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ON PARTIALLY ORDERED SEMIGROUPS WITH GLOBALLY IDEMPOTENT TWO-SIDED IDEALS

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Abstract

The main purpose of this paper is to prove that there are fourteen classes of partially ordered semigroups, out of which a class consists of partially ordered semigroups with its two-sided ideals globally idempotent. The result obtained generalizes the result of Kuroki [8]. Moreover, it is shown that the result can be applied to characterize a partially ordered semigroup to be simple, which in turn, extends a result of Kuroki [9].

1. Preliminaries

A semigroup (without order) S is said to be *regular* [4] if for any $a \in S$

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there exists $x \in S$ such that $a = axa$. In [7], Iséki proved that a commutative semigroup S is regular if and only if every ideal of the semigroup is globally idempotent (i.e., if and only if every ideal A of S, $A = A^2$). A similar result for normal semigroups can be found in [10]. Both these results lead to study a semigroup whose proper two-sided ideals are globally idempotent. Results in this direction were obtained by Kuroki [8], Lajos [10], Venkatesan [11], and Courter [5, p. 418, Theorem 1.2].

The main purpose of this paper is to extend the results of Kuroki [8] to partially ordered semigroups. Moreover, as in [9], we apply the result to characterize simple partially ordered semigroups, and partially ordered semigroups having no two-sided ideals other than itself.

A semigroup $(S, \cdot)$ together with a partial order that is *compatible* with the semigroup operation, meaning that

$$x \leq y \Rightarrow zx \leq zy, \quad xz \leq yz$$

for all $x, y, z \in S$, is called a *partially ordered semigroup*, or simply a *po-semigroup* (cf. [1, 2, 6]). For non-empty subsets A and B of a partially ordered semigroup $(S, \cdot, \leq)$, the set product AB and the subset $(A]$ of S are defined by:

$$AB := \{xy \mid x \in A, y \in B\};$$

and

$$(A] := \{x \in S \mid \exists a \in A(x \leq a)\}.$$

It is observed that the following conditions hold:

(1) $A \subseteq (A]$;

(2) $A \subseteq B \Rightarrow (A] \subseteq (B]$;

(3) $(A](B] \subseteq (AB]$;

(4) $((A]] = (A]$;

(5) $((A](B]] = (AB]$;

(6) $(A \cap B] \subseteq (A] \cap (B]$.

The following can be found in [3]. A non-empty subset A of a partially ordered semigroup $(S, \cdot, \leq)$ is called a *left* (resp. *right*) *ideal* of S if

(i) $SA \subseteq A$ (resp. $AS \subseteq A$);

(ii) $A = (A]$.

And A is called a *two-sided ideal* (or simply an *ideal*) of S if A is both a left and a right ideal of S. Note that

(1) the condition $(A] = A$ is equivalent to for any $x \in A$ and $y \in S$, $y \leq x$ implies $y \in A$;

(2) if R (resp. L) is a right (resp. a left) ideal of S, then $(LR]$ is an ideal of S;

(3) if A and B are ideals of S, then $(AB]$ is an ideal of S.

Let a be an element of a partially ordered semigroup $(S, \cdot, \leq)$. Then the intersection of all left ideals of S containing a is a left ideal of S containing a; this will be denoted by $L(a)$ and it is called the *principal left ideal* of S generated by a. It is observed that

$$L(a) = (a \cup Sa].$$

Similarly, the *principal right* (resp. *two-sided*) *ideal* of S generated by a is of the form

$$R(a) = (a \cup Sa] \quad (\text{resp. } I(a) = (a \cup aS \cup Sa \cup SaS]).$$

2. Po-semigroups with all Ideals are Idempotent

We begin this section with the following definition:

Definition 2.1. An ideal A of a partially ordered semigroup $(S, \cdot, \leq)$ is said to be *globally idempotent* if $(A^2] = A$.

To prove our main theorem we need the following: for non-empty subsets X, Y of a partially ordered semigroup $(S, \cdot, \leq)$, we let

$$(Y : X)_r := \{s \in S \mid Xs \subseteq Y\};$$

$$(Y : X)_l := \{s \in S \mid sX \subseteq Y\}.$$

Lemma 2.2. *Let X, Y be two-sided ideals of a partially ordered semigroup $(S, \cdot, \leq)$. Then*

(1) $X(Y : X)_r \subseteq Y$.

(2) $(Y : X)_r$ *is a two-sided ideal of S containing Y.*

Proof. (1) If $s \in X$ and $t \in (Y : X)_r$, then $Xt \subseteq Y$; hence $st \in Xt \subseteq Y$.

(2) Since $XY \subseteq Y$, $Y \subseteq (Y : X)_r$. Let $s \in S$ and $t \in (Y : X)_r$. Then $Xt \subseteq Y$, and so

$$X(ts) = (Xt)s \subseteq Ys \subseteq Y$$

and

$$X(st) = (Xs)t \subseteq Xt \subseteq Y.$$

Assume $s \leq t$. If $z \in Xs$, then $z = xs$ for some $x \in X$. Since

$$z = xs \leq xt \in Xt \subseteq Y,$$

we have $z \in (Y] = Y$. Thus, $Xs \subseteq Y$, and $s \in (Y : X)_r$. □

We have the following theorem extended Theorem 1 in [8]:

Theorem 2.3. *The following statements are equivalent for a partially ordered semigroup $(S, \cdot, \leq)$:*

(1) $(X^2] = X$ *for every ideal X of S;*

(2) $X \cap Y = (XY]$ *for every ideal X, Y of S;*

(3) $(Y : X)_r \cap X = X \cap Y$ *for every ideal X, Y of S;*

(4) $(Y : X)_l \cap X = X \cap Y$ *for every ideal* X, Y *of* S;

(5) $(Y : X)_r \cap Z = Y \cap Z$ *for every ideal* X, Y, Z *of* S *such that* $Z \subseteq X$;

(6) $(Y : X)_l \cap Z = Y \cap Z$ *for every ideal* X, Y, Z *of* S *such that* $Z \subseteq X$;

(7) $R \cap X \subseteq (XR]$ *for every right ideal* R *of* S *and every ideal* X *of* S;

(8) $L \cap X \subseteq (LX]$ *for every left ideal* L *of* S *and every ideal* X *of* S;

(9) $R \subseteq (XR]$ *for every right ideal* R *of* S *and every ideal* X *of* S *such that* $R \subseteq X$;

(10) $L \subseteq (LX]$ *for every left ideal* L *of* S *and every ideal* X *of* S *such that* $L \subseteq X$;

(11) $(R : X)_r \cap X \subseteq X \cap R$ *for every right ideal* R *of* S *and every ideal* X *of* S;

(12) $(L : X)_l \cap X \subseteq L \cap X$ *for every left ideal* L *of* S *and every ideal* X *of* S;

(13) $(R : X)_r \cap X \subseteq R$ *for every right ideal* R *of* S *and every ideal* X *of* S *such that* $R \subseteq X$;

(14) $(L : X)_l \cap X \subseteq L$ *for every left ideal* L *of* S *and every ideal* X *of* S *such that* $L \subseteq X$.

Proof. (1) $\Leftrightarrow$ (2) If (1) holds, then for any ideals X, Y of S, we have

$$\begin{aligned}(XY] &\subseteq X \cap Y \\ &= ((X \cap Y)(X \cap Y)] \\ &\subseteq (XY].\end{aligned}$$

Hence (2) holds. That (2) $\Rightarrow$ (1) follows by setting $X = Y$.

(2) $\Leftrightarrow$ (3) Assume that (2) holds. Let X, Y be ideals of S. By Lemma

2.2(2), $(Y : X)_r$ is an ideal of *S*. By assumption and Lemma 2.2(1),

$$\begin{aligned} X \cap (Y : X)_r &= (X(Y : X)_r] \\ &\subseteq (X \cap Y] \\ &\subseteq (X] \cap (Y] \\ &= X \cap Y. \end{aligned}$$

The reverse inclusion is clear. Conversely, assume that (3) holds. Let *X*, *Y* be ideals of *S*. By assumption and $Y \subseteq ((XY] : X)_r$,

$$\begin{aligned} Y \cap X &\subseteq ((XY] : X)_r \cap X \\ &= (XY] \cap X \\ &= (XY]. \end{aligned}$$

Since $(XY] \subseteq X \cap Y$, it follows that $X \cap Y = (XY]$; thus (2) holds.

(3) $\Leftrightarrow$ (5) Setting $Z = X$, we obtain (5) $\Rightarrow$ (3). Assume (3) holds. If *X*, *Y*, *Z* are ideals of *S* such that $Z \subseteq X$, then $Z \cap X = Z$; hence

$$\begin{aligned} (Y : X)_r \cap Z &= (Y : X)_r \cap Z \cap X \\ &= X \cap Y \cap Z \\ &= Y \cap Z. \end{aligned}$$

Thus (5) holds.

By the left-right dual of (3) and (4), (5) and (6) we obtain that (1)-(6) are equivalent.

Now, we prove (1) $\Leftrightarrow$ (9). If (9) is true, then setting $R = X$ in (9) we obtain (1). Assume that (1) holds, and let *R* be a right ideal of *S* and *X* be an ideal of *S* such that $R \subseteq X$. Since $(XR]$ is an ideal of *S*, using (3) we have

$$\begin{aligned} R &= R \cap X \\ &\subseteq ((XR] : X)_r \cap X \end{aligned}$$

$$= X \cap (XR]$$
$$\subseteq (XR].$$

Hence (9) holds.

Again, by the left-right dual of (7) and (8), (9) and (10) we obtain that (1)-(10) are equivalent.

Finally, we shall prove (9) ⇔ (13). And by the left-right dual of (11) and (12), (13) and (14) we obtain that (1)-(14) are equivalent. Therefore, the theorem is proved. Now, assume that (13) holds true. Let R be a right ideal of S and X be an ideal of S such that $R \subseteq X$. Then $(XR] \subseteq X$. We have

$$\begin{aligned} R &= R \cap X \\ &\subseteq ((XR] : X)_r \cap X \\ &\subseteq (XR]. \end{aligned}$$

Hence, (9) follows. Conversely, assume that (9) true; then (1)-(10) are true. Let R be a right ideal of S and X be an ideal of S such that $R \subseteq X$. By (7),

$$\begin{aligned} X \cap ((XR] : X)_r &\subseteq X((XR] : X)_r \\ &\subseteq X \cap R. \end{aligned}$$

Hence, (11) follows. This implies (13). □

Theorem 2.4. *For a partially ordered semigroup* $(S, \cdot, \leq)$, *the following statements are equivalent*:

(1) $((I(x))^2] = I(x)$ *for all* $x \in S$;

(2) $I(x) \cap I(y) = (I(x)I(y)]$ *for all* $x, y \in S$;

(3) $(I(y) : I(x))_r \cap I(x) = I(x) \cap I(y)$ *for all* $x, y \in S$;

(4) $(I(y) : I(x))_l \cap I(x) = I(x) \cap I(y)$ *for all* $x, y \in S$;

(5) $(I(y) : I(x))_r \cap I(z) = I(y) \cap I(y)$ *for all* $x, y, z \in S$ *such that* $z \in I(x)$;

(6) $(I(y) : I(x))_l \cap I(z) = I(y) \cap I(z)$ *for all* $x, y, z \in S$ *such that* $z \in I(x)$;

(7) $R(a) \cap I(x) \subseteq (I(x)R(a)]$ *for all* $a, x \in S$;

(8) $L(b) \cap I(x) \subseteq (L(b)I(x)]$ *for all* $b, x \in S$;

(9) $R(a) \subseteq (I(x)R(a)]$ *for all* $a, x \in S$ *such that* $R(a) \subseteq I(x)$;

(10) $L(b) \subseteq (L(b)I(x)]$ *for all* $b, x \in S$ *such that* $L(b) \subseteq I(x)$;

(11) $(R(a) : I(x)) \cap I(x) \subseteq I(x) \cap R(a)$ *for all* $a, x \in S$;

(12) $(L(b) : I(x))_l \cap I(x) \subseteq L(b) \cap I(x)$ *for all* $b, x \in S$;

(13) $(R(a) : I(x))_r \cap I(x) \subseteq R(a)$ *for all* $a, x \in S$ *such that* $R(a) \subseteq I(x)$;

(14) $(L(b) : I(x))_l \cap I(x) \subseteq L(a)$ *for all* $b, x \in S$ *such that* $L(b) \subseteq I(x)$.

Proof. This can be proved in the same manner as Theorem 2.3. □

Theorem 2.5. *For a partially ordered semigroup* $(S; \cdot, \leq)$, *the statements in Theorem* 2.3 *and Theorem* 2.4 *are equivalent*.

Proof. We shall show that (1) in Theorem 2.3 and (1) in Theorem 2.4 are equivalent. Assume that $((I(x))^2] = x$ for all $x \in S$. Let X be an ideal of S. We have $(X^2] \subseteq X$. If $x \in X$, then by assumption,

$$x = ((I(x))^2] \subseteq (X^2].$$

Hence $X \subseteq (X^2]$. The converse statement is clear. □

A partially ordered semigroup $(S, \cdot, \leq)$ is said to be *regular* if for any $a \in S$ there exists $x \in S$ such that $a \leq axa$. The following theorem shows that Theorem 2.5 (also, Theorems 2.3-2.4) is useful.

Theorem 2.6. *A commutative partially ordered semigroup* $(S, \cdot, \leq)$ *is regular if and only if* $X = (X^2]$ *for any ideal X of S*.

Proof. Assume that S is regular. Let X be an ideal of S. We have

$$(X^2] \subseteq (X] = X.$$

If $a \in X$, then by assumption there exists $x \in S$ such that $a \leq axa$. By

$$axa \in XSX \subseteq XX = X^2$$

it follows that $a \in (X^2]$. Hence $X = (X^2]$.

Conversely, assume that $X = (X^2]$ for any ideal X of S. Let $a \in S$. Consider

$$\begin{aligned} a &\in I(a) \\ &= (I(a)I(a)] \\ &= ((a \cup aS \cup Sa \cup SaS](a \cup aS \cup Sa \cup SaS]] \\ &= ((a \cup aS \cup Sa \cup SaS)(a \cup aS \cup Sa \cup SaS)] \\ &\subseteq (a^2 \cup aSa]. \end{aligned}$$

Hence S is regular. □

3. On Simple Partially Ordered Semigroups

Let $(S, \cdot, \leq)$ be a partially ordered semigroup, and let $P(S)$ be the set of all non-empty subsets of S. Under the inclusion for sets, it is observed that $(P(S), \circ, \subseteq)$ is a partially ordered semigroup with the multiplication defined by

$$A \circ B = (AB]$$

for any $A, B \in P(S)$. Moreover, let

$\mathcal{I}(S)$ denote the set of all two-sided ideals of S.

It is easy to see that $\mathcal{I}(S)$ is a subsemigroup of $P(S)$.

A partially ordered semigroup $(S, \cdot, \leq)$ is said to be a *left* (resp. *right*) *zero partially ordered semigroup* if, for any $x, y \in S$, $xy = x$ (resp. $xy = y$). And S is said to be *simple* if it contains no two-sided ideals.

Theorem 3.1. *Let* $(S, \cdot, \leq)$ *be a partially ordered semigroup. Then* S *is simple if and only if one of the following conditions holds*:

(1) $\mathcal{I}(S)$ *is a left zero semigroup*;

(2) $\mathcal{I}(S)$ *is a right zero semigroup*.

Proof. It is clear that S is simple implies $\mathcal{I}(S)$ is left zero. Assume that $\mathcal{I}(S)$ is left zero. If X is an ideal of S, then $X \circ X = X$; hence $(X^2] = X$ for every ideal X of S. By Theorem 2.3(2) and assumption,

$$X = (XY] = X \cap Y \subseteq Y$$

for every ideal X, Y of S. This implies S is simple. The second assertion can be proved similarly. □

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