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TOPOLOGICAL COMPLEXITY (WITHIN 1) OF THE SPACE OF ISOMETRY CLASSES OF PLANAR *n*-GONS FOR SUFFICIENTLY LARGE *n*

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Abstract

Hausmann and Rodriguez classified spaces of isometry classes of planar n-gons according to their genetic code which is a collection of sets (called genes) containing n. Omitting the n yields what we call gees. We prove that, for a set of gees with largest gee of size $k > 0$, the topological complexity (TC) of the associated space of n-gons is either $2n - 5$ or $2n - 6$ if $n \geq 2k + 3$. We present evidence that suggests that it is very rare that the TC is not equal to $2n - 5$ or $2n - 6$.

1. Introduction

The topological complexity, $TC(X)$, of a topological space X is, roughly, the number of rules required to specify how to move between any two points of X. A "rule" must be such that the choice of path varies continuously with the choice of endpoints (see [3, Section 4]). We continue our study, begun in

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[2], of $TC(X)$, where $X = \overline{M}(\ell)$ is the space of equivalence classes of oriented n-gons in the plane with consecutive sides of lengths $\ell_1, ..., \ell_n$, identified under translation, rotation, and reflection (see, e.g., [5, Section 6]). Here $\ell = (\ell_1, ..., \ell_n)$ is an n-tuple of positive real numbers. Thus

$$\overline{M}(\ell) = \{(z_1, ..., z_n) \in (S^1)^n : \ell_1 z_1 + \cdots + \ell_n z_n = 0\}/O(2).$$

We can think of the sides of the polygon as linked arms of a robot, and then $TC(X)$ is the number of rules required to program the robot to move from any configuration to any other.

We assume that ℓ is *generic*, which means that there is no subset $S \subset [\![n]\!]$ with $\sum_{i\in S} \ell_i = \sum_{i\notin S} \ell_i$. Here $[\![n]\!] = \{1, ..., n\}$, notation that will be used throughout the paper. When ℓ is generic, $\overline{M}(\ell)$ is a connected $(n-3)$-manifold [5, p. 314], and hence, by [3, Corollary 4.15], satisfies

$$TC(\overline{M}(\ell)) \le 2n - 5. \tag{1.1}$$

The mod-2 cohomology ring $H^*(\overline{M}(\ell))$ was determined in [5]. See Theorem 1.5 for our interpretation. All of our cohomology groups have coefficients in $\mathbb{Z}_2$, omitted from the notation. We shall prove that for most length-n vectors ℓ, there is a nonzero product in $H^*(\overline{M}(\ell) \times \overline{M}(\ell))$ of $2n-7$ classes of the form $z \otimes 1 + 1 \otimes z$, which implies $TC(\overline{M}(\ell)) \ge 2n - 6$ by [3, Corollary 4.40], within 1 of being optimal by (1.1). We say that this lower bound for $TC(\overline{M}(\ell))$ is obtained by zcl (zero-divisor cup length) consideration, or that $zcl(\overline{M}(\ell)) \ge 2n - 7$. We write $\bar{z} = z \otimes 1 + 1 \otimes z$.

To formulate our result, we recall the concepts of genetic code and gees. Since permuting the ℓ_i's does not affect the space up to homeomorphism, we may assume $\ell_1 \le \cdots \le \ell_n$. We also assume that $\ell_n < \ell_1 + \cdots + \ell_{n-1}$, so that $\overline{M}(\ell)$ is nonempty. It is well-understood (e.g., [5, Section 2]) that the homeomorphism type of $\overline{M}(\ell)$ is determined by which subsets S of $[\![n]\!]$ are *short*, which means that $\sum_{i\in S}\ell_i < \sum_{i\notin S}\ell_i$. Define a partial order on the power set of $[\![n]\!]$ by $S \le T$ if $S = \{s_1, ..., s_\ell\}$ and $T \supset \{t_1, ..., t_\ell\}$ with $s_i \le t_i$ for all i. As introduced in [6], the *genetic code* of ℓ is the set of maximal elements (called *genes*) in the set of short subsets of $[\![n]\!]$ which contain n. The homeomorphism type of $\overline{M}(\ell)$ is determined by the genetic code of ℓ. A list of all genetic codes for $n \le 9$ appears in [7]. For $n = 6, 7$ and 8, there are 20, 134, and 2469 genetic codes, respectively.

In [2], we introduced the term "gee," to refer to a gene without listing the n. Also, a *subgee* is any set which is $\le G$ for some gee G, under the partial ordering just described. Thus the subgees are just the sets $S \subset [\![n-1]\!]$ for which $S \cup \{n\}$ is short. Our main theorem is as follows. Let $\mathrm{lg}(-) = \lfloor \log_2(-) \rfloor$.

Theorem 1.2. *For a set of gees with largest gee of size $k > 0$, the associated space of n-gons $\overline{M}(\ell)$ satisfies*

$$2n - 6 \le TC(\overline{M}(\ell)) \le 2n - 5 \tag{1.3}$$

if $n \ge k + 2^{\mathrm{lg}(k)} + 3$.

Here we mean that ℓ is a length-n vector whose genetic code has the given set as its gees, with n appended to form its genes.

The stipulation $k > 0$ excludes the n-gon space whose genetic code is $\langle\{n\}\rangle$. One length vector with this genetic code is $(1^{n-1}, n-2)$. A polygon space with this genetic code is homeomorphic to real projective space RP^{n-3}, and it is known that, except for RP^1, RP^3, and RP^7, $TC(RP^{n-3})$ equals the immersion dimension plus 1, which is usually much less than $2n - 6$ [4].

We will prove two results, Theorems 1.4 and 4.10, which suggest that

(1.3) fails only very rarely. As noted above, it usually fails if the genetic code is $\langle\{n\}\rangle$. At the other extreme, it fails if the genetic code is $\langle\{n, n-3, n-4, ..., 1\}\rangle$, in which case $\overline{M}(\ell)$ is a torus T^{n-3} of topological complexity $n-2$. We showed in [2] that, of the 132 equivalence classes of 7-gon spaces excluding RP^4 and T^4, there are only two, namely those with a single gene, 7321 or 7521, for which we cannot prove that they satisfy (1.3). Here, we have begun the usual practice of writing genes or gees consisting of single-digit numbers by concatenating. In Theorem 1.4, we obtain a similar result for 8-gon spaces, but this time with just two exceptions out of 2467 equivalence classes.

Theorem 1.4. *Excluding RP^5 and T^5, the only spaces $\overline{M}(\ell)$ with $n = 8$ which might not satisfy* (1.3) *are those with a single gene*, 84321 *or* 86321.

In Section 4, we specialize to monogenic codes and prove in Theorem 4.10 that the only genetic codes with a single gene of size 5 (for *any n*) which do not necessarily satisfy (1.3) are those noted above.

Theorem 1.2 is an immediate consequence of Theorems 1.7 and 1.8. We introduce those theorems by giving our interpretation [2, Theorem 2.1] of [5, Corollary 9.2], the complete structure of the mod 2 cohomology ring of $\overline{M}(\ell)$.

Theorem 1.5. *If ℓ has length n, the ring $H^*(\overline{M}(\ell))$ is generated by classes*

$$R, V_1, ..., V_{n-1} \in H^1(\overline{M}(\ell))$$

subject to only the following relations:

(1) *All monomials of the same degree which are divisible by the same V_i's are equal. Hence, letting $V_S := \prod_{i\in S} V_i$, monomials $R^{d-|S|}V_S$ for $S \subset [\![n-1]\!]$ span $H^d(\overline{M}(\ell))$.*

(2) $V_S = 0$ *unless S is a subgee of* ℓ.

(3) *For every subgee S with* $|S| \geq n-2-d$, *there is a relation* $\mathcal{R}_S$ *in* $H^d(\overline{M}(\ell))$, *which says*

$$\sum_{T \not\supset S} R^{d-|T|} V_T = 0. \tag{1.6}$$

It is convenient to let $m = n-3$, which we do throughout. Note that $\overline{M}(\ell)$ is an m-manifold. The proof of Theorem 1.2 is split into two cases depending upon whether or not $R^m = 0$.

Theorem 1.7. *Suppose* $R^m = 0$. *Then there exists a positive integer r and distinct integers* $i_1, ..., i_r$ *such that* $R^{m-r} V_{i_1} \cdots V_{i_r} \neq 0 \in H^m(\overline{M}(\ell)) \approx \mathbb{Z}_2$, *but for all proper subsets* $S \subsetneq \{i_1, ..., i_r\}$, $R^{m-|S|} \prod_{i \in S} V_i = 0$. *Assume* $m \geq r + 2^{\lg r}$, *and let* $f = \lg(m-r) \geq \lg(r)$. *Let* $A = 2m - 2r - 2^{f+1} + 3$. *Then*

$$\prod_{j=1}^{r-1} \overline{V}_{i_r}^3 \cdot \overline{V}_{i_r}^A \cdot \overline{R}^{2m+2-A-3r} \neq 0 \in H^{2m-1}(\overline{M}(\ell) \times \overline{M}(\ell)),$$

and hence $TC(\overline{M}(\ell)) \geq 2m = 2n-6$.

Theorem 1.8. *Suppose* $R^m \neq 0$. *If m is a* 2*-power, then* $\overline{R}^{2m-1} \neq 0$. *If m is not a* 2*-power, then there exist positive integers t and A and distinct integers* $i_1, ..., i_{t+1}$ *such that*

$$\overline{V}_{i_1} \cdots \overline{V}_{i_t} \overline{V}_{i_{t+1}}^A \overline{R}^{2m-A-t-1} \neq 0 \in H^{2m-1}(\overline{M}(\ell) \times \overline{M}(\ell)).$$

Hence in either case $TC(\overline{M}(\ell)) \geq 2m = 2n-6$.

In Theorem 1.8, any m large enough, with respect to the given gees, to yield a valid genetic code works in the theorem.

In Section 2, we prove Theorems 1.7, 1.8, and 1.4. In Section 3, we discuss the effect on the length vectors and the cohomology ring of increasing n while leaving the gees unchanged, and give some examples regarding the sharpness of the bound on how large m must be in Theorem 1.7. In Section 4, we give several explicit families of gees of arbitrarily large size to which Theorem 1.7 applies. We also prove in Theorem 4.10 that there are only three size-5 genes for which we cannot prove (1.3).

2. Proofs of Theorems 1.7, 1.8, and 1.4

In this section, which we feel is the heart of the paper, we prove Theorems 1.7, 1.8, and 1.4.

Proof of Theorem 1.7. We begin with the simple observation that if $R^m = 0$, then r can be chosen as

$$r = \min\{t : \exists \text{ distinct } i_1, ..., i_t \text{ with } R^{m-t}V_{i_1} \cdots V_{i_t} \neq 0\}.$$

First observe that $A \geq 3$ and the exponent of $\overline{R}$ is $2^{f+1} - 1 - r \geq 0$, since $f \geq \lg(r)$. By minimality of r, in the expansion of the product, factors $V_{i_j}^3 \otimes 1$, $1 \otimes V_{i_j}^3$, $V_{i_r}^A \otimes 1$, and $1 \otimes V_{i_r}^A$ will yield 0 in products. A product of s of the $V_{i_j}^2$'s and $(r-1-s)$ of the V_{i_j}'s can be written as $R^s P$, where $P := \prod_{j=1}^{r-1} V_{i_j}$. Thus our product expands in bidegree $(m, m-1)$ as

$$\sum_{s=0}^{r-1}\binom{r-1}{s}R^s P\sum_{j=1}^{A-1}\binom{A}{j}V_{i_r}^j\binom{2m+2-A-3r}{m-s-r-j+1}R^{m-s-r-j+1}$$

$$\otimes R^{r-1-s}PV_{i_r}^{A-j}R^{m+1-A-2r+s+j}. \tag{2.1}$$

Let $\phi : H^m(\overline{M}(\ell)) \to \mathbb{Z}_2$ be the Poincaré duality isomorphism. Let V_I denote any product of distinct classes V_i. There is a homomorphism ψ :

$H^{m-1}(\overline{M}(\ell)) \to \mathbb{Z}_2$ satisfying that $\psi(R^{m-1-|I|}V_I) \neq 0$ iff $\phi(R^{m-|I|}V_I) \neq 0$. This follows from Theorem 1.5 since the relations in $H^{m-1}(\overline{M}(\ell))$ are also relations in $H^m(\overline{M}(\ell))$ (with a dimension shift). Thus $\phi \otimes \psi$ applied to any summand of (2.1) which has a nonzero coefficient, mod 2, is nonzero, and so $\phi \otimes \psi$ applied to (2.1) equals

$$\sum_{s=0}^{r-1}\sum_{j=1}^{A-1}\binom{r-1}{s}\binom{A}{j}\binom{2m+2-A-3r}{m-s-r-j+1}$$

$$= \sum_{s=0}^{r-1}\sum_{j=0}^{A}\binom{r-1}{s}\binom{A}{j}\binom{2m+2-A-3r}{m-s-r-j+1} + \sum_{s=0}^{r-1}\binom{r-1}{s}\binom{2m+2-A-3r}{m-s-r+1}$$

$$+ \sum_{s=0}^{r-1}\binom{r-1}{s}\binom{2m+2-A-3r}{m-s-r-A+1}.$$

By Vandermonde's identity, this equals

$$\binom{2m-2r+1}{m-r+1} + \binom{2m-2r+1-A}{m-r+1} + \binom{2m-2r+1-A}{m-r+1-A}. \qquad (2.2)$$

The last binomial coefficient equals $\binom{2m-2r+1-A}{m-r}$, and so the sum of the last two equals $\binom{2m-2r+2-A}{m-r+1}$. Inserting now the value of A, we find that the image of our class equals

$$\binom{2m-2r+1}{m-r+1} + \binom{2^{f+1}-1}{m-r+1}.$$

For $2^f \leq m-r \leq 2^{f+1}-1$, this expression equals 1, coming from the first term if $m-r = 2^{f+1}-1$ and from the second term otherwise. □

The proof of Theorem 1.8 is a bit more elaborate. We will always be using the homomorphism $\psi : H^{m-1}(\overline{M}(\ell)) \to \mathbb{Z}_2$ which equals the Poincaré duality isomorphism $\phi : H^m(\overline{M}(\ell)) \to \mathbb{Z}_2$ in the sense of the preceding proof. We first observe that if $\phi(R^m) \neq 0$ and $\psi(R^{m-1}) \neq 0$ and $m = 2^e$, then Theorem 1.8 is true since

$$(\phi \otimes \psi)(\overline{R}^{2m-1}) = \binom{2m-1}{m}\phi(R^m)\psi(R^{m-1}) \neq 0.$$

In the rest of this section, we assume m is not a 2-power.

The following key lemma rules out certain possibilities for ϕ.

Lemma 2.3. *It cannot happen that there is a subset $S \subset [\![m+2]\!]$ such that*

$$\phi(R^{m-|I|}V_I) = \begin{cases} 1 & I \subset S, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We assume such a set S exists and will derive a contradiction. Let k denote the size of the largest subgee. By [1, Corollary 1.6], $\phi(R^{m-k}V_{i_1} \cdots V_{i_k}) = 1$ whenever $\{i_1, ..., i_k\}$ is a subgee. Although the result in [1] is apparently only referring to monogenic codes, the proof applies more generally. By the assumption, we conclude that there can only be one subgee of size k, and it must be $[\![k]\!]$. The sum $\mathcal{R}_{[\![k]\!]} + \mathcal{R}_{[\![k-1]\!]}$ of relations from Theorem 1.5 implies that the sum of $\phi(R^{m-|J|}V_J)$ taken over all subgees J for which $J \cap [\![k]\!] = \{k\}$ must be 0. This sum includes the term $\phi(R^{m-1}V_k) = 1$, while all other terms in the sum have $J \not\subset [\![k]\!]$ and hence have $\phi(R^{m-|J|}V_J) = 0$, contradicting that the sum is 0. $\square$

The next two propositions are special cases of the theorem. If $S \subset [\![t]\!]$, let $\widetilde{S} = [\![t]\!] - S$. We repeat that $\psi(R^{m-1-|I|}V_I) = \phi(R^{m-|I|}V_I)$ is assumed.

Proposition 2.4. *Let $T \subset [\![m+2]\!]$ with $|T| \leq m$, and reindex as $T =$*

$[\![t+1]\!]$. *Suppose* $\phi(R^m) = 1$, $\phi(R^{m-|T|}V_T) = 1$, *and* $\phi(R^{m-|I|}V_I) = 0$ *for all* $I \subsetneq T$ *such that* $t+1 \in I$. *Then*

$$(\phi \otimes \psi)(\overline{V}_1 \cdots \overline{V}_t \overline{V}_{t+1}^{m-t} \overline{R}^{m-1}) = 1.$$

Proof. The expression expands as

$$\sum_{S \subset [\![t]\!]} \sum_{i=0}^{m-t} \binom{m-t}{i} \binom{m-1}{m-|S|-i} \phi(V_S V_{t+1}^i R^{m-|S|-i}) \psi(V_{\tilde{S}} V_{t+1}^{m-t-i} R^{|S|+i-1}).$$

The only terms with $\phi \cdot \psi \neq 0$ are those with $(S, i) = ([\![t]\!], m-t)$ or $(\varnothing, 0)$, and the coefficients of these are 1 and 0, respectively. □

Proposition 2.5. *Let* $T \subset [\![m+2]\!]$ *with* $|T| \leq m$, *and reindex as* $T = [\![t+1]\!]$. *Suppose* $\phi(V_I) = 1$ *for* $I \subsetneq T$, *and* $\phi(V_T) = 0$. *Let* $m = 2^e + \Delta$, $1 \leq \Delta < 2^e$. *Then*

(1) $(\phi \otimes \psi)(\overline{V}_1 \cdots \overline{V}_t \overline{V}_{t+1}^A \overline{R}^{2m-A-t-1}) = \binom{2m-t}{m} + \binom{2m-A-t}{m-t}$.

(2) $\binom{2m-t}{m} + \binom{2m-A-t}{m-t}$ *is odd for the following values of* A:

(a) *If* $1 \leq t \leq 2\Delta$, *use* $A = 2\Delta + 1 - t$.

(b) *If* $2\Delta + 1 \leq t \leq 2^e$ *and* $\binom{t-\Delta-1}{\Delta}$ *is even, use* $A = \Delta$.

(c) *If* $2\Delta + 1 \leq t \leq 2^e$ *and* $\binom{t-\Delta-1}{\Delta}$ *is odd, use* $A = 2^{\lg(2\Delta)}$.

(d) *If* $2^e < t \leq m-1$ *and* $\binom{m-t+\Delta}{\Delta}$ *is even, use* $A = \Delta$.

(e) *If* $2^e < t \leq m-1$ *and* $\binom{m-t+\Delta}{\Delta}$ *is odd, use* $A = \Delta - 2^{\nu(m-t)}$.

Proof. (1) The expression expands as

$$\sum_{S\subset[\![t]\!]}\sum_{i=0}^{A}\binom{A}{i}\binom{2m-A-t-1}{m-i-|S|}\phi(V_SV_{t+1}^iR^{m-i-|S|})\psi(V_{\tilde S}V_{t+1}^{A-i}R^{m-A-t-1+i+|S|}).$$

If it was the case that all $\phi(-)=1$, this would equal $\sum\binom{t}{s}\binom{2m-t-1}{m-s}=\binom{2m-1}{m}\equiv 0$, since we assume m is not a 2-power. Our given values of ϕ differ from this only for $S=[\![t]\!]$ and $i>0$, or $S=\varnothing$ and $i<A$. Thus the sum becomes

$$\sum_{i=1}^{A}\binom{A}{i}\binom{2m-A-t-1}{m-i-t}+\sum_{i=0}^{A-1}\binom{A}{i}\binom{2m-A-t-1}{m-i}$$
$$=\binom{2m-t-1}{m-t}+\binom{2m-A-t-1}{m-t}+\binom{2m-t-1}{m}+\binom{2m-A-t-1}{m-A}$$
$$=\binom{2m-t}{m}+\binom{2m-A-t}{m-t}.$$

(2) (a) The expression equals $\binom{2^{e+1}+2\Delta-t}{2^e+\Delta}+\binom{2^{e+1}-t}{2^e-\Delta+t-1}=0+1.$

(b) The expression equals $\binom{2^{e+1}+2\Delta-t}{2^e+\Delta}+\binom{2^{e+1}\Delta-t}{2^e}$. The first of these is congruent to $\binom{-(t-2\Delta)}{\Delta}\equiv\binom{t-\Delta-1}{\Delta}\equiv 0$ by assumption, while the second is 1 since $t\le 2^e$.

(c) The first term is as in (b), except now it is 1 by assumption. Let $\Delta=2^\nu+\delta$ with $0\le\delta<2^\nu$. The second term becomes $\binom{2^{e+1}+2\delta-t}{2^e-2^\nu+\delta}$, which is 0 since the top has a 0 in the 2^{v+1} position, while the bottom has a 1 there.

(d) The expression is the same as in (b). Now the first term is of the form $\binom{2^e+m+\Delta-t}{2^e+\Delta} \equiv 0$ since $\binom{m+\Delta-t}{\Delta} \equiv 0$ by assumption. The second term is 1 since its top is between 2^e and $2^e+\Delta$.

(e) Let $m-t=2^w\alpha$ with α odd. The first term is $\binom{2^e+2^w\alpha+\Delta}{2^w\alpha}$ $\equiv \binom{2^w\alpha+\Delta}{2^w\alpha} \equiv 1$ by assumption. The second term is $\binom{2^e+2^w\alpha+2^w}{2^w\alpha}$, yielding 0 due to the 2^w position. □

We need one more lemma, in which $\mathcal{P}(S)$ denotes the power set of S.

Lemma 2.6. *If U is a set, and $\mathcal{C} \subset \mathcal{P}(U)$ with $\varnothing \in \mathcal{C}$, then either*

(a) $\mathcal{C} = \mathcal{P}(X)$ *for some* $X \subset U$, *or*

(b) *there exists* $T \subset U$ *with* $|T| \geq 2$ *such that* $\mathcal{C} \cap \mathcal{P}(T) = \mathcal{P}(T) - \{T\}$, *or*

(c) *there exists* $s \in S \subset U$ *such that* $\{C \in \mathcal{C} : s \in C\} \cap \mathcal{P}(S) = \{S\}$, *and* $|S| \geq 2$.

Proof. Let $X = \{t \in U : \{t\} \in \mathcal{C}\}$. If $\mathcal{P}(X) \not\subset \mathcal{C}$, then a minimal element T of $\mathcal{P}(X) - \mathcal{C}$ is of type (b), and we are done. If $\mathcal{C} = \mathcal{P}(X)$, we are done by (a).

Thus, we may assume that $\mathcal{P}(X) \subsetneq \mathcal{C}$. Choose $S' \in \mathcal{C} - \mathcal{P}(X)$ and $s \in S' - X$. Note that $\{s\} \notin \mathcal{C}$. Let S be a minimal element in $\{C \in \mathcal{C} : s \in \mathcal{C}\}$. This S is of type (c), so we are done. □

Proof of Theorem 1.8. Let U denote the set of i's such that V_i is a factor of some nonzero monomial in $H^m(\overline{M}(\ell))$. Let $\mathcal{C} = \{I \subset U : \phi(R^{m-|I|}V_I) \neq 0\}$. By Lemma 2.3, either (b) or (c) of Lemma 2.6 is true of $\mathcal{C}$. If (b), then the theorem is true by Proposition 2.5, and if (c), then the theorem is true by Proposition 2.4. □

Proof of Theorem 1.4. By Theorem 1.8, (1.3) holds whenever $R^m \neq 0$, and by Theorem 1.7, it holds for $n = 8$ (hence $m = 5$) whenever $R^m = 0$ and there is some nonzero monomial $R^{m-r}V_{i_1} \cdots V_{i_r}$ with $1 \le r \le 3$. The only genetic code with $n = 8$ having a gee of length 5 is that of the 5-torus, 854321. The majority of the genetic codes with $n = 8$ do not have any gees of size 4, and the theorem follows immediately for these. However, there are many genetic codes with $n = 8$ having a 4-gee of the form 4321, 5321, 6321, 7321, 5421, 5431, or 5432, plus perhaps other, shorter, gees. This can be seen in [7], or deduced from the definitions. We will show that, except in the excluded cases, if all monomials M_S in $H^m(\overline{M}(\ell))$ corresponding to subgees S of size ≤ 3 are 0, then so are the monomials in $H^m(-)$ corresponding to the 4-gees, contradicting that $H^m(-) \approx \mathbb{Z}_2$.

If the 4-gee is 4321 and there are any other gees, then 5 is a subgee, and $\mathcal{R}_5$ (see Theorem 1.5(3)) in $H^m(\overline{M}(\ell))$ is a sum of the monomial M_{4321} corresponding to 4321 plus monomials M_S corresponding to certain subgees of size ≤ 3. If all these $M_S = 0$, then the relation implies that $M_{4321} = 0$ and hence $H^m(-) = 0$.

If the only gees and subgees of size 4 are one or more of 5321, 5421, 5431, or 5432, let T denote any one of them, and let j denote the element of $[\![5]\!]$ not contained in T. The relation $\mathcal{R}_j$ in $H^m(-)$ is a sum of the monomial M_T corresponding to T plus monomials M_S corresponding to subgees S of size ≤ 3. If all M_S are 0, then so is M_T, and since this holds for all T, we deduce $H^m(-) = 0$.

If 6321 is the 4-gee and there is another, shorter, gee, then the shorter gee must contain 7. This can be seen at [7], or can be deduced as follows. [Any gee of size ≤ 3 which does not contain 7 and is not ≤ 6321 would have to be ≥ 54. But if 86321 is short, then 754 is long and hence so is 854.] Thus 7 is a subgee. The gees and subgees of size 4 are 6321, 5321, and 4321. If all monomials M_S in $H^m(-)$ corresponding to subgees S of size ≤ 3 are 0, then the relations $\mathcal{R}_7$, $\mathcal{R}_6$, and $\mathcal{R}_5$ imply, respectively, $M_{6321} + M_{5321} + M_{4321} = 0$, $M_{5321} + M_{4321} = 0$, and $M_{6321} + M_{4321} = 0$, and hence $H^m(-) = 0$.

If 6321 is the only gee, then we do not have $\mathcal{R}_7$ to work with, and in fact $M_{6321} = M_{5321} = M_{4321} \ne 0$ with all monomials corresponding to shorter gees being 0. Hence in this case, we cannot use Theorem 1.7 to deduce (1.3) when $n = 8$.

If 7321 is a gee, there can be no other gees, as can be seen from [7] or deduced similarly to the deduction involving 6321. If all monomials M_S corresponding to subgees of size ≤ 3 are 0, then the relations $\mathcal{R}_7$, $\mathcal{R}_6$, $\mathcal{R}_5$ and $\mathcal{R}_4$ imply that $M_{7321} = M_{6321} = M_{5321} = M_{4321} = 0$ and hence $H^m(-) = 0$. □

3. The Effect of Increasing *n*

In this section, we discuss the effect of increasing *n*, while leaving the gees fixed, on the length vectors and the cohomology ring.

The operation of increasing the number of edges by 1 while leaving the gees unchanged has the following nice interpretation. Two length vectors are said to be *equivalent* if they have the same genetic code, or equivalently their moduli spaces $\overline{M}(\ell)$ are homeomorphic.

Proposition 3.1. *Any generic length-n vector is equivalent to one,* $(\ell_1, \ldots, \ell_n)$ *with* $\ell_1 \le \cdots \le \ell_n$, *in which all lengths are positive integers with odd sum* $|\ell|$, *and*

$$\ell_n + \ell_{n-1} \le \ell_1 + \cdots + \ell_{n-2} + 1. \tag{3.2}$$

For such an ℓ, *define a new vector* ℓ' *of length* $n+1$ *by*

$$\left(\ell_1, \ldots, \ell_{n-1}, \frac{|\ell|+1}{2} - \ell_n, \frac{|\ell|+1}{2}\right). \tag{3.3}$$

Then ℓ *and* ℓ' *have the same gees.*

Proof. It is shown in [6, Section 4] that any length vector is equivalent to one with rational entries and hence to one with integral entries. In [7], it is shown that this is equivalent to one with $|\ell|$ odd. If it has $\ell_n + \ell_{n-1} = \ell_1 + \cdots + \ell_{n-2} + 2d + 1$ for $d \ge 1$, we can find an equivalent length vector with smaller d, and hence eventually satisfy (3.2), as follows: (a) If $\ell_{n-1} > \ell_{n-2}$, then decrease ℓ_n and ℓ_{n-1} by 1. (b) If $\ell_n > \ell_{n-1} = \cdots = \ell_{n-t} > \ell_{n-t-1}$ for some $t \ge 2$, decrease $\ell_{n-1}, \ldots, \ell_{n-t}$ by 1, and decrease ℓ_n by t. (c) If $\ell_n = \ell_{n-1} = \ell_{n-2}$, decrease each of them by 2.

We explain briefly why each of these changes does not affect the genetic code. (a) The only short subsets containing n, either before or after the change, are of the form $T = S \cup \{n\}$ with $S \subset [\![n-2]\!]$, and $\sum_{i\in T} \ell_i - \sum_{i\notin T} \ell_i$ is invariant under the change. (b) First we show that the new vector has nondecreasing entries. We need $\ell_n - \ell_{n-1} \ge t-1$ for $t \ge 3$, and this follows easily from $\ell_n + \ell_{n-1} \ge (t-1)\ell_{n-1} + 3$. Because in either the old or new vector, the sum of the last two components is greater than half the total, the only short subsets containing n will be of the form $T = S \cup \{n\}$ with $S \subset [\![n-t-1]\!]$, and the change subtracts t from both T and its complement.

(c) The hypothesis implies that $\displaystyle\sum_{j=1}^{n-3} \ell_j \le \ell_n - 3$, and so the new vector has nondecreasing entries, and both the old and new vectors have $[\![n-3]\!]$ as the only gee.

Now we compare the gees of ℓ satisfying (3.2) with those of (3.3), which satisfies its own analogue of (3.2), in fact with equality.

If equality holds in (3.2), then for ℓ (resp. ℓ'), the only short subsets containing n (resp. $n+1$) are of the form $T = S \cup \{n\}$ (resp. $S \cup \{n+1\}$) with $S \subset [\![n-2]\!]$, and the change adds $\frac{|\ell|+1}{2} - \ell_n$ to both $\sum_{j \in T} \ell_j$ and $\sum_{j \notin T} \ell_j$, thus leaving the gees unaffected. If, on the other hand, $\ell_n + \ell_{n-1} \leq \ell_1 + \cdots + \ell_{n-2} - 1$, then ℓ has some short subsets of the form $T_1 = S_1 \cup \{n-1, n\}$ with $S_1 \subset [\![n-2]\!]$, and others of the form $T_2 = S_2 \cup \{n\}$ with $S_2 \subset [\![n-2]\!]$. The vector ℓ' of (3.3) will have exactly the sets $S_1 \cup \{n-1, n+1\}$ and $S_2 \cup \{n+1\}$ as its short subsets containing $n+1$, since again we add $\frac{|\ell|+1}{2} - \ell_n$ to both $\sum_{j \in T_\varepsilon} \ell_j$ and $\sum_{j \notin T_\varepsilon} \ell_j$, $\varepsilon = 1, 2$. □

Next we point out the effect on $H^*(-)$ of increasing *n* while leaving the gees unchanged. Recall that $m = n-3$. If *s* denotes the size of the largest gee and $m \geq 2s$, then for $d \leq m-s$, $H^d(\overline{M}(\ell))$ has a basis

$$\{R^{d-|S|}V_S : S \text{ is a subgee } |S| \leq d\},$$

so in this range increasing *m* leaves the cohomology groups unchanged, while for $m - s < d \leq m$, the relations among the subgees depend on $m-d$, and so in this range increasing *m* acts like suspending.

For our strong TC results, we only use $H^{m-1}(-)$ and $H^m(-)$. We illustrate with a simple example: monogenic codes 7321 and 8321, so the gee is 321 and $m = 4$ and 5. One can verify that in this case $H^m(-) \approx$

$\langle R^{m-3}V_1V_2V_3\rangle$ and

$$H^{m-1}(-) \approx \langle R^{m-3}V_1V_2,\ R^{m-3}V_1V_3,\ R^{m-3}V_2V_3,\ R^{m-4}V_1V_2V_3\rangle.$$

So $r = 3$ in Theorem 1.7, and the theorem applies for $m \geq 5$. Note that in Theorem 1.7, we are only considering homomorphisms $\psi : H^{m-1}(-) \to \mathbb{Z}_2$ which are essentially equal to $\phi : H^m(-) \to \mathbb{Z}_2$, so that in this case the theorem is only utilizing the class $R^{m-4}V_1V_2V_3$ in $H^{m-1}(-)$.

When $m = 5$, $(\phi \otimes \psi)(\overline{V}_1^3\overline{V}_2^3\overline{V}_3^3) \neq 0$ in bidegree $(5, 4)$ as it equals

$$\phi(V_1V_2^2V_3^2)\psi(V_1^2V_2V_3) + \phi(V_1^2V_2V_3^2)\psi(V_1V_2^2V_3)$$
$$+ \phi(V_1^2V_2^2V_3)\psi(V_1V_2V_3^2) \neq 0.$$

(Keep in mind part (1) of Theorem 1.5.)

When $m = 4$, we would need

$$\overline{V}_1^i\overline{V}_2^j\overline{V}_3^k\overline{R}^{7-i-j-k} \neq 0 \tag{3.4}$$

in bidegree $(4, 3)$. The only way to get $RV_1V_2V_3 \otimes V_1V_2V_3$ would be with none of *i*, *j*, or *k* being a 2-power, and this is impossible with $i + j + k \leq 7$. We could get $RV_1V_2V_3 \otimes RV_1V_2$ from $i = j = 3$, $k = 1$, but this would have even coefficient from

$$V_1^2V_2V_3 \otimes V_1V_2^2 + V_1V_2^2V_3 \otimes V_1^2V_2.$$

We conclude that (3.4) is impossible, and so we cannot deduce $TC(\overline{M}(\ell)) \geq 8$ for this ℓ when $m = 4$ by cohomological considerations. Thus Theorem 1.7 is optimal for the gee 321 in obtaining the strong lower bound for its TC when $m \geq 5$.

If the only gee is $[\![r]\!]$, we easily see, using Theorem 1.5, that the only nonzero monomial in $H^m(\overline{M}(\ell))$ is $R^{m-r}V_{[\![r]\!]}$, while those in $H^{m-1}(\overline{M}(\ell))$

are $R^{m-1-r}V_{[\![r]\!]}$ and $R^{m-2-r}V_{[\![r]\!]-\{i\}}$ for $1 \le i \le r$. We can sometimes improve upon Theorem 1.7 because of this explicit information about $H^{m-1}(-)$.

For example, if the only gee is $[\![4]\!]$, Theorem 1.7 said $TC \ge 2m$ for $m \ge 8$, but we can also deduce $TC \ge 2m$ for $m = 7$ and $m = 6$ using

$$\overline{V}_1^3\overline{V}_2^3\overline{V}_3^3\overline{V}_4^4 = V_1V_2V_3V_4^4 \otimes V_1^2V_2^2V_3^2 \text{ in bidegree } (7,\ 6) \tag{3.5}$$

and

$$\overline{V}_1^3\overline{V}_2^3\overline{V}_3^3\overline{V}_4^2 = V_1^2V_2V_3V_4^2 \otimes V_1V_2^2V_3^2 \text{ in bidegree } (6,\ 5).$$

When $m = 5$, we cannot deduce $TC \ge 10$ using zcl. When $m = 4$, so the genetic code is 74321, $\overline{M}(\ell)$ is homeomorphic to a 4-torus with $TC = 5$.

If the only gee is $[\![r]\!]$ for $r = 5,\ 6,$ or 7, Theorem 1.7 says $TC \ge 2m$ for $m \ge r + 4$, and one can check that this is all we can do, in the sense that, for $m < r + 4$, $\overline{V}_1^{i_1} \cdots \overline{V}_r^{i_r}\,\overline{R}^{2m-1-i_1-\cdots-i_r} = 0$.

4. Specific Results for Monogenic Codes of Arbitrary Length

In this section, we discuss some families of monogenic genetic codes of arbitrary length in which we can show that $R^m = 0$ and find the value of r that works in Theorem 1.7. At the end, we discuss evidence suggesting that it is quite rare for a monogenic code to have $zcl < 2n - 7$ (and hence not be able to deduce $TC \ge 2n - 6$ from zcl).

Definition 4.1. Let $\mathcal{S}_k$ denote the set of k-tuples of nonnegative integers such that, for all j, the sum of the first j components is $\le j$.

For $B = (b_1, ..., b_k)$, let $|B| = \sum b_i$. The following theorem is the main result of [1].

Theorem 4.2. *Suppose* ℓ *has a single gee,* $\{g_1, ..., g_k\}$, *with* $a_i = g_i - g_{i+1} > 0$. $(a_k = g_k.)$ *If J is a set of distinct integers* $\le g_1$, *let* $\theta(J) =$

$(\theta_1, ..., \theta_k)$, *where* θ_i *is the number of elements* $j \in J$ *satisfying* $g_{i+1} < j \le g_i$. *Then, if* ℓ *has length* $m+3$, *the Poincaré duality isomorphism* $\phi : H^m(\overline{M}(\ell)) \to \mathbb{Z}_2$ *satisfies*

$$\phi(R^{m-r}V_{j_1}\cdots V_{j_r}) = \sum_B \prod_{i=1}^{k} \binom{a_i+b_i-2}{b_i}, \tag{4.3}$$

where B *ranges over all* $(b_1, ..., b_k)$ *for which* $|B| = k-r$ *and* $B + \theta(\{j_1, ..., j_r\}) \in \mathcal{S}_k$.

The following corollary is useful.

Corollary 4.4. *In Theorem* 4.2, (4.3) *depends only on the reductions* $a_i \bmod 2^{\lg(2i)}$.

Proof. For a B-summand in (4.3) to be nonzero, it is necessary that each b_i be $\le i$. Binomial coefficients $\binom{x}{b}$ depend only on $x \bmod 2^{\lg(2b)}$. □

We often write $\overline{a}_i$ for the $\bmod 2^{\lg(2i)}$-reduction of a_i. In Theorems 4.8 and 4.9, we describe two infinite families of $(\overline{a}_1, ..., \overline{a}_k)$ for which Theorem 1.7 applies, and so we can deduce the strong lower bound for $TC(\overline{M}(\ell))$.

Let $w_i = V_{g_i}$ (or any V_j satisfying $g_{i+1} < j \le g_i$). If $I = (\varepsilon_1, ..., \varepsilon_k)$ with each $\varepsilon_j \in \{0, 1\}$, we let

$$Y_I = R^t w_1^{\varepsilon_1} \cdots w_k^{\varepsilon_k} \tag{4.5}$$

for any t. This notation for Y_I could be extended to include products of distinct V_j with j-values in the same subinterval $(g_{i+1}, g_i]$, but the consideration of such Y_I seems not to be useful. The total grading of our classes Y_I will be implicit, usually m or $m-1$, and the value of t in (4.5) is chosen to make the class have the desired grading. Then (4.3) could be restated as

$$\phi(Y_I) = \sum_B \prod_{i=1}^{k} \binom{a_i + b_i - 2}{b_i} \tag{4.6}$$

in grading *m*, where *B* ranges over all $(b_1, ..., b_k)$ such that $B + I \in \mathcal{S}_k$ and $|B + I| = k$.

For monogenic codes as in Theorem 4.2, we can easily check whether $R^m = 0$, (or equivalently $\phi(R^m) = 0$), since R^m is just Y_I with *I* consisting of all 0's. Thus $\phi(R^m) = \sum_B \binom{a_i + b_i - 2}{b_i}$ with the sum taken over all $B = (b_1, ..., b_k)$ satisfying $|B| = k$ and $B \in \mathcal{S}_k$. For example, when $k = 3$,

$$\phi(R^m) = \binom{a_3 + 1}{2} + \binom{a_2}{2} a_3' + \binom{a_3}{2}(a_1' + a_2') + a_1' a_2' a_3',$$

where $a_i' = a_i - 1$.

For a given *k*, there are $\prod_{i=1}^{k} 2^{\lg(2i)}$ possibilities for $(\bar{a}_i, ..., \bar{a}_k)$. `Maple` determined the information in Table 4.7 regarding how many of these have $R^m = 0$.

Table 4.7

k	# $\bar{a}$'s	# $R^m = 0$
3	32	20
4	256	128
5	2048	1216
6	16384	9600

Our simplest result follows. Keep in mind that in the rest of this section $\bar{a}_i$ refers to the reduction mod $2^{\lg(2i)}$ of $a_i = g_i - g_{i+1}$, where ℓ has a single gene $\{m + 3, g_1, ..., g_k\}$.

Theorem 4.8. *If there is a set* $Z \subset [\![k]\!]$ *with* $|Z| = r$ *such that*

$$\bar{a}_i = \begin{cases} 1 & i \in Z, \\ 0 & i \notin Z, \end{cases}$$

then the hypothesis of Theorem 1.7 *holds for* $R^{m-r} \prod_{i \in Z} w_i$, *and hence* $TC(\overline{M}(\ell)) \geq 2m$ *if* $m \geq r + 2^{\lg(r)}$.

Proof. For $\bar{a}_i \in \{0, 1\}$, the only times that $\binom{a_i + b_i - 2}{b_i}$ is odd are when $b_i = 0$ or ($b_i = 1$ and $\bar{a}_i = 0$). Thus a nonzero term in (4.6) requires $|B| \leq k - r$. For $|I| \leq r$, $\phi(Y_I)$ can have nonzero terms only if $|B| = k - r$ and $|I| = r$, and moreover only for the single B given by

$$b_i = \begin{cases} 1 & i \notin Z, \\ 0 & i \in Z. \end{cases}$$

If I has 1's in the positions in Z, and 0's elsewhere, $\phi(Y_I) = 1$ from the single term with B as above and $B + I = (1, ..., 1)$. □

For example, the case $(\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4) = (1, 1, 0, 1)$ applies to any ℓ whose genetic code is $\langle\{n, g_1, g_2, g_3, g_4\}\rangle$ with $n > g_1 > g_2 > g_3 > g_4$ and $g_4 \equiv 1(8)$, $g_3 \equiv 1(4)$, $g_2 \equiv 2(4)$, and $g_1 \equiv 1(2)$. Theorem 4.8 says that $TC(\overline{M}(\ell)) \geq 2n - 6$ if $n - 3 \geq 3 + 2$, so if $n \geq 8$. But the smallest values that satisfy the conditions on n and g_i are 87651, and so the condition in the theorem that $m \geq r + 2^{\lg r}$ covers all possibilities here. Moreover, there is no ℓ having this as genetic code because 432 and its complement would both be short. This suggests that the condition in Theorem 1.7 that $m \geq r + 2^{\lg r}$ will usually cover all possible values of m, and so the "sufficiently large" in our title will rarely need to be invoked. We will discuss this in more detail at the end of this section.

A result similar to Theorem 4.8 holds if some of the 0's in $\langle \overline{a}_i \rangle$ are immediately followed by a 2.

Theorem 4.9. *If there are disjoint subsets* $T,\ Z \subset [\![k]\!]$ *such that*

$$\overline{a}_i = \begin{cases} 2, & i \in T, \\ 1, & i \in Z, \\ 0, & i \in C := [\![k]\!] - (T \cup Z) \end{cases}$$

and $\overline{a}_{i-1} = 0$ *whenever* $i \in T$, *then, with* $r = |T| + |Z|$, *the hypothesis of Theorem* 1.7 *holds for* $R^{m-r} \prod_{i \in Z} w_i \cdot \prod_{i \in T} w_{i-1}$, *and hence* $TC(\overline{M}(\ell)) \geq 2m$ *if* $m \geq r + 2^{\lg(r)}$.

Proof. Let $T' = \{t - 1 : t \in T\}$. Then $T' \subset C$. Let $C' = C - T'$.

Let $I = (\varepsilon_1, \ldots, \varepsilon_k)$ with $\varepsilon_j \in \{0, 1\}$ be given. We will show that $\phi(Y_I) = 0$ if $|I| < r$, and $\phi(Y_I) = 1$ if $\varepsilon_i = 1$ if $i \in Z \cup T'$, and $\varepsilon_i = 0$ otherwise.

If U is a k-tuple of nonnegative integers, define

$$\chi(U) = \begin{cases} 1 & \text{if } U \in \mathcal{S}_k, \\ 0 & \text{if } U \notin \mathcal{S}_k. \end{cases}$$

Let $\mathcal{B}$ denote the set of k-tuples $B = (b_1, \ldots, b_k)$ such that $|B + I| = k$ and

$$b_i \begin{cases} \in \mathbb{N} & i \in T, \\ = 0 & i \in Z, \\ \in \{0, 1\} & i \in C. \end{cases}$$

Here $\mathbb{N}$ denotes the set of nonnegative integers. These b_i are the values for which $\binom{a_i + b_i - 2}{b_i} \equiv 1 \bmod 2$. By (4.6),

$$\phi(Y_I) = \sum_{B \in \mathcal{B}} \chi(B + I) \in \mathbb{Z}_2.$$

Let

$$\mathcal{P} = \{B \in \mathcal{B} : b_{t-1} + b_t \geq 2 \text{ for some } t \in T\}.$$

For an element B of $\mathcal{P}$, choose the minimal t with $b_{t-1} + b_t \geq 2$, and pair B with the element B' which has $b'_{t-1} = 1 - b_{t-1}$, $b'_t = b_t + 2b_{t-1} - 1$, with other entries equal to those of B. We show in the next paragraph that $\chi(B + I) = \chi(B' + I)$ for all $B \in \mathcal{P}$. Thus

$$\sum_{B \in \mathcal{P}} \chi(B + I) \equiv 0 \bmod 2.$$

Assume, without loss of generality, that $b_{t-1} = 0$. The only value of h for which

$$\sum_{i=1}^{h} (B + I)_i \neq \sum_{i=1}^{h} (B' + I)_i$$

is $h = t - 1$. The only conceivable way to have $\chi(B + I) \neq \chi(B' + I)$ is if

$$\sum_{i=1}^{t-2} (B + I)_i = t - 2$$

and $\varepsilon_{t-1} = 1$, for then, since $b'_{t-1} = 1$, $B' + I$ fails the condition to be in $\mathcal{S}_k$ at position $t - 1$, but $B + I$ does not. However, since $b_t \geq 2$, $B + I$ fails at position t. Thus $\chi(B + I) = \chi(B' + I)$.

Now let $\mathcal{Q} = \mathcal{B} - \mathcal{P}$. If $B \in \mathcal{Q}$, then

$$|B| = \sum_{T' \cup T} b_i + \sum_{C'} b_i + \sum_{Z} b_i \leq |T| + |C'| + 0 = |C| = k - r.$$

Since $|I| \leq r$, we must have $|B| = k - r$ and $|I| = r$ in order to get a nonzero term in (4.6). Thus $\phi(Y_I) = 0$ whenever $|I| < r$.

The Y_I being considered has $\varepsilon_i = 1$ for $i \in Z \cup T'$. The B's in $\mathcal{Q}$ which might give a nonzero term in (4.6) must have $(b_{t-1}, b_t) = (0, 1)$ or $(1, 0)$ for all $t \in T$, and $b_i = 1$ if $i \in C'$. Thus $B + I$ has 1's in $Z \cup C'$, and $(1, 1)$ or $(2, 0)$ in each position pair $(t-1, t)$. If any $(2, 0)$ occurs, then $B + I$ fails to be in $\mathcal{S}_k$, and so (4.6) has only one nonzero term, namely where $(b_{t-1}, b_t) = (0, 1)$ for all $t \in T$. Thus $B + I \in \mathcal{S}_k$, and $\phi(Y_I) = 1$. □

A similar result and proof holds when $\langle \overline{a}_i \rangle$ contains subsequences of the form $(0, 0, 3)$ or $(0, 1, 2)$. Moreover, all these can be combined, so that (1.3) holds for $m \geq r + 2^{\lg(r)}$, where r is the number of 1's plus the number of $(0, 2)$, $(0, 0, 3)$, and $(0, 1, 2)$ sequences in $\langle \overline{a}_i \rangle$, provided that $\langle \overline{a}_i \rangle$ contains only 0's and 1's and these sequences. However, there are many other sequences $\langle \overline{a}_i \rangle$ not of this type for which the result holds.

In the remainder of the paper, we present more evidence that it is very rare that we cannot prove (1.3). For genetic codes with a single gene of size 4, it was shown in [2] that $zcl(\overline{M}(\ell)) \geq 2n - 7$ unless the gene is 6321, 7321, or 7521. Here we prove a totally analogous result for genes of size 5.

Theorem 4.10. *For a genetic code with a single gene of size* 5, $zcl(\overline{M}(\ell)) \geq 2n - 7$ *except when the gene is* 74321, 84321, *or* 86321.

The proof will use the following general lemma. Notation is as in (4.5).

Lemma 4.11. *For a monogenic code with gene* $\{m + 3, g_1, ..., g_k\}$ *and* $a_i = g_i - g_{i+1} > 0$, *all monomials* $R^{m-t} w_{i_1} \cdots w_{i_t}$ *for* $t < k$ *are* 0 *if and only if* $a_i \equiv 1 \bmod 2^{\lg(2i)}$ *for all* i.

Proof. Let $\hat{Y}_J$ denote Y_I, where I has 0's in the positions in J and 1's elsewhere. Let $a_i' = a_i - 1$. Using (4.6), $\phi(\hat{Y}_j) = a_j' + \cdots + a_k'$, and hence all $\phi(\hat{Y}_j)$ are 0 iff a_i is odd for all i. Similarly, $\phi(\hat{Y}_{j_1, j_2})$ with $j_1 < j_2$ has terms with factors a_i', which are 0 by the above, plus terms in (4.6) with $B = 2\varepsilon_j$

for $j \geq j_2$. For $\phi(\hat{Y}_{j_1,j_2})$ to be 0, we must have $\binom{a_{j_2}}{2} + \cdots + \binom{a_k}{2} = 0$. For all these to be 0, we must now have $a_i \equiv 1 \bmod 4$ for all $i > 1$. Similarly, $\phi(\hat{Y}_{j_1,j_2,j_3,j_4})$ will be a sum of terms already shown to be 0 plus $\binom{a_{j_4}}{4} + \cdots + \binom{a_k}{4}$. For all these to be 0, we must have $a_i \equiv 1 \bmod 8$ for $i \geq 4$. Continuing in this way implies the result. □

Proof of Theorem 4.10. The theorem holds for those gees having $R^m \neq 0$ by Theorem 1.8. For those with $R^m = 0$ and $(\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4) \neq (1, 1, 1, 1)$, the result holds for $m \geq 5$ (hence for $n \geq 8$) by Lemma 4.11 and Theorem 1.7. But the only length-5 gene with $n < 8$ is 74321.

It remains to consider codes with $(\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4) = (1, 1, 1, 1)$. Theorem 1.7 implies the result for $m \geq 8$, so for $n \geq 11$. We must consider the gees with $g_1 \leq 9$ (and all $\bar{a}_i = 1$). These are 4321, 6321, 8321, 8721, and 8761. As noted at the end of the previous section, for the gee 4321, we have

$$H^{m-1}(\overline{M}(\ell)) = \langle R^{m-5}V_1V_2V_3V_4,\ R^{m-4}V_1V_2V_3,$$

$$R^{m-4}V_1V_2V_4,\ R^{m-4}V_1V_3V_4,\ R^{m-4}V_2V_3V_4\rangle \tag{4.12}$$

and $zcl(\overline{M}(\ell)) \geq 2n - 7$ for $n \geq 9$ but not for $n = 7$ or 8.

We now study the case when the gee is 6321. From (4.12), we deduce that there is a uniform homomorphism $\psi : H^{m-1}(\overline{M}(\ell)) \to \mathbb{Z}_2$ sending only $R^{m-4}V_1V_2V_4$, $R^{m-4}V_1V_2V_5$, and $R^{m-4}V_1V_2V_6$ nontrivially. “Uniform,” as discussed in [2], means here that ψ treats V_4, V_5, and V_6 identically because of the interval from 4 to 6, inclusive, in the gee. This dependence of uniform homomorphisms only on $\bar{a}_i$ was observed in [2], but can be seen in this case directly as follows. [The only relations $\mathcal{R}_J$ corresponding to gees of size ≥ 2

which involve any of these three monomials are $\mathcal{R}_{3,\,j}$ for $j \in \{4, 5, 6\}$. Each of these relations involves only two of the three monomials and so would be sent to 0 by our ψ.] Using this ψ, we have for $m = 7$, similarly to (3.5),

$$(\phi \otimes \psi)(\overline{V}_1^3\overline{V}_2^3\overline{V}_3^4\overline{V}_6^3) = \phi(V_1V_2V_3^4V_6)\psi(V_1^2V_2^2V_6^2) \neq 0,$$

and for $m = 6$,

$$(\phi \otimes \psi)(\overline{V}_1^3\overline{V}_2^3\overline{V}_3^2\overline{V}_6^3) = \phi(V_1^2V_2V_3^2V_6)\psi(V_1V_2^2V_6^2) \neq 0.$$

One easily checks that nothing works when $m = 5$. This establishes the claim when the gee is 6321.

A similar analysis works for gees 8321, 8721, and 8761. For each, there is a uniform homomorphism $\psi : H^{m-1}(\overline{M}(\ell)) \to \mathbb{Z}_2$ sending five classes $R^{m-4}V_1V_jV_k$ nontrivially, where k ranges over the gap in the gee (for example 3, 4, 5, 6, and 7 in the second), and $j = 2, 2,$ and 7 in the three cases. There are products similar to those of the previous paragraph mapped nontrivially by $\phi \otimes \psi$ when $m = 7$ or 6. For example, for 8721 and $m = 7$,

$$(\phi \otimes \psi)(\overline{V}_1^3\overline{V}_2^3\overline{V}_7^3\overline{V}_8^4) = \phi(V_1V_2V_7V_8^4)\psi(V_1^2V_2^2V_7^2) \neq 0.$$

Thus $zcl \geq 2m - 1$ for these when $m = 7$ or 6. □

We have performed a similar analysis for single genes of size 6, and found that again the only exceptions to $zcl \geq 2n - 7$ occur when all $\overline{a}_i = 1$. However, this time there are twelve such exceptional genes: (using T for 10, and E for 11), 854321, 954321, T54321, E54321, 974321, T74321, E74321, T94321, E94321, T98321, E98321, E98721. Details are available from the author upon request.

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