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# GENERIC LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE TRANS-SASAKIAN MANIFOLD

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## Abstract

We study generic lightlike submanifolds $M$ of an indefinite trans-Sasakian manifold $\bar{M}$. The purpose of this paper is to prove several classification theorems of such a generic lightlike submanifold subject to the condition that the structure vector field $\zeta$ of $\bar{M}$ is tangent to $M$.

## 1. Introduction

In the theory of Riemannian submanifold, there exists a class of submanifolds of an almost contact manifold $\bar{M}$. A submanifold $M$ of $\bar{M}$ is called *generic* [12, 13] if the normal bundle $TM^{\perp}$ of $M$ is mapped into the tangent bundle $TM$ by action of the almost contact structure tensor $J$ of $\bar{M}$, that is,

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$$J(TM^{\perp}) \subset TM.$$

We extended the concept of generic submanifold in case *M* is a lightlike submanifold of an indefinite almost contact manifold $\overline{M}$. In case *M* is lightlike submanifold, the radical distribution $Rad(TM) = TM \bigcap TM^{\perp}$ is non-trivial vector bundle of *M* and *TM* is lightlike vector bundle. Thus, we have

$$T\overline{M} \neq TM \oplus_{orth} TM^{\perp}.$$

Consider a complementary vector bundle $S(TM)$ of $Rad(TM)$ in *TM*, i.e.,

$$TM = Rad(TM) \oplus_{orth} S(TM).$$

We call $S(TM)$ a *screen distribution* of *M*. It is immediate from the last equation that $S(TM)$ is non-degenerate. Moreover, if *M* is para-compact, then there always exists a screen distribution $S(TM)$. Along *M*, we have

$$T\overline{M}_{|M} = S(TM) \oplus_{orth} S(TM)^{\perp}, \; S(TM) \bigcap S(TM)^{\perp} \neq \{0\},$$

where $S(TM)^{\perp}$ is orthogonal complement to $S(TM)$ in $T\overline{M}_{|M}$. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $S(TM)^{*} = TM/Rad(TM)$ [10]. Thus, all $S(TM)s$ are mutually isomorphic. Moreover, while *TM* is lightlike, all $S(TM)s$ are non-degenerate. Due to these reasons, we defined generic lightlike submanifold as follows [6-8]:

A lightlike submanifold *M* of an indefinite almost contact manifold $\overline{M}$ is called *generic* if there exists a screen distribution $S(TM)$ of *M* such that

$$J(S(TM)^{\perp}) \subset S(TM). \tag{1.1}$$

The geometry of generic lightlike submanifold is an extension of the geometry of lightlike hypersurface or 1-lightlike submanifold. Much of its theory will be immediately generalized in a formal way to general lightlike submanifolds.

Oubina [11] introduced the notion of trans-Sasakian manifold of type $(\alpha, \beta)$, where $\alpha$ and $\beta$ are smooth functions. Sasakian, Kenmotsu and cosymplectic manifolds are three important kinds of trans-Sasakian manifold such that

$$\alpha = \varepsilon, \beta = 0; \quad \alpha = 0, \beta = \varepsilon; \quad \alpha = \beta = 0,$$

respectively, where $\varepsilon = \pm 1$. In this case, if $\overline{M}$ is a semi-Riemannian manifold, then we say that $\overline{M}$ is an *indefinite trans-Sasakian manifold of type* $(\alpha, \beta)$.

Alegre et al. [2] introduced generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$. Sasakian, Kenmotsu and cosymplectic space forms are three important kinds of generalized Sasakian space forms such that

$$f_1 = \frac{c+3}{4},\ f_2 = f_3 = \frac{c-1}{4}; \quad f_1 = \frac{c-3}{4},\ f_2 = f_3 = \frac{c+1}{4};$$

$$f_1 = f_2 = f_3 = \frac{c}{4},$$

respectively, where $c$ is a constant $J$-sectional curvature of each space forms.

We study generic lightlike submanifolds of an indefinite trans-Sasakian manifold $\overline{M}$ or an indefinite generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$. We prove several classification theorems of such a generic lightlike submanifold subject such that the structure vector field $\zeta$ of $\overline{M}$ is tangent to $M$.

## 2. Preliminaries

An odd-dimensional semi-Riemannian manifold $(\overline{M}, \bar{g})$ is called an *indefinite trans-Sasakian manifold* if there exists a structure set $\{J, \zeta, \theta, \bar{g}\}$, where $J$ is a (1, 1)-type tensor field, $\zeta$ is a vector field which is called the *structure vector field* and $\theta$ is a 1-form, a Levi-Civita connection $\overline{\nabla}$ on $\overline{M}$ and two smooth functions $\alpha$ and $\beta$ on $\overline{M}$, such that

$$J^2\overline{X} = -\overline{X} + \theta(\overline{X})\zeta,\ \theta(\zeta) = 1,\ \theta(\overline{X}) = \epsilon\overline{g}(\overline{X}, \zeta),$$

$$\theta \circ J = 0,\ \overline{g}(J\overline{X}, J\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) - \epsilon\theta(\overline{X})\theta(\overline{Y}), \tag{2.1}$$

$$(\overline{\nabla}_{\overline{X}}J)\overline{Y} = \alpha\{\overline{g}(\overline{X}, \overline{Y})\zeta - \epsilon\theta(\overline{Y})\overline{X}\} + \beta\{\overline{g}(J\overline{X}, \overline{Y})\zeta - \epsilon\theta(\overline{Y})J\overline{X}\}, \tag{2.2}$$

for any vector fields $\overline{X}, \overline{Y}$ and $\overline{Z}$ on $\overline{M}$, where $\epsilon = 1$ or $-1$ according as the vector field $\zeta$ is spacelike or timelike, respectively. In this case, the set $\{J, \zeta, \theta, \overline{g}\}$ is called an *indefinite trans-Sasakian structure of type* $(\alpha, \beta)$.

Throughout this paper, we may assume that the structure vector field $\zeta$ is unit spacelike, i.e., $\epsilon = 1$, no loss of generality. From (2.1) and (2.2), we get

$$\overline{\nabla}_{\overline{X}}\zeta = -\alpha J\overline{X} + \beta\{\overline{X} - \theta(\overline{X})\zeta\}, \quad d\theta(\overline{X}, \overline{Y}) = \alpha\overline{g}(\overline{X}, J\overline{Y}). \tag{2.3}$$

An indefinite trans-Sasakian manifold $\overline{M}$ is called an *indefinite generalized Sasakian space form* and denote it by $\overline{M}(f_1, f_2, f_3)$ if it admits a curvature tensor $\overline{R}$ and three smooth functions $f_1$, $f_2$ and $f_3$ satisfying

$$\begin{aligned}\overline{R}(\overline{X}, \overline{Y})\overline{Z} = {} & f_1\{\overline{g}(\overline{Y}, \overline{Z})\overline{X} - \overline{g}(\overline{X}, \overline{Z})\overline{Y}\} \\ & + f_2\{\overline{g}(\overline{X}, J\overline{Z})J\overline{Y} - \overline{g}(\overline{Y}, J\overline{Z})J\overline{X} + 2\overline{g}(\overline{X}, J\overline{Y})J\overline{Z}\} \\ & + f_3\{\theta(\overline{X})\theta(\overline{Z})\overline{Y} - \theta(\overline{Y})\theta(\overline{Z})\overline{X} \\ & + \overline{g}(\overline{X}, \overline{Z})\theta(\overline{Y})\zeta - \overline{g}(\overline{Y}, \overline{Z})\theta(\overline{X})\zeta\}.\end{aligned} \tag{2.4}$$

Let $(M, g)$ be an $m$-dimensional lightlike submanifold of an indefinite trans-Sasakian manifold $(\overline{M}, \overline{g})$, of dimension $(m + n)$. Then the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ of $M$ is a subbundle of the tangent bundle $TM$ and the normal bundle $TM^{\perp}$, of rank $r(1 \le r \le \min\{m, n\})$. In general, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in $TM$ and $TM^{\perp}$, respectively, which are called the *screen distribution* and the *co-screen distribution* of $M$, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on $M$ and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle $E$. Also denoted by $(2.1)_i$ the $i$th equation of (2.1). We use the same notations for any others. We use the following range of indices:

$$i, j, k, \ldots \in \{1, \ldots, r\}, \quad a, b, c, \ldots \in \{r+1, \ldots, n\}.$$

Let $tr(TM)$ and $ltr(TM)$ be complementary vector bundles to $TM$ in $T\overline{M}_{|M}$ and $TM^{\perp}$ in $S(TM)^{\perp}$, respectively and let $\{N_1, \ldots, N_r\}$ be a lightlike basis of $ltr(TM)_{|\mathcal{U}}$, where $\mathcal{U}$ is a coordinate neighborhood of $M$, such that

$$\overline{g}(N_i, \xi_j) = \delta_{ij}, \quad \overline{g}(N_i, N_j) = 0,$$

where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $Rad(TM)_{|\mathcal{U}}$. Then we have

$$T\overline{M} = TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM)$$

$$= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}).$$

We say that a lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\overline{M}$ is

(1) *r-lightlike submanifold* if $1 \leq r < \min\{m, n\}$;

(2) *co-isotropic submanifold* if $1 \leq r = n < m$;

(3) *isotropic submanifold* if $1 \leq r = m < n$;

(4) *totally lightlike submanifold* if $1 \leq r = m = n$.

The above three classes (2)~(4) are particular cases of the class (1) as follows:

$$S(TM^{\perp}) = \{0\}, \quad S(TM) = \{0\}, \quad S(TM) = S(TM^{\perp}) = \{0\},$$

respectively. The geometry of $r$-lightlike submanifolds is more general form than that of the others. For this reason, we consider only $r$-lightlike submanifolds $M$, with following local quasi-orthonormal field of frames of $\overline{M}$ :

$$\{\xi_1, ..., \xi_r, N_1, ..., N_r, F_{r+1}, ..., F_m, E_{r+1}, ..., E_n\},$$

where $\{F_{r+1}, ..., F_m\}$ and $\{E_{r+1}, ..., E_n\}$ are orthonormal bases of $S(TM)$ and $S(TM^{\perp})$, respectively. Denote $\varepsilon_a = \overline{g}(E_a, E_a)$. Then $\varepsilon_a\delta_{ab} = \overline{g}(E_a, E_b)$.

In the following, let $X$, $Y$, $Z$ and $W$ be the vector fields on $M$, unless otherwise specified. Let $P$ be the projection morphism of $TM$ on $S(TM)$. Then the local Gauss-Weingarten formulae of $M$ and $S(TM)$ are given by

$$\overline{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^{\ell}(X, Y)N_i + \sum_{a=r+1}^{n} h_a^s(X, Y)E_a, \tag{2.5}$$

$$\overline{\nabla}_X N_i = -A_{N_i}X + \sum_{j=1}^{r} \tau_{ij}(X)N_j + \sum_{a=r+1}^{n} \rho_{ia}(X)E_a, \tag{2.6}$$

$$\overline{\nabla}_X E_a = -A_{E_a}X + \sum_{i=1}^{r} \phi_{ai}(X)N_i + \sum_{a=r+1}^{n} \sigma_{ab}(X)E_b, \tag{2.7}$$

$$\nabla_X PY = \nabla^*_X PY + \sum_{i=1}^{r} h_i^*(X, PY)\xi_i, \tag{2.8}$$

$$\nabla_X \xi_i = -A^*_{\xi_i}X - \sum_{j=1}^{r} \tau_{ji}(X)\xi_j, \tag{2.9}$$

respectively, where $\nabla$ and $\nabla^*$ are induced linear connections on $TM$ and $S(TM)$, respectively, $h_i^{\ell}$ and $h_a^s$ are called the *local second fundamental forms* on $TM$, $h_i^*$ are called the *local second fundamental forms* on $S(TM)$.

$A_{N_i}$, $A_{E_a}$ and $A^*_{\xi_i}$ are the shape operators and $\tau_{ij}$, $\rho_{ia}$, $\phi_{ai}$ and $\sigma_{\alpha\beta}$ are 1-forms.

Since $\overline{\nabla}$ is torsion-free, $\nabla$ is also torsion-free, and both $h_i^\ell$ and $h_a^s$ are symmetric. From the fact that $h_i^\ell(X, Y) = \overline{g}(\overline{\nabla}_X Y, \xi_i)$, we know that each $h_i^\ell$ is independent of the choice of the screen distribution $S(TM)$. The above three local second fundamental forms are related to their shape operators by

$$g(A^*_{\xi_i} X, Y) = h_i^\ell(X, Y) + \sum_{j=1}^{r} h_j^\ell(X, \xi_i)\eta_j(Y), \tag{2.10}$$

$$g(A_{E_a} X, Y) = \varepsilon_a h_a^s(X, Y) + \sum_{i=1}^{r} \phi_{ai}(X)\eta_i(Y), \tag{2.11}$$

$$g(A_{N_i} X, PY) = h_i^*(X, PY), \eta_k(A^*_{\xi_i} X) = 0, \tag{2.12}$$

where $\eta_i$ is the 1-forms given by $\eta_i(X) = \overline{g}(X, N_i)$. Applying $\overline{\nabla}_X$ to $g(\xi_i, \xi_j) = 0$, $\overline{g}(\xi_i, E_a) = 0$, $\overline{g}(N_i, N_j) = 0$, $\overline{g}(N_i, E_a) = 0$ and $\overline{g}(E_a, E_b) = \varepsilon\delta_{ab}$, we get

$$h_i^\ell(X, \xi_j) + h_j^\ell(X, \xi_i) = 0, h_a^s(X, \xi_i) = -\varepsilon_a\phi_{ai}(X),$$

$$\eta_j(A_{N_i} X) + \eta_i(A_{N_j} X) = 0, \overline{g}(A_{E_a} X, N_i) = \varepsilon_a\rho_{ia}(X),$$

$$\varepsilon_b\sigma_{ab} + \varepsilon_a\sigma_{ba} = 0 \text{ and } h_i^\ell(X, \xi_i) = 0, h_i^\ell(\xi_j, \xi_k) = 0. \tag{2.13}$$

Denote by $\overline{R}$, $R$ and $R^*$ the curvature tensors of the Levi-Civita connection $\overline{\nabla}$ on $\overline{M}$ and the linear connections $\nabla$ and $\nabla^*$ on $M$ and $S(TM)$, respectively. By using the Gauss-Weingarten formulae (2.5)~(2.9) for $M$ and $S(TM)$, we obtain the Gauss equations for $M$ and $S(TM)$ such that

$$\begin{aligned}
\overline{R}(X, Y)Z = R(X, Y)Z &+ \sum_{i=1}^{r} \{h_i^{\ell}(X, Z)A_{N_i}Y - h_i^{\ell}(Y, Z)A_{N_i}X\} \\
&+ \sum_{a=r+1}^{n} \{h_a^s(X, Z)A_{E_a}Y - h_a^s(Y, Z)A_{E_a}X\} \\
&+ \sum_{i=1}^{r} \Big\{(\nabla_X h_i^{\ell})(Y, Z) - (\nabla_Y h_i^{\ell})(X, Z) \\
&+ \sum_{j=1}^{r} [\tau_{ji}(X)h_j^{\ell}(Y, Z) - \tau_{ji}(Y)h_j^{\ell}(X, Z)] \\
&+ \sum_{a=r+1}^{n} [\phi_{ai}(X)h_a^s(Y, Z) - \phi_{ai}(Y)h_a^s(X, Z)]\Big\} N_i \\
&+ \sum_{a=r+1}^{n} \Big\{(\nabla_X h_a^s)(Y, Z) - (\nabla_Y h_a^s)(X, Z) \\
&+ \sum_{i=1}^{r} [\rho_{ia}(X)h_i^{\ell}(Y, Z) - \rho_{ia}(Y)h_a^s(X, Z)] \\
&+ \sum_{b=r+1}^{n} [\sigma_{ba}(X)h_b^s(Y, Z) - \sigma_{ba}(Y)h_b^s(X, Z)]\Big\} E_a. \qquad (2.14)
\end{aligned}$$

$$\begin{aligned}
R(X, Y)PZ = R^*(X, Y)PZ &+ \sum_{i=1}^{r} \{h_i^*(X, PZ)A^*_{\xi_i}Y - h_i^*(Y, PZ)A^*_{\xi_i}X\} \\
&+ \sum_{i=1}^{r} \Big\{(\nabla_X h_i^*)(Y, PZ) - (\nabla_Y h_i^*)(X, PZ) \\
&+ \sum_{j=1}^{r} [\tau_{ij}(Y)h_j^*(X, PZ) - \tau_{ij}(X)h_j^*(Y, PZ)]\Big\} \xi_i. \qquad (2.15)
\end{aligned}$$

In the case $R = 0$, we say that $M$ is *flat*.

## 3. Generic Lightlike Submanifolds

For a generic lightlike submanifold *M*, from (1.1) we see that $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^{\perp}))$ are subbundles of $S(TM)$. In the following, we shall assume that the vector field $\zeta$ is tangent to *M*. Călin [3] proved that if $\zeta$ is tangent to *M*, then it belongs to $S(TM)$. Using this result, there exists a non-degenerate almost complex distribution $H_o$, i.e., $J(H_o) = H_o$, such that

$$S(TM) = \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{orth} J(S(TM^{\perp})) \oplus_{orth} H_o.$$

Denote by *H* the almost complex distribution with respect to *J* such that

$$H = Rad(TM) \oplus_{orth} J(Rad(TM)) \oplus_{orth} H_o.$$

Therefore, the general decomposition form of *TM* in Section 2 is reduced to

$$TM = H \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^{\perp})). \tag{3.1}$$

Consider $2r$-local null vector fields $U_i$ and $V_i$, $(n-r)$-local non-null unit vector fields $W_a$, and their associated 1-forms $u_i$, $v_i$ and $w_a$ defined by

$$U_i = -JN_i, \quad V_i = -J\xi_i, \quad W_a = -JE_a, \tag{3.2}$$

$$u_i(X) = g(X, V_i), \quad v_i(X) = g(X, U_i), \quad w_a(X) = \varepsilon_a g(X, W_a). \tag{3.3}$$

Denote by *S* the projection morphism of *TM* on *H* and *F* a tensor field of type (1, 1) globally defined on *M* by $F = J \circ S$. Then *JX* is expressed as

$$JX = FX + \sum_{i=1}^{r} u_i(X)N_i + \sum_{a=r+1}^{n} w_a(X)E_a. \tag{3.4}$$

Applying $\overline{\nabla}_X$ to (3.2)~(3.4) by turns and using (2.2), (2.5)~(2.13) and (3.2)~(3.4), we have

$$h_j^{\ell}(X, U_i) = h_i^{*}(X, V_j),\ \varepsilon_a h_i^{*}(X, W_a) = h_a^{s}(X, U_i),$$

$$h_j^{\ell}(X, V_i) = h_i^{\ell}(X, V_j),\ \varepsilon_a h_i^{\ell}(X, W_a) = h_a^{s}(X, V_i),$$

$$\varepsilon_b h_b^{s}(X, W_a) = \varepsilon_a h_a^{s}(X, W_b), \tag{3.5}$$

$$\nabla_X U_i = F(A_{N_i}X) + \sum_{j=1}^{r} \tau_{ij}(X)U_j + \sum_{a=r+1}^{n} \rho_{ia}(X)W_a - \{\alpha\eta_i(X) + \beta v_i(X)\}\zeta, \tag{3.6}$$

$$\nabla_X V_i = F(A^*_{\xi_i}X) - \sum_{j=1}^{r} \tau_{ji}(X)V_j + \sum_{j=1}^{r} h_j^{\ell}(X, \xi_i)U_j - \sum_{a=r+1}^{n} \epsilon_a \phi_{ai}(X)W_a - \beta u_i(X)\zeta, \tag{3.7}$$

$$\nabla_X W_a = F(A_{E_a}X) + \sum_{i=1}^{r} \phi_{ai}(X)U_i + \sum_{b=r+1}^{n} \sigma_{ab}(X)W_b - \epsilon_a \beta w_a(X)\zeta, \tag{3.8}$$

$$(\nabla_X u_i)(Y) = -\sum_{j=1}^{r} u_j(Y)\tau_{ji}(X) - \sum_{a=r+1}^{n} w_a(Y)\phi_{ai}(X) - \beta\theta(Y)u_i(X) - h_i^{\ell}(X, FY), \tag{3.9}$$

$$(\nabla_X v_i)(Y) = \sum_{j=1}^{r} v_j(Y)\tau_{ij}(X) + \sum_{a=r+1}^{n} \epsilon_a w_a(Y)\rho_{ia}(X) - \sum_{j=r+1}^{r} u_j(Y)\eta_j(A_{N_i}X) - g(A_{N_i}X, FY) - \theta(Y)\{\alpha\eta_i(X) + \beta v_i(X)\}, \tag{3.10}$$

$$(\nabla_X F)(Y) = \sum_{i=1}^{r} u_i(Y)A_{N_i}X + \sum_{a=r+1}^{n} w_a(Y)A_{E_a}X - \sum_{i=1}^{r} h_i^{\ell}(X, Y)U_i - \sum_{a=r+1}^{n} h_a^{s}(X, Y)W_a + \alpha\{g(X, Y)\zeta - \theta(Y)X\} + \beta\{\bar{g}(JX, Y)\zeta - \theta(Y)FX\}. \tag{3.11}$$

Applying $\overline{\nabla}_X$ to $\overline{g}(\zeta, \xi_i) = 0, \overline{g}(\zeta, E_a) = 0$ and $\overline{g}(\zeta, N_i) = 0$ by turns and using (2.1), (2.3), (2.5)~(2.12), (3.2) and (3.3), we have

$$h_i^{\ell}(X, \zeta) = -\alpha u_i(X), \quad h_a^s(X, \zeta) = -\alpha w_a(X),$$

$$h_i^*(X, \zeta) = -\alpha v_i(X) + \beta\eta_i(X). \tag{3.12}$$

Substituting (3.4) into (2.3) and using (2.5), we have

$$\nabla_X \zeta = -\alpha FX + \beta(X - \theta(X)\zeta). \tag{3.13}$$

We denote by $\lambda_{ij}$, $\mu_{ia}$, $\nu_{ia}$, $\kappa_{ab}$ and $\chi_{ij}$ the 1-forms such that

$$\lambda_{ij}(X) = h_i^{\ell}(X, U_j) = h_j^*(X, V_i), \quad \kappa_{ab}(X) = \varepsilon_a h_a^s(X, W_b),$$

$$\mu_{ia}(X) = h_i^{\ell}(X, W_a) = \varepsilon_a h_a^s(X, V_i), \quad \chi_{ij}(X) = h_i^{\ell}(X, V_j),$$

$$\nu_{ai}(X) = h_i^*(X, W_a) = \varepsilon_a h_a^s(X, U_i). \tag{3.14}$$

Note that, from $(3.5)_{3,5}$ and $(3.14)_{2,4}$, we see that $\chi_{ij}$ and $\kappa_{ab}$ are symmetric.

Denote by $H'$ the distribution on $H(TM)$ such that

$$H' = J(ltr(TM)) \oplus_{orth} J(S(TM^{\perp})), \quad TM = H \oplus H'.$$

**Definition.** We say that a lightlike submanifold $M$ of $\overline{M}$ is called

(1) *irrotational* [10] if $\overline{\nabla}_X \xi_i \in \Gamma(TM)$ for all $i \in \{1, ..., r\}$,

(2) *solenoidal* [9] if $A_{W_a}$ and $A_{N_i}$ are $S(TM)$-valued,

(3) *statical* [9] if $M$ is both irrotational and solenoidal.

**Remark.** From (2.5) and $(3.12)_2$, the item (1) is equivalent to

$$h_i^{\ell}(X, \xi_i) = 0, \quad h_a^s(X, \xi_i) = \phi_{ai}(X) = 0.$$

By using $(3.12)_4$, the item (2) is equivalent to

$$\eta_j(A_{N_i}X) = 0, \; \rho_{ia}(X) = \eta_j(A_{E_a}X) = 0.$$

**Theorem 3.1.** *Let M be a generic lightlike submanifold of an indefinite trans-Sasakian manifold* $\overline{M}$. *If F is parallel with respect to the induced connection* $\nabla$, *then the following statements are satisfied*:

(1) $\overline{M}$ *is an indefinite cosymplectic manifold*, *i.e.*, $\alpha = \beta = 0$,

(2) *M is statical*,

(3) $J(ltr(TM))$, $J(S(TM^{\perp}))$ *and H are parallel distributions on M*,

(4) *M is locally a product manifold* $M_r \times M_{n-r} \times M^{\#}$, *where* $M_r$, $M_{n-r}$ *and* $M^{\#}$ *are leaves of* $J(ltr(TM))$, $J(S(TM^{\perp}))$ *and H*, *respectively*.

**Proof.** (1) As $\nabla_X F = 0$, taking the scalar product with $U_j$ to (3.11), we get

$$\sum_{i=1}^{r} u_i(Y) g(A_{N_i} X, U_j) + \sum_{\alpha=r+1}^{n} w_a(Y) g(A_{E_a} X, U_j)$$
$$- \theta(Y)\{\alpha v_j(X) - \beta\eta_j(Y)\} = 0.$$

Replacing $Y$ by $\zeta$ to this equation, we have $\alpha v_j(X) - \beta\eta_j(X) = 0$. Therefore, $\alpha = \beta = 0$ and $\overline{M}$ is an indefinite cosymplectic manifold.

(2) From the last equation, we obtain

$$\bar{g}(A_{N_i} X, U_j) = 0, \quad \nu_{ai}(X) = \bar{g}(A_{E_a} X, U_j) = 0. \tag{3.15}$$

Replacing $Y$ by $\xi_j$ to (3.11) such that $\alpha = \beta = 0$, we get

$$\sum_{i=1}^{r} h_i^{\ell}(X, \xi_j) U_i + \sum_{a=r+1}^{n} h_a^s(X, \xi_j) W_a = 0.$$

From this equation and $(2.13)_2$, we obtain

$$h_i^{\ell}(X, \xi_j) = 0, \quad \phi_{ai}(X) = h_a^s(X, \xi_i) = 0. \tag{3.16}$$

Taking the scalar product with $N_j$ to (3.11) such that $\alpha = \beta = 0$, we have

$$\sum_{i=1}^{r} u_i(Y)\bar{g}(A_{N_i}X,\, N_j) + \sum_{\alpha=r+1}^{n} w_a(Y)\bar{g}(A_{E_a}X,\, N_j) = 0.$$

From this equation and $(2.13)_4$, we obtain

$$\bar{g}(A_{N_i}X,\, N_j) = 0, \quad \rho_{ia}(X) = \bar{g}(A_{E_a}X,\, N_j) = 0. \tag{3.17}$$

From (3.16) and (3.17), we see that $M$ is statical.

(3) Taking the scalar product with $V_j$ and $W_b$ to (3.11) by turns, we get

$$h_j^{\ell}(X,\, Y) = \sum_{i=1}^{r} u_i(Y)\lambda_{ji}(X) + \sum_{a=r+1}^{n} w_a(Y)\mu_{ja}(X),$$

$$h_b^{s}(X,\, Y) = \sum_{a=r+1}^{n} w_a(Y)\kappa_{ab}(X),$$

due to $\nu_{ai}(X) = 0$. Replacing $Y$ by $V_i$ to the second equation, we have

$$\mu_{ia}(X) = h_i^{\ell}(X,\, W_a) = \varepsilon_a h_a^{s}(X,\, V_i) = 0.$$

As $\rho_{ia} = 0$, $A_{W_a}X$ belongs to $S(TM)$ by $(2.13)_4$. As $A^*_{\xi_i}X$ and $A_{W_a}X$ belong to $S(TM)$ and $S(TM)$ is non-degenerate, from the last three equations we get

$$A^*_{\xi_i}X = \sum_{j=1}^{r} \lambda_{ij}(X)V_j, \quad A_{E_a}X = \sum_{b=r+1}^{n} \kappa_{ab}(X)W_b. \tag{3.18}$$

As $\nu_{aj}(X) = h_a^{s}(X,\, U_j) = 0$, taking $Y = U_j$ to (3.11), we get

$$A_{N_i}X = \sum_{j=1}^{r} \lambda_{ji}(X)U_j. \tag{3.19}$$

Taking $Y \in \Gamma(H)$ and then, taking $Y = V_j$ to (3.11) by turns, we have

$$\sum_{i=1}^{r} h_i^{\ell}(X, Y)U_i + \sum_{a=r+1}^{n} h_a^s(X, Y)W_a = 0,$$

$$\sum_{i=1}^{r} h_i^{\ell}(X, V_j)U_i + \sum_{a=r+1}^{n} h_a^s(X, V_j)W_a = 0,$$

respectively. Taking the scalar product with $U_j$ and $W_b$ to these two equations by turns, for any $X \in \Gamma(TM)$ and $Y \in \Gamma(H)$, we have

$$h_i^{\ell}(X, Y) = 0, \quad h_a^s(X, Y) = 0, \quad h_i^{\ell}(X, V_j) = 0, \quad h_a^s(X, V_j) = 0, \quad (3.20)$$

respectively. Taking the scalar product with $Z \in \Gamma(H_o)$ to (3.11), we get

$$\sum_{i=1}^{r} u_i(Y)h_i^*(X, Z) + \sum_{a=r+1}^{n} \varepsilon_a w_a(Y)h_a^s(X, Z) = 0.$$

Taking $Y = U_k$ to this equation, we have

$$h_i^*(X, Y) = 0, \quad \forall X \in \Gamma(TM), \quad Y \in \Gamma(H_o). \quad (3.21)$$

By directed calculations from (3.1) and by using (2.5), (3.5), (3.7), (3.8), (3.16), (3.17), (3.20) and the fact that $\phi_{ai} = \rho_{ia} = 0$, we derive

$$g(\nabla_X \xi_i, V_j) = g(\nabla_X V_i, V_j) = g(\nabla_X Y, V_i) = 0,$$

$$g(\nabla_X \xi_i, W_a) = g(\nabla_X V_i, W_a) = g(\nabla_X Y, W_a) = 0,$$

for all $X \in \Gamma(TM)$ and $Y \in \Gamma(H_o)$, or equivalently, we get

$$\nabla_X Y \in \Gamma(H), \quad \forall X \in \Gamma(TM), \quad \forall Y \in \Gamma(H).$$

This result implies that $H$ is a parallel distribution on $M$.

By using (3.4), (3.6), (3.15), (3.17), (3.20), (3.21) and $\rho_{ia} = 0$, we derive

$$g(\nabla_X U_i, N_j) = g(\nabla_X U_i, U_j) = g(\nabla_X U_i, Y) = 0,$$

$$g(\nabla_X W_a, N_j) = g(\nabla_X W_a, U_j) = g(\nabla_X W_a, Y) = 0,$$

for all $X \in \Gamma(TM)$ and $Y \in \Gamma(H_o)$, or equivalently, we get

$$\nabla_X Z \in \Gamma(H'), \quad \forall X \in \Gamma(TM), \quad Z \in \Gamma(H').$$

Thus, $H'$ is also a parallel distribution of $M$.

(4) As $TM = H \oplus H'$, and $H$ and $H'$ are parallel distributions, by the decomposition theorem of de Rham [4], $M$ is locally a product manifold $M_1 \times M_2$, where $M_1$ and $M_2$ are leaves of the distributions $H'$ and $H$, respectively.

**Theorem 3.2.** *Let M be a generic lightlike submanifold of an indefinite trans-Sasakian manifold* $\overline{M}$. *If* $U_i$ *or* $V_i$ *is parallel with respect to* $\nabla$, *then* $\alpha = \beta = 0$, *i.e.*, $\overline{M}$ *is an indefinite cosymplectic manifold.*

**Proof.** (1) If $U_i$ is parallel with respect to $\nabla$, then, taking the scalar product with $\zeta$ to (3.6), we have $\alpha\eta_i(X) + \beta v_i(X) = 0$. Thus, $\alpha = \beta = 0$ and $\overline{M}$ is an indefinite cosymplectic manifold. Taking the scalar product with $V_k$ and $W_b$ to (3.6) by turns, we get $\tau_{ij} = 0$ and $\rho_{ia} = 0$, respectively. Applying $J$ to (3.6) and using $(2.1)_1$, $(3.12)_3$, we obtain

$$A_{N_i} X = \sum_{j=1}^{r} \lambda_{ji}(X) U_j + \sum_{a=r+1}^{n} \nu_{ai}(X) W_a. \tag{3.22}$$

Taking the scalar product with $N_j$ to (3.22), we obtain $\eta_j(A_{N_i} X) = 0$.

(2) If $V_i$ is parallel with respect $\nabla$, then, taking the scalar product with $\zeta$, $U_k$, $V_k$ and $W_b$ to (3.7) by turns, we have $\beta = 0$, $\tau_{ji} = 0$, $h_j^{\ell}(X, \xi_i) = 0$ and $\phi_{ai} = 0$, respectively. Applying $J$ to (3.7) and using (2.1) and $(3.12)_1$, we get

$$A^*_{\xi_i} X = -\alpha u_i(X)\zeta + \sum_{j=1}^{r} \chi_{ij}(X) U_j + \sum_{a=r+1}^{n} \mu_{ia}(X) W_a.$$

Taking the scalar product with $U_k$, we get $h_i^\ell(X, U_j) = 0$. Taking $X = U_i$ to $(3.12)_1$, we have $-\alpha = -\alpha u_i(U_i) = h_i^\ell(U_i, \zeta) = 0$. As $\alpha = 0$, we obtain

$$A^*_{\xi_i} X = \sum_{j=1}^{r} \chi_{ij}(X) U_j + \sum_{a=r+1}^{n} \mu_{ia}(X) W_a. \tag{3.23}$$

## 4. Submanifolds of Space Forms

**Theorem 4.1.** *Let $M$ be a generic lightlike submanifold of an indefinite generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$. Then we have the following*:

(1) $\alpha$ *is a constant*,

(2) $\alpha\beta = 0$,

(3) $f_1 - f_2 = \alpha^2 - \beta^2$ *and* $f_1 - f_3 = (\alpha^2 - \beta^2) - \zeta\beta$.

**Proof.** Comparing the tangential, lightlike transversal and co-screen components of the two equations (2.4) and (2.14), and using (3.4), we get

$$\begin{aligned} R(X, Y)Z = {} & f_1\{g(Y, Z)X - g(X, Z)Y\} \\ & + f_2\{\bar{g}(X, JZ)FY - \bar{g}(Y, JZ)FX + 2\bar{g}(X, JY)FZ\} \\ & + f_3\{\theta(X)\theta(Z)Y - \theta(Y)\theta(Z)X \\ & + \bar{g}(X, Z)\theta(Y)\zeta - \bar{g}(Y, Z)\theta(X)\zeta\} \\ & + \sum_{i=1}^{r}\{h_i^\ell(Y, Z)A_{N_i}X - h_i^\ell(X, Z)A_{N_i}Y\} \\ & + \sum_{a=r+1}^{n}\{h_a^s(Y, Z)A_{E_a}X - h_a^s(X, Z)A_{E_a}Y\}, \end{aligned} \tag{4.1}$$

$$
\begin{aligned}
&(\nabla_X h_i^{\ell})(Y, Z) - (\nabla_Y h_i^{\ell})(X, Z) \\
&+ \sum_{j=1}^{r} \{\tau_{ji}(X) h_j^{\ell}(Y, Z) - \tau_{ji}(Y) h_j^{\ell}(X, Z)\} \\
&+ \sum_{a=r+1}^{n} \{\phi_{ai}(X) h_a^{s}(Y, Z) - \phi_{ai}(Y) h_a^{s}(X, Z)\} \\
= \ & f_2\{u_i(Y)\bar{g}(X, JZ) - u_i(X)\bar{g}(Y, JZ) + 2u_i(Z)\bar{g}(X, JY)\}. \quad (4.2)
\end{aligned}
$$

Taking the scalar product with $N_i$ to (2.15), and then, substituting (4.1) into the resulting equation and using $(2.13)_4$, we obtain

$$
\begin{aligned}
&(\nabla_X h_i^{*})(Y, PZ) - (\nabla_Y h_i^{*})(X, PZ) \\
&+ \sum_{j=1}^{r} \{\tau_{ij}(Y) h_j^{*}(X, PZ) - \tau_{ij}(X) h_j^{*}(Y, PZ)\} \\
&+ \sum_{a=r+1}^{n} \varepsilon_a \{\rho_{ia}(Y) h_a^{s}(X, PZ) - \rho_{ia}(X) h_a^{s}(Y, PZ)\} \\
&+ \sum_{j=1}^{r} \{h_j^{\ell}(X, PZ)\eta_i(A_{N_j}Y) - h_j^{\ell}(Y, PZ)\eta_i(A_{N_j}X)\} \\
= \ & f_1\{g(Y, PZ)\eta_i(X) - g(X, PZ)\eta_i(Y)\} \\
&+ f_2\{v_i(Y)\bar{g}(X, JPZ) - v_i(X)\bar{g}(Y, JPZ) + 2v_i(PZ)\bar{g}(X, JY)\} \\
&+ f_3\{\theta(X)\eta_i(Y) - \theta(Y)\eta_i(X)\}\theta(PZ). \quad (4.3)
\end{aligned}
$$

Applying $\nabla_Y$ to $(3.5)_1$ and using (2.10), (2.12), (3.4) and (3.5), we obtain

$$
\begin{aligned}
(\nabla_X h_j^{\ell})(Y, U_i) = & (\nabla_X h_i^{*})(Y, V_j) + g(A_{N_i}Y, \nabla_X V_j) - g(A^{*}_{\xi_j}Y, \nabla_X U_i) \\
&+ \sum_{k=1}^{r} h_i^{*}(X, U_k) h_k^{\ell}(Y, \xi_j).
\end{aligned}
$$

Using (2.1), (3.2)~(3.7) and (3.12), we have

$$
\begin{aligned}
(\nabla_X h^\ell_j)(Y, U_i) = (\nabla_X h^*_i)(Y, V_j)
&- \sum_{k=1}^{r} \{\tau_{kj}(X) h^\ell_k(Y, U_i) + \tau_{ik}(X) h^*_k(Y, V_j)\} \\
&- \sum_{a=r+1}^{n} \{\phi_{aj}(X) h^s_a(Y, U_i) + \epsilon_a \rho_{ia}(X) h^s_a(Y, V_j)\} \\
&- g(A^*_{\xi_j} X, F(A_{N_i} Y)) - g(A^*_{\xi_j} Y, F(A_{N_i} X)) \\
&+ \sum_{k=1}^{r} \{h^*_i(Y, U_k) h^\ell_k(X, \xi_j) + h^*_i(X, U_k) h^\ell_k(Y, \xi_j)\} \\
&- \sum_{k=1}^{r} h^\ell_j(X, V_k) \eta_k(A_{N_i} Y) - \alpha^2 u_j(Y) \eta_i(X) \\
&- \beta^2 u_j(X) \eta_i(Y) + \alpha\beta\{u_j(X) v_i(Y) - u_j(Y) v_i(X)\}.
\end{aligned}
$$

Substituting this into (4.2) such that replace $i$ by $j$ and take $Z = U_i$, we have

$$
\begin{aligned}
&(\nabla_X h^*_i)(Y, V_j) - (\nabla_Y h^*_i)(X, V_j) \\
&- \sum_{k=1}^{r} \{\tau_{ik}(X) h^*_k(Y, V_j) - \tau_{ik}(Y) h^*_k(X, V_j)\} \\
&- \sum_{a=r+1}^{n} \epsilon_a \{h^s_a(Y, V_j) \rho_{ia}(X) - h^s_a(X, V_j) \rho_{ia}(Y)\} \\
&- \sum_{k=1}^{r} \{h^\ell_k(Y, V_j) \eta_i(A_{N_k} X) - h^\ell_k(X, V_j) \eta_i(A_{N_k} Y)\} \\
&+ (\alpha^2 - \beta^2)\{u_j(X) \eta_i(Y) - u_j(Y) \eta_i(X)\}
\end{aligned}
$$

$$+ 2\alpha\beta\{u_j(X)v_i(Y) - u_j(Y)v_i(X)\}$$
$$= f_2\{u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y) + 2\delta_{ij}\bar{g}(X, JY)\}.$$

Comparing this equation with (4.3) such that $PZ = V_j$ and using the facts that $h_i^{\ell}(X, V_j)$ are symmetric and $\eta_i(A_{N_j}X)$ are skew-symmetric with respect to *i* and *j* due to $(2.13)_3$ and $(3.5)_3$, we get

$$\{f_1 - f_2 - (\alpha^2 - \beta^2)\}[u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y)]$$
$$= 2\alpha\beta\{u_j(Y)v_i(X) - u_j(X)v_i(Y)\}.$$

Taking $X = \xi_i$, $Y = U_j$ and $X = V_i$, $Y = U_j$, respectively, we have

$$f_1 - f_2 = \alpha^2 - \beta^2, \quad \alpha\beta = 0.$$

Applying $\bar{\nabla}_X$ to $\eta_i(Y) = \bar{g}(Y, N_i)$ and using (2.5) and (2.6), we have

$$(\nabla_X \eta_i)Y = -g(A_{N_i}X, Y) + \sum_{j=1}^{r} \tau_{ij}(X)\eta_j(Y).$$

Applying $\nabla_Y$ to $(3.12)_3$ and using (3.10), (3.14) and the last equation, we have

$$\begin{aligned}(\nabla_X h_i^*)(Y, \zeta) = {} & -(X\alpha)v_i(Y) + (X\beta)\eta_i(Y) \\ & + \alpha^2\theta(Y)\eta_i(X) + \beta^2\theta(X)\eta_i(Y) \\ & + \alpha\Bigg\{g(A_{N_i}X, FY) + g(A_{N_i}Y, FX) - \sum_{j=1}^{r} v_j(Y)\tau_{ij}(X) \\ & - \sum_{a=r+1}^{n} \epsilon_a w_a(Y)\rho_{ia}(X) - \sum_{j=1}^{r} u_j(Y)\eta_i(A_{N_j}X)\Bigg\} \\ & - \beta\Bigg\{g(A_{N_i}X, Y) + g(A_{N_i}Y, X) - \sum_{j=1}^{r} \tau_{ij}(X)\eta_j(Y)\Bigg\}.\end{aligned}$$

Substituting this and (3.12) into (4.3) such that $PZ = \zeta$, we get

$$\{X\beta + [f_1 - f_3 - (\alpha^2 - \beta^2)]\theta(X)\}\eta_i(Y)$$
$$- \{Y\beta + [f_1 - f_3 - (\alpha^2 - \beta^2)]\theta(Y)\}\eta_i(X)$$
$$= (X\alpha)v_i(Y) - (Y\alpha)v_i(X).$$

Taking $X = \zeta$ and $Y = \xi_i$, and taking $X = U_k$ and $Y = V_i$ by turns, we get

$$f_1 - f_3 = \alpha^2 - \beta^2 - \zeta\beta, \quad U_i\alpha = 0, \quad \forall i.$$

Applying $\nabla_X$ to $h_i^\ell(Y, \zeta) = -\alpha u_i(Y)$ and using (3.9) and (3.13), we get

$$(\nabla_X h_i^\ell)(Y, \zeta) = -(X\alpha)u_i(Y) - \beta h_i^\ell(X, Y)$$
$$+ \alpha\left\{\sum_{j=1}^{r} u_j(Y)\tau_{ji}(X) + \sum_{a=r+1}^{n} w_a(Y)\phi_{ai}(X)\right.$$
$$\left. + h_i^\ell(X, FY) + h_i^\ell(Y, FX)\right\}.$$

Substituting this and (3.12) into (4.2) such that $Z = \zeta$, we obtain

$$(X\alpha)u_i(Y) = (Y\alpha)u_i(X).$$

Replacing $Y$ by $U_i$ to this, we obtain $X\alpha = 0$. Thus, $\alpha$ is a constant.

**Theorem 4.2.** *Let $M$ be a generic lightlike submanifold of an indefinite generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$. If one of $\{F, U_i, V_i\}$ is parallel with respect to the induced connection $\nabla$, then $\overline{M}(f_1, f_2, f_3)$ is flat.*

**Proof.** (1) If $F$ is parallel with respect to $\nabla$, then, by Theorem 3.1, we get (3.19) and the results: $\alpha = \beta = 0$ and $\phi_{ia} = \rho_{ai} = \eta_i(A_{N_j}X) = 0$. As $\alpha = 0$, we see that $f_1 = f_2 = f_3$ by Theorem 4.1.

Taking the scalar product with $U_j$ to (3.19), we get

$$h_i^*(X, U_j) = 0.$$

Using this result, (3.6), (3.19) and the facts that $\rho_{ai} = 0$ and $FU_i = 0$, we get

$$(\nabla_X h_i^*)(Y, U_j) = 0.$$

Substituting the last two equations into (4.3) with $PZ = U_j$, we have

$$f_1\{v_j(Y)\eta_i(X) - v_j(X)\eta_i(Y)\} + f_2\{v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)\} = 0,$$

due to $\rho_{ai} = \eta_i(A_{N_j}X) = 0$. Taking $X = \xi_i$ and $Y = V_j$ to this equation, we obtain $f_1 = 0$. Therefore, $f_1 = f_2 = f_3 = 0$ and $\overline{M}(f_1, f_2, f_3)$ is flat.

(2) If $U_i$ is parallel with respect to $\nabla$, then we get (3.22), $\alpha = \beta = 0$ and $\tau_{ij} = \rho_{ia} = \eta_j(A_{N_i}X) = 0$ by (1) of Theorem 3.2. As $\alpha = 0$, by Theorem 4.1, we have $f_1 = f_2 = f_3$. Taking the scalar product with $U_j$ to (3.22), we get

$$h_i^*(X, U_j) = 0.$$

Applying $F$ to (3.22) and using the facts that $FU_i = FW_a = 0$, we see that $F(A_{N_i}X) = 0$. Applying $\nabla_X$ to $h_i^*(Y, U_j) = 0$ and using (3.6), we obtain

$$(\nabla_X h_i^*)(Y, U_j) = 0.$$

Substituting the last two equations into (4.3) with $PZ = U_j$, we have

$$f_1\{v_j(Y)\eta_i(X) - v_j(X)\eta_i(Y)\} + f_2\{v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)\} = 0,$$

due to $\tau_{ij} = \rho_{ia} = \eta_j(A_{N_i}X) = 0$. Taking $X = \xi_i$ and $Y = V_j$ to this equation, we obtain $f_1 = 0$. Therefore, $f_1 = f_2 = f_3 = 0$ and $\overline{M}(f_1, f_2, f_3)$ is flat.

(3) If $V_i$ is parallel with respect to $\nabla$, then we get (3.23), $\alpha = \beta = 0$ and $\tau_{ij} = \phi_{ia} = h_j^{\ell}(X, \xi_i) = 0$ by (2) of Theorem 3.2. As $\alpha = \beta = 0$ by Theorem 4.1, $f_1 = f_2 = f_3$. Taking the scalar product with $U_k$ to (3.23), we get

$$h_i^{\ell}(Y, U_j) = 0.$$

Applying $\nabla_X$ to this equation and using (3.6) and (3.12), we have

$$(\nabla_X h_i^{\ell})(Y, U_j) = -g(A^*_{\xi_i}Y, F(A_{N_j}X)) - \sum_{a=r+1}^{n} \rho_{ja}(X)h_i^{\ell}(Y, W_a).$$

Substituting the last two equations into (4.2) with $Z = U_j$, we obtain

$$\begin{aligned} &g(A^*_{\xi_i}X, F(A_{N_j}Y)) - g(A^*_{\xi_i}Y, F(A_{N_j}X)) \\ &+ \sum_{a=r+1}^{n} \{\rho_{ja}(Y)h_i^{\ell}(X, W_a) - \rho_{ja}(X)h_i^{\ell}(Y, W_a)\} \\ &= f_2\{u_i(Y)\eta_j(X) - u_i(X)\eta_j(Y) + 2\delta_{ij}\bar{g}(X, JY)\}. \end{aligned} \tag{4.4}$$

As $h_i^{\ell}(\xi_j, X) = 0$ and $h_i^{\ell}(U_j, X) = 0$, we have $A^*_{\xi_i}\xi_j = 0$ and $A^*_{\xi_i}U_j = 0$. Taking $X = \xi_j$ and $Y = U_i$ to (4.4), we obtain $f_2 = 0$. Therefore $f_1 = f_2 = f_3 = 0$.

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