{3.7}$$

Using (3.7) in (3.5), we see that the subordination becomes

$$(n-1)\,p(z)^{1+a} - z(p(z))^a\, p'(z) \prec h(z).$$

Define the function $q(z)$ by

$$q(z) := \frac{1+Az}{1-z}.$$

It is clear that $q(z)$ is univalent in $\mathbf{U}$ and $q(\mathbf{U})$ is the region $\Re q(z) > (1-A)/2.$ Define the functions ϑ and φ by

$$\vartheta(\omega) = (n-1)\omega^{a+1} \quad \text{and} \quad \varphi(\omega) = \omega^a.$$

We observe that (3.5) can be written as (2.17). Note that φ and ϑ are analytic in $\mathbb{C}\backslash 0.$ Also, we see that

$$Q(z) := zq'(z)\varphi(q(z)) = \frac{z(1+A)(1+Az)^a}{(1-z)^{2+a}}$$

and

$$h(z) := \vartheta(q(z)) + Q(z) = \left(\frac{1+Az}{1-z}\right)^{a+1}\left((n-1) + \frac{(1+A)z}{(1+Az)(1-z)}\right).$$

By our assumptions, we have

$$\Re\frac{zQ'(z)}{Q(z)} = \Re\left[1 + a\frac{Az}{1+Az} + (2+a)\frac{z}{1-z}\right]$$

$$> 1 + \frac{a|A|}{1+|A|} - \frac{2+a}{2} = \frac{-a(1-|A|)}{2(1+|A|)} > 0,$$

and

$$\Re\frac{zh'(z)}{Q(z)} = \Re\left[\frac{\vartheta'(q(z))}{\varphi(q(z))} + \frac{zQ'(z)}{Q(z)}\right] = (n-1)(1+a) + \Re\frac{zQ'(z)}{Q(z)} \geq 0.$$

The results now follow by an application of Lemma 2.1. □

Theorem 3.3. *Let* $a \geq -1$, $\gamma > 1$, $f \in \Sigma$ *and* $I^k L_\mu^\lambda(a_l, b_m, q) f(z)/z \neq 0$ *in* $\mathbf{U}^*$. *If f satisfies*

$$\left(\frac{I^k L_\mu^\lambda(a_l, b_m, q) f(z)}{z}\right)^a \left(\frac{n+1}{z^2}\right)$$

$$\prec \left(\frac{\gamma(1-z)}{(\gamma-z)}\right)^{1+a}\left((n-1) - \frac{z(1-\gamma)}{(1-z)(\gamma-z)}\right), \tag{3.8}$$

then

$$\frac{I^k L_\mu^\lambda(a_l, b_m, q) f(z)}{z} \prec \frac{\gamma(1-z)}{\gamma-z} \quad (z \in U^*).$$

Proof. Define the function $p(z)$ by

$$p(z) = \frac{I^k L_\mu^\lambda(a_l, b_m, q) f(z)}{z}. \tag{3.9}$$

Clearly, $p(z)$ is analytic in $\mathbf{U}^*$, we can compute to show

$$zp'(z) = \frac{z(I^k L_\mu^\lambda(a_l, b_m, q) f(z))' - I^k L_\mu^\lambda(a_l, b_m, q) f(z)}{z}. \tag{3.10}$$

By using the identity (2.16), we can get from (3.10),

$$\frac{n+1}{z^2} = (n-1)p(z) - zp'(z). \tag{3.11}$$

Using (3.11) in (3.8), we obtain

$$(n-1)(p(z))^{a+1} - zp'(z)(p(z))^a$$

$$\prec \left(\frac{\gamma(1-z)}{(\gamma-z)}\right)^{1+a}\left((n-1) - \frac{z(1-\gamma)}{(1-z)(\gamma-z)}\right). \tag{3.12}$$

Define the function $q(z)$ by

$$q(z) = \frac{\gamma(1-z)}{\gamma - z}$$

which is univalent in $\mathbf{U}^*$. We see that (3.12) can be written as (2.17) when ϑ and φ are given by $\vartheta(\omega) = (n-1)\omega^{a+1}$, $\varphi(\omega) = -\omega^a$ such that ϑ and φ are analytic in $\mathbf{C}\backslash 0$. Now

$$Q(z) = zq'(z)\varphi(q(z)) = \frac{-z\gamma^{a+1}(1-\gamma)(1-z)^a}{(\gamma-z)^{a+z}},$$

$$h(z) = \vartheta(q(z)) + Q(z) = \left(\frac{\gamma(1-z)}{(\gamma-z)}\right)^{1+a}\left((n-1) - \frac{z(1-\gamma)}{(1-z)(\gamma-z)}\right).$$

By our assumptions, we have

$$\Re\frac{zQ'(z)}{Q(z)} = \Re\left(1 + \frac{z(1-a)}{1-z} + (1+a)\frac{z}{\gamma-z}\right)$$

$$> 1 - \frac{1}{2}(1-a) - \frac{(1+a)\gamma}{1+\gamma}$$

$$= \frac{(1+a)(\gamma-1)}{2(1+\gamma)} > 0,$$

hence $Q(z)$ is starlike. Now

$$\Re\frac{zh'(z)}{Q(z)} = \Re\left[\frac{\vartheta'(q(z))}{\varphi(q(z))} + \frac{zQ'(z)}{Q(z)}\right] = (1-n)(1+a) + \Re\frac{zQ'(z)}{Q(z)} \geq 0.$$

Now we can apply Lemma 2.1 to get $p(z) \prec q(z)$. We have

$$\frac{I^k L_\mu^\lambda(a_l + 1, b_m, q) f(z)}{z} \prec \frac{\gamma(1-z)}{\gamma - z}.$$

This completes the proof of Theorem 3.3. $\square$

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