Far East Journal of Mathematical Sciences (FJMS)
© 2017 Pushpa Publishing House, Allahabad, India
http://www.pphmj.com
http://dx.doi.org/10.17654/MS101092067
Volume 101, Number 9, 2017, Pages 2067-2077 **ISSN: 0972-0871**

ON γ-OPERATIONS IN SOFT TOPOLOGICAL SPACES

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Abstract

In this paper, the notion of soft γ-open sets in soft topological spaces together with its corresponding interior and closure operators are introduced.

Received: January 4, 2017; Accepted: March 7, 2017
2010 Mathematics Subject Classification: 54A05, 54A10, 54C08.
Keywords and phrases: soft γ-open set, soft γ-interior, soft γ-closure and soft $\gamma\text{-}T_i$ spaces $\left(i = 0, \frac{1}{2}, 1, 2\right)$.

1. Introduction

Uncertain or imprecise data are inherent and pervasive in many important applications in the areas such as economics, engineering, environment and business management. Due to the importance of those applications and the rapidly increasing amount of uncertain data collected, and accumulated, research on effective and efficient techniques that are dedicated to modeling uncertain data and tackling uncertainties has attracted much interest in recent years.

Soft set theory was introduced by Molodtsov [9] in 1999 as a general mathematical tool for dealing with uncertain fuzzy, not clearly defined objects. Maji et al. [7, 8], Chen [2], Chen at al. [3, 4], Kong et al. [6], Xiao et al. [13], and Pei and Miao [10] contributed many concepts to the soft set theory and applications.

Shabir and Naz [11] introduced the notion of soft topological spaces which are defined over an initial universe with a fixed set of parameters. They also studied some of basic concepts of soft topological spaces. Aygunoglu and Aygun [1], Zorlutuna et al. [14] and Hussain and Ahmad [12] studied the properties of soft topological spaces.

In Section 3, we introduce the notion of $\tau_{s\gamma}$ which is the collection of all soft γ-open sets in a soft topological space (X, τ, E). Further, we introduce the concept of $\tau_{s\gamma}$ interior and $\tau_{s\gamma}$ closure operators and study some of their properties.

In Section 4, we characterize soft $\gamma\text{-}T_i$ spaces $\left(i = 0, \frac{1}{2}, 1, 2\right)$ using the notion of soft γ-closed or soft γ-open sets and study the relationship between them.

2. Preliminaries

In this section, we recall some of the basic definitions and theorems.

Let U be an initial universe set and E_U be a collection of all possible parameters with respect to U, where parameters are the characteristic or properties of objects in U. We will call E_U the *universe set* of parameters with respect to U.

Definition 2.1 [7]**.** A pair (F, A) is called a *soft set* over U if $A \subset E_U$ and $F : A \to P(U)$, where $P(U)$ is the set of all subsets of U.

Definition 2.2 [4]**.** Let (F, A) and (G, B) be soft sets over a common universe set U and $A, B \subseteq E$. Then (F, A) is a subset of (G, B), denoted by $(F, A) \tilde{\subset} (G, B)$, if (i) $A \subset B$; (ii) for all $e \in A$, $F(e) \subset G(e)$.

$(F, A) = (G, B)$, if $(F, A) \tilde{\subset} (G, B)$ and $(G, B) \tilde{\subset} (F, A)$.

Definition 2.3 [5]**.** A soft set (F, A) over U is called a *null soft set*, denoted by ϕ, if $e \in A$, $F(e) = \phi$.

Definition 2.4 [5]**.** A soft set (F, A) over U is called an *absolute soft set*, denoted by $\widetilde{A}$, if $e \in A$, $F(e) = U$.

Definition 2.5 [5]**.** The union of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C), where $C = A \bigcup B$, and $\forall e \in C$,

$$A^{\gamma} = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ F(e) \bigcup G(e), & \text{if } e \in B \bigcap A. \end{cases}$$

We write $(F, A) \bigcup (G, B) = (H, C)$.

Definition 2.6 [4]**.** The intersection of two soft sets of (F, A) and (G, B) over common universe U is the soft set (H, C), where $C = A \bigcap B$, and $\forall e \in C$, $H(e) = F(e) \bigcap G(e)$. We write $(F, A) \bigcap (G, B) = (H, C)$.

We recall some definitions and results defined and discussed in [10-12].

Henceforth, let X be an initial universe set and E be the fixed non-empty set of parameter with respect to X unless otherwise specified.

Definition 2.7 [10]**.** For a soft set (F, A) over U, the relative complement of (F, A) is denoted by $(F, A)'$ and is defined by $(F, A)' = (F', A)$, where $F' : A \to P(U)$ is a mapping given by $F'(e) = U - F(e)$ for all $e \in A$.

Definition 2.8 [10]**.** Let τ be the collection of soft sets over X, then τ is called *soft topology* on X if τ satisfies the following axioms:

(i) $\phi, \widetilde{X}$ belong to τ.

(ii) The union of any number of soft sets in τ belongs to τ.

(iii) The intersection of any two soft sets in τ belongs to τ.

The triplet (X, τ, E) is called a *soft topological space* over X.

Definition 2.9 [10]**.** Let (X, τ, E) be a soft topological space over X. Then the members of τ are said to be *soft open sets* in X.

Definition 2.10 [10]**.** Let (X, τ, E) be a soft topological space over X. A soft set (F, E) over X is said to be a soft closed set in X, if its relative complement $(F, E)'$ belongs to τ.

Definition 2.11 [10]**.** Let (X, τ, E) be a soft topological space and (A, E) be a soft set over X.

(i) The soft interior of (A, E) is the soft set $\operatorname{int}(A, E) = \bigcup\{(O, E) : (O, E)$ is soft open and $(O, E) \,\widetilde{\subseteq}\, (A, E)\}$.

(ii) The soft closure of (A, E) is the soft set $\operatorname{cl}(A, E) = \bigcap\{(F, E) : (F, E)$ is soft closed and $(A, E) \,\widetilde{\subseteq}\, (F, E)\}$.

Proposition 2.12 [10]**.** *Let (X, τ, E) be a soft topological space and let (F, E) and (G, E) be soft sets over X. Then*

(i) $\operatorname{int}(\operatorname{int}(F, E)) = \operatorname{int}(F, E)$,

(ii) $(F, E) \tilde{\subseteq} (G, E)$ *implies* $\operatorname{int}(F, E) \tilde{\subseteq} \operatorname{int}(G, E)$,

(iii) $\operatorname{cl}(\operatorname{cl}(F, E)) = \operatorname{cl}(F, E)$,

(iv) $(F, E) \tilde{\subseteq} (G, E)$ *implies* $\operatorname{cl}(F, E) \tilde{\subseteq} \operatorname{cl}(G, E)$.

3. Soft γ-open Set

Definition 3.1. Let (X, τ, E) be a soft topological space. An operation γ on the soft topology τ is a mapping from τ into the power set $P_S(X)$ of X such that $(V, E) \tilde{\subseteq} (V, E)^{\gamma}$ for each $(V, E) \in \tau$. $(V, E)^{\gamma}$ denotes the value of γ at (V, E). It is denoted by $\gamma_S : \tau \to P_S(X)$.

Definition 3.2. A soft subset (A, E) of a soft topological space (X, τ, E) is said to be a *soft γ-open set* if and only if for each $x \in (A, E)$, there exists a soft open set (U, E) such that $x \in (U, E)$ and $(U, E)^{\gamma} \tilde{\subseteq} (A, E)$. $\tau_{S\gamma}$ denotes the set of all γ-open sets. We have $\tau_{S\gamma} \tilde{\subseteq} \tau$. A subset (B, E) of (X, τ, E) is said to be *soft γ-closed set* in (X, τ, E) if $X - (B, E)$ is soft γ-open set in (X, τ, E).

Example 3.3. (i) Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\tau = \{\phi, \tilde{X}, (F_1, E), (F_2, E), (F_1, F_2, E)\}$, where $(F_1, E), (F_2, E)$ and (F_3, E) are soft sets over X, defined as follows: $F_1(e_1) = \{h_3, h_2\}$, $F_1(e_2) = \{h_2, h_3\}$, $F_2(e_1) = \{h_2\}$, $F_2(e_2) = \{h_2, h_3\}$, $F_3(e_1) = \{h_3, h_1\}$ and $F_3(e_2) = \{h_2\}$. Let $\gamma : \tau \to P(X)$ be defined by $\gamma(B) = \operatorname{int}(\operatorname{cl}(B))$. Then $\tau_{S\gamma} = \{\phi, \tilde{X}, (F_1, E), (F_2, E), (F_1, F_2, E)\}$.

(ii) Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\tau = \{\phi, \tilde{X}, (F_1, E), (F_2, E), (F_1, F_2, E)\}$, where $(F_1, E), (F_2, E)$ and (F_3, E) are soft sets over X, defined as follows: $F_1(e_1) = \{h_3, h_2\}$, $F_1(e_2) = \{h_2, h_3\}$, $F_2(e_1) = \{h_2\}$, $F_2(e_2) = \{h_2, h_3\}$, $F_3(e_1) = \{h_3, h_1\}$ and $F_3(e_2) = \{h_2\}$. Let $\gamma : \tau \to p(X)$ be defined by $\gamma(B) = \mathrm{cl}(B)$. Then $\tau_{s\gamma} = \{\phi, \widetilde{X}\}$.

Proposition 3.4. *The family* $\tau_{s\gamma}$ *is a generalized topology of X in the sense of Lugojan* [5].

Definition 3.5. An operation γ on τ is said to be *open* if, for every open neighbourhood (U, E) of *X*, there exists a soft γ open set *V* such that $x \in (V, E)$ and $(V, E) \widetilde{\subseteq} (U, E)^{\gamma}$.

Definition 3.6. Let (X, τ, E) be a soft topological space and (A, E) be a subset of (X, τ, E). Then soft τ_{γ}-interior of (A, E) is the union of all soft γ-open sets contained in (A, E) and it is denoted by $\tau_{s\gamma}\text{-int}(A, E)$. That is

$$\tau_{s\gamma}\text{-int}(A, E) = \bigcup\{(U, E) : (U, E) \text{ is a soft } \gamma\text{-open set and } (U, E) \widetilde{\subseteq} (A, E)\}.$$

Definition 3.7. Let (X, τ, E) be a soft topological space and (A, E) be a soft subset of (X, τ, E). Then $\tau_{s\gamma}$-closure of (A, E) is the intersection of soft γ-closed sets containing (A, E) and it is denoted by $\tau_{s\gamma}\text{-cl}(A, E)$. That is

$$\tau_{s\gamma}\text{-cl}(A, E) = \bigcap\{(F, E) : (F, E) \text{ is a soft } \gamma\text{-closed set and } (A, E) \widetilde{\subseteq} (F, E)\}.$$

Proposition 3.8. *For a point* $x \in X$, $x \in \tau_{s\gamma}\text{-}cl(A, E)$ *if and only if* $(V, E) \cap (A, E) \neq \phi$ *for any* $(V, E) \in \tau_{s\gamma}$ *such that* $x \in (V, E)$.

Proof. Let (F_0, E) be the set of all $y \in X$ such that $(V, E) \cap (A, E) \neq \phi$ for any $(V, E) \in \tau_{s\gamma}$ and $y \in (V, E)$. It is enough to show that $(F_0, E) = \tau_{s\gamma}\text{-cl}(A, E)$. It is easily seen that $X - (F_0, E)$ is a soft γ-open set and $(A, E) \widetilde{\subseteq} (F_0, E)$. This means that $\tau_{s\gamma}\text{-cl}(A, E) \widetilde{\subseteq} (F_0, E)$.

Conversely, let (F, E) be a soft set such that $(A, E) \tilde{\subseteq} (F, E)$ and $X - F \in \tau_{S\gamma}$. If $x \notin (F, E)$, then we have $x \in X - F (\in \tau_{S\gamma})$ and $(X - F) \bigcap (A, E) = \phi$. This means $x \notin (F_0, E)$. Then we have $F_0 \tilde{\subseteq} F$ and $F_0 \tilde{\subseteq} \tau_{S\gamma}\text{-cl}(A, E)$.

It is easily shown that for any soft subset (A, E) of $(X, \tau, E)(A, E) \tilde{\subseteq} \text{cl}(A, E) \tilde{\subseteq} \tau_{S\gamma}\text{-cl}(A, E)$.

Theorem 3.9. *For a soft subset* (A, E) *of* (X, τ, E), *the following statements are equivalent*:

(i) (A, E) *is soft* γ*-open set in* (X, τ, E).

(ii) $X - (A, E)$ *is a soft* γ*-closed set in* (X, τ, E).

(iii) $\tau_{S\gamma}\text{-cl}(X - (A, E)) = X - (A, E)$ *holds*.

Proof. (i) $\rightarrow$ (ii) The proof follows from Definition 3.2.

(ii) $\rightarrow$ (iii) The proof follows from Definition 3.7.

4. Soft γ-T_i Spaces

In this section, we investigate soft γ-T_i spaces where $i = 0, \frac{1}{2}, 1, 2$.

Definition 4.1. A space (X, τ, E) is called a *soft* γ-T_0 *space* if for each distinct points $x, y \in X$ there exists a soft γ-open set (U, E) such that $x \in (U, E)$ and $y \notin (U, E)^{\gamma}$ or $y \in (U, E)$ and $x \notin (V, E)^{\gamma}$.

Definition 4.2. A space (X, τ, E) is called a *soft* γ-T_1 *space* if for each distinct points $x, y \in X$ there exist soft γ-open sets $(U, E), (V, E)$ containing *x* and *y*, respectively, such that $y \notin (U, E)^{\gamma}$ and $x \notin (V, E)^{\gamma}$.

Definition 4.3. A space (X, τ, E) is called a *soft* γ-T_2 *space* if for each distinct points $x, y \in X$ there exist soft γ-open sets $(U, E), (V, E)$ such that $x \in (U, E), y \in (V, E)$ and $(U, E)^{\gamma} \cap (V, E)^{\gamma} = \phi$.

Definition 4.4. Let (X, τ, E) be a soft topological space. Then a soft subset (A, E) of X is said to be *soft* γ *g-closed* if $\tau_{s\gamma}$-cl$(A, E) \tilde{\subseteq} (U, E)$ whenever $(A, E) \tilde{\subseteq} (U, E)$ and (U, E) is a soft γ-open set in (X, τ, E).

Remark 4.5. From Definition 4.4, every soft γ-closed set in (X, τ, E) is soft α g-closed set. However, the converse need not be true.

Definition 4.6. A soft topological space (X, τ, E) is called a *soft* γ-$T_{\frac{1}{2}}$ space if every soft γ g-closed set in (X, τ, E) is soft γ-closed.

Theorem 4.7. *Let* (X, τ, E) *be a soft topological space. Then a subset* (A, E) *of* X *is said to be soft* γ *g-closed set if and only if* $\tau_{s\gamma}$-cl$(\{x\}) \cap (A, E) \neq \phi$ *holds for every* $x \in \tau_{s\gamma}$-cl(A, E).

Proof. Let (U, E) be a soft γ-open set such that $(A, E) \tilde{\subseteq} (U, E)$. Let $x \in \tau_{s\gamma}$-cl(A, E). Then by assumption there exists a $z \in \tau_{s\gamma}$-cl$(\{x\})$ and $z \in (A, E) \tilde{\subseteq} (U, E)$. Hence by Proposition 3.8 $(U, E) \cap \{x\} \neq \phi$, this implies that $x \in U$. Therefore, $\tau_{s\gamma}$-cl$(A, E) \tilde{\subseteq} (U, E)$ and (A, E) is a soft γ g-closed set in (X, τ, E).

Conversely, suppose $x \in \tau_{s\gamma}$-cl(A, E) such that $\tau_{s\gamma}$-cl$(\{x\}) \cap (A, E) = \phi$, then $A \tilde{\subseteq} (X - \tau_{s\gamma}$-cl$(\{x\}))$. Then by Theorem 3.9 and assumption it follows that $\tau_{s\gamma}$-cl$(A, E) \tilde{\subseteq} (X - \tau_{s\gamma}$-cl$(\{x\}))$. Hence $x \notin \tau_{s\gamma}$-cl$(\{x\})$. Hence $x \notin \tau_{s\gamma}$-cl(A, E). This is a contradiction. Therefore, $\tau_{s\gamma}$-cl$(\{x\}) \cap (A, E) \neq \phi$.

Theorem 4.8. *Let (X, τ, E) be a soft topological space. If a soft subset (A, E) of X is said to be soft γ g-closed, then $\tau_{s\gamma}$-cl$(A, E) - (A, E)$ does not contain a non-empty soft γ-closed set.*

Proof. Suppose that there exists a non-empty soft γ-closed set (F, E) such that $F \tilde{\subseteq} \tau_{s\gamma}$-cl$(A, E) - (A, E) (\tilde{\subseteq} \tau_{s\gamma}$-cl$(A, E))$. Let $x \in F$. Then $x \in \tau_{s\gamma}$-cl(A, E), hence it follows that from Theorem 3.9 that $(F, E) \cap (A, E) = \tau_{s\gamma}$-cl$(F, E) \tilde{\cap} (A, E) \tilde{\subseteq} \tau_{s\gamma}$-cl$(\{x\}) \tilde{\cap} (A, E) \neq \phi$. This implies that $(F, E) \cap (A, E) \neq \phi$. This is a contradiction. Hence $\tau_{s\gamma}$-cl$(A, E) - (A, E)$ does not contain a non-empty soft γ-closed set.

Theorem 4.9. *Let (X, τ, E) be a soft topological space. Then for each $x \in X$, $\{x\}$ is a soft γ-closed or $X - \{x\}$ is soft γ g-closed set in (X, τ, E).*

Proof. Suppose that $\{x\}$ is not soft γ-closed. Then $X - \{x\}$ is not soft γ-open set. This implies that X is the only soft γ-open set containing $X - \{x\}$. Therefore, $X - \{x\}$ is a soft γ g-closed set.

Theorem 4.10. *A soft space (X, τ, E) is soft γ-$T_{\frac{1}{2}}$ if and only if for each $x \in X$, $\{x\}$ is soft γ-closed set or soft γ-open set in (X, τ, E).*

Proof. Suppose $\{x\}$ is not soft γ-closed. Then it follows from the assumption and Theorem 4.9 that $\{x\}$ is soft γ-open. Conversely, let (F, E) is a soft γ g-closed set in (X, τ, E) and $x \in \tau_{s\alpha}$-cl$(F, E) (\tilde{\subseteq} X)$. Then by assumption $\{x\}$ is soft γ-open set or soft γ-closed set.

Case (i). Suppose $\{x\}$ is soft γ-open set. Then by Proposition 3.8, $\{x\} \cap (F, E) \neq \phi$, hence $x \in F$.

Case (ii). Suppose $\{x\}$ is a soft γ-closed set. Assume $x \notin F$, then $\{x\} \in \tau_{s\gamma}$-cl$(F, E) - F$. This is not possible by Theorem 4.8. Thus, we have $x \in$

(F, E). Therefore $\tau_{s\gamma}$-cl$(F, E) = (F, E)$ and hence (F, E) is soft γ-closed set.

Theorem 4.11. *A soft space (X, τ, E) is soft γ-T_1 space if and only if for every $x \in X$, $\{x\}$ is soft γ-closed set.*

Proof. The proof follows from Definition 4.2.

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