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# NOTE ON APOSTOL-DAEHEE POLYNOMIALS AND NUMBERS

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### Abstract

A variety of polynomials, including their diverse generalizations, have been investigated. In this paper, we aim to present some (presumably) new identities and relations for the $\lambda$-Apostol-Daehee polynomials $\mathfrak{D}_n(x; \lambda)$ and the $\lambda$-Apostol-Daehee polynomials $\mathfrak{D}_n^{(k)}(x; \lambda)$ of order $k$, which have been very recently introduced, with (possibly) other polynomials and numbers.

## 1. Introduction and Preliminaries

Special polynomials and numbers such as Bernoulli polynomials and numbers, Euler polynomials and numbers, Genocchi polynomials and numbers, Apostol-Bernoulli polynomials and numbers, Stirling numbers of the first and second kinds, Daehee polynomials and numbers, Apostol-Daehee polynomials and numbers with their generalizations have applied in a wide range of areas including mathematics, mathematical physics,

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probability, statistics and engineering. Very recently, Simsek [2, 4] introduced Apostol-Daehee numbers and polynomials and their extensions to give certain interesting identities and relations among them and some other polynomials and numbers (see also [3, 5]).

In this sequel, we also aim to present certain (presumably) new identities and relations among Apostol-Daehee polynomials and numbers and some other polynomials and numbers.

For our purpose, we choose to recall some known polynomials and numbers. The Bernoulli polynomials $B_n(x)$ are defined by the generating function:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \qquad (|t| < 2\pi). \tag{1.1}$$

The numbers $B_n := B_n(0)$ are called the *Bernoulli numbers* generated by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \qquad (|t| < 2\pi). \tag{1.2}$$

The generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ of degree $\alpha$ in $x$ are defined by the generating function:

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!} \qquad (|t| < 2\pi;\ 1^{\alpha} := 1) \tag{1.3}$$

for arbitrary (real or complex) parameter $\alpha$. Clearly, we have

$$B_n^{(\alpha)}(x) = (-1)^n B_n^{(\alpha)}(\alpha - x), \tag{1.4}$$

so that

$$B_n^{(\alpha)}(x) = (-1)^n B_n^{(\alpha)}(0) =: (-1)^n B_n^{(\alpha)} \tag{1.5}$$

in terms of the generalized Bernoulli numbers $B_n^{(\alpha)}$ defined by the generating function:

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} = \sum_{n=0}^{\infty} B_n^{(\alpha)} \frac{t^n}{n!} \qquad (|t| < 2\pi;\ 1^{\alpha} := 1). \tag{1.6}$$

The Apostol-Bernoulli polynomials $\mathcal{B}_n(x; \lambda)$ are defined by means of the generating function (see [1, 6]):

$$\frac{te^{xt}}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x; \lambda) \frac{t^n}{n!} \tag{1.7}$$

$$(|t| < 2\pi \text{ when } \lambda = 1;\ |t| < |\log \lambda| \text{ when } \lambda \neq 1)$$

with, of course,

$$B_n(x) = \mathcal{B}_n(x; 1) \text{ and } \mathcal{B}_n(\lambda) := \mathcal{B}_n(0; \lambda), \tag{1.8}$$

where $\mathcal{B}_n(\lambda)$ denotes the so-called Apostol-Bernoulli numbers. The Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x; \lambda)$ of (real or complex) order $\alpha$ are defined by means of the following generating function:

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} \cdot e^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!} \tag{1.9}$$

$$(|t| < 2\pi \text{ when } \lambda = 1;\ |t| < |\log \lambda| \text{ when } \lambda \neq 1;\ 1^{\alpha} := 1)$$

with, of course,

$$B_n^{(\alpha)}(x) = \mathcal{B}_n^{(\alpha)}(x; 1) \text{ and } \mathcal{B}_n^{(\alpha)}(\lambda) := \mathcal{B}_n^{(\alpha)}(0; \lambda), \tag{1.10}$$

where $\mathcal{B}_n^{(\alpha)}(\lambda)$ denotes the so-called Apostol-Bernoulli numbers of order $\alpha$.

The Stirling numbers $s(n, k)$ of the first kind are defined by the generating functions:

$$z(z-1)\cdots(z-n+1)=\sum_{k=0}^{n}s(n,\,k)z^{k} \tag{1.11}$$

and

$$\{\log(1+z)\}^{k}=k!\sum_{n=k}^{\infty}s(n,\,k)\frac{z^{n}}{n!} \qquad (|\,z\,|<1). \tag{1.12}$$

The Stirling numbers $S(n,\,k)$ of the second kind are defined by the generating functions:

$$z^{n}=\sum_{k=0}^{n}S(n,\,k)z(z-1)\cdots(z-k+1), \tag{1.13}$$

$$(e^{z}-1)^{k}=k!\sum_{n=k}^{\infty}S(n,\,k)\frac{z^{n}}{n!}, \tag{1.14}$$

and

$$(1-z)^{-1}(1-2z)^{-1}\cdots(1-kz)^{-1}=\sum_{n=k}^{\infty}S(n,\,k)z^{n-k} \qquad (|\,z\,|<k^{-1}), \tag{1.15}$$

where $S(n,\,k)$ denotes the number of ways of partitioning a set of $n$ elements into $k$ non-empty subsets.

For more details of the above-recalled polynomials and numbers and other ones, one may refer to [7, Sections 1.6-1.8].

Simsek [2, 4] introduced to investigate the $\lambda$-Apostol-Daehee polynomials $\mathfrak{D}_{n}(x;\,\lambda)$ by means of the following generating function:

$$G(t,\,x;\,\lambda)=\frac{\log\lambda+\log(1+\lambda t)}{\lambda(1+\lambda t)-1}(1+\lambda t)^{x}:=\sum_{n=0}^{\infty}\mathfrak{D}_{n}(x;\,\lambda)\frac{t^{n}}{n!}. \tag{1.16}$$

Simsek [4] defined the $\lambda$-Apostol-Daehee polynomials $\mathfrak{D}_{n}^{(k)}(x;\,\lambda)$ of order $k$

by means of the following generating function:

$$F_{\mathfrak{D}}(t,\ x;\ \lambda,\ k) = \left(\frac{\log\lambda + \log(1+\lambda t)}{\lambda(1+\lambda t)-1}\right)^k (1+\lambda t)^x := \sum_{n=0}^{\infty} \mathfrak{D}_n^{(k)}(x;\ \lambda)\frac{t^n}{n!}. \tag{1.17}$$

It is easy to see that

$$\mathfrak{D}_n^{(1)}(x;\ \lambda) = \mathfrak{D}_n(x;\ \lambda). \tag{1.18}$$

**Remark 1.** It is easy to see that

$$\mathfrak{D}_n^{(1)}(x;\ \lambda) = \mathfrak{D}_n(x;\ \lambda). \tag{1.19}$$

Define a function $f(t)$ as follows:

$$f(t) := \begin{cases} 1 & (t = 0), \\ \dfrac{t}{\log(1+t)} & (t \neq 0), \end{cases} \tag{1.20}$$

where $\log(1+t)$ is assumed to have the principal branch. Since $\lim_{t\to 0} f(t) = 1 = f(0)$, $f$ is analytic at $t = 0$. So we have

$$f(t) := \sum_{n=0}^{\infty} p_n t^n \qquad (|\, t\,| < 1). \tag{1.21}$$

Then we obtain

$$\begin{aligned} t = \log(1+t)\sum_{n=0}^{\infty} p_n t^n &= \left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}t^{n+1}\right)\left(\sum_{n=0}^{\infty} p_n t^n\right) \\ &= t\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-1)^{n-k}}{n-k+1}p_k\right)t^n. \end{aligned}$$

Equating the coefficients of $t^n$, we get a recurrence formula for the $p_n$ as follows:

$$p_0 = 1 \text{ and } p_n = \sum_{k=0}^{n-1} \frac{(-1)^{n-k+1}}{n-k+1} p_k \qquad (n \in \mathbb{N}). \tag{1.22}$$

The first few of $p_n$ are

$$p_0 = 1,\ p_1 = \frac{1}{2},\ p_2 = -\frac{1}{12},\ p_3 = \frac{1}{24},\ p_4 = -\frac{19}{720},$$

$$p_5 = \frac{3}{160},\ p_6 = -\frac{863}{60480},\ \cdots.$$

Define a function $g(t;\, \alpha)(\alpha \neq 0)$ as follows:

$$g(t;\, \alpha) := \begin{cases} \dfrac{1}{\alpha} & (t = 0), \\ \dfrac{t}{(1+t)^{\alpha} - 1} & (t \neq 0). \end{cases} \tag{1.23}$$

It is easy to see that $g(t;\, \alpha)$ is analytic at $t = 0$. Let

$$g(t;\, \alpha) := \sum_{n=0}^{\infty} q_n(\alpha) t^n \qquad (|\, t\,| < 1). \tag{1.24}$$

Likewise as in getting (1.22), we obtain a recurrence formula for the $q_n(\alpha)$:

$$q_0(\alpha) = \frac{1}{\alpha} \text{ and } q_n(\alpha) = -\frac{1}{\alpha} \sum_{j=0}^{n-1} \binom{\alpha}{n-j+1} q_j(\alpha) \qquad (n \in \mathbb{N}). \tag{1.25}$$

The first few of $q_n(\alpha)$ are

$$q_0(\alpha) = \frac{1}{\alpha},\ q_1(\alpha) = \frac{1-\alpha}{2\alpha},\ q_2(\alpha) = \frac{(\alpha-1)(\alpha+1)}{12\alpha},$$

$$q_3(\alpha) = -\frac{(\alpha-1)(\alpha+1)}{24\alpha},\ q_4(\alpha) = -\frac{(\alpha-1)(\alpha+1)(\alpha^2-19)}{720\alpha},\ \cdots.$$

## 2. Main Results

Here we present some (presumably) new identities and relations for the $\lambda$-Apostol-Daehee polynomials $\mathfrak{D}_n(x;\lambda)$ and the $\lambda$-Apostol-Daehee polynomials $\mathfrak{D}_n^{(k)}(x;\lambda)$ of order $k$ with (possibly) other polynomials and numbers. Here and in the following, let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

**Theorem 1.** *The following difference formula for the* $\lambda$*-Apostol-Daehee polynomials* $\mathfrak{D}_n(x;\lambda)$ *holds true*:

$$\lambda\mathfrak{D}_n(x+1;\lambda) - \mathfrak{D}_n(x;\lambda) = \lambda^n \log\lambda \sum_{j=0}^{n} s(n, j)x^j$$

$$+ \lambda^n \sum_{k=0}^{n-1}\sum_{j=0}^{k}(-1)^{n-1-k}(n-1-k)!\binom{n}{k}s(k, j)x^j \qquad (n \in \mathbb{N}), \quad (2.1)$$

*where* $s(k, j)$ *are Stirling numbers of the first kind given in* (1.11) *and* (1.12).

**Proof.** We find from (1.16) that

$$\sum_{n=0}^{\infty}\{\lambda\mathfrak{D}_n(x+1;\lambda) - \mathfrak{D}_n(x;\lambda)\}\frac{t^n}{n!} = \lambda G(t, x+1;\lambda) - G(t, x;\lambda)$$

$$= \{\log\lambda + \log(1+\lambda t)\}(1+\lambda t)^x. \quad (2.2)$$

The Maclaurin series of $\log\lambda + \log(1+\lambda t)$ is given as follows:

$$\log\lambda + \log(1+\lambda t) = \sum_{n=0}^{\infty}\alpha_n t^n \qquad (|\lambda t| < 1), \quad (2.3)$$

where

$$\alpha_0 = \log\lambda \text{ and } \alpha_n = \frac{(-1)^{n-1}}{n}\lambda^n \qquad (n \in \mathbb{N}).$$

The binomial series of $(1+\lambda t)^x$ is given as follows:

$$(1+\lambda t)^x = \sum_{n=0}^{\infty} \binom{x}{n} \lambda^n t^n \qquad (|\lambda t| < 1), \tag{2.4}$$

where

$$\binom{x}{n} := \begin{cases} 1 & (n = 0), \\ \dfrac{x(x-1)\cdots(x-n+1)}{n!} & (n \in \mathbb{N}). \end{cases} \tag{2.5}$$

Applying (1.11) to (2.5), we have

$$(1+\lambda t)^x = \sum_{n=0}^{\infty} \beta_n(x;\lambda) t^n \qquad (|\lambda t| < 1), \tag{2.6}$$

where

$$\beta_n(x;\lambda) = \frac{\lambda^n}{n!} \sum_{j=0}^{n} s(n, j) x^j \qquad (n \in \mathbb{N}_0).$$

We find from (2.2), (2.4) and (2.6) that

$$\sum_{n=0}^{\infty} \{\lambda \mathfrak{D}_n(x+1;\lambda) - \mathfrak{D}_n(x;\lambda)\} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \alpha_{n-k} \beta_k(x;\lambda) \right) t^n,$$

which, upon equating the coefficients of $t^n$, gives the desired identity (2.1).

□

**Theorem 2.** *The following explicit formula for the* λ*-Apostol-Daehee polynomials* $\mathfrak{D}_n(x;\lambda)$ *holds true*:

$$\mathfrak{D}_n(x;\lambda) = \frac{n!}{\lambda - 1} \sum_{k=0}^{n} \left\{ (\log \lambda)(-1)^{n-k} \frac{\lambda^k}{k!} \left( \frac{\lambda^2}{\lambda - 1} \right)^{n-k} \sum_{j=0}^{k} s(k, j) x^j \right.$$

$$+\sum_{m=0}^{n-k-1}\frac{(-1)^{n-k-1}}{(n-k-m)k!}\lambda^{n-m}\left(\frac{\lambda^2}{\lambda-1}\right)^m\sum_{j=0}^{k}s(k,\ j)x^j\Bigg\}\qquad(n\in\mathbb{N}_0),\quad(2.7)$$

*where* $s(k,\ j)$ *are Stirling numbers of the first kind given in* (1.11) *and* (1.12).

**Proof.** We see that

$$\frac{1}{\lambda(1+\lambda t)-1}=\sum_{n=0}^{\infty}\gamma_n t^n\qquad\left(\left|\frac{\lambda^2}{\lambda-1}t\right|<1\right),\tag{2.8}$$

where

$$\gamma_n=\frac{(-1)^n}{\lambda-1}\left(\frac{\lambda^2}{\lambda-1}\right)^n\qquad(n\in\mathbb{N}_0).$$

We find from (2.4), (2.6) and (2.8) that

$$\sum_{n=0}^{\infty}\mathfrak{D}_n(x;\ \lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\sum_{m=0}^{n-k}\alpha_{n-k-m}\gamma_m\beta_k(x;\ \lambda)\right\}t^n,$$

which, upon equating the coefficients of $t^n$, yields the desired identity (2.7).

□

**Theorem 3.** *The following higher-order derivative formula for the* $\lambda$*-Apostol-Daehee polynomials* $\mathfrak{D}_n^{(k)}(x;\ \lambda)$ *of order k holds true*:

$$\frac{\partial^\ell}{\partial x^\ell}\mathfrak{D}_n^{(k)}(x;\ \lambda)=\ell!\sum_{j=\ell}^{n}\binom{n}{j}\mathfrak{D}_{n-j}^{(k)}(x;\ \lambda)\lambda^j s(j,\ \ell)\qquad(\ell\in\mathbb{N}_0),\tag{2.9}$$

*where* $s(j,\ \ell)$ *are Stirling numbers of the first kind given in* (1.11) *and* (1.12).

**Proof.** Differentiating both sides of (1.17) $\ell$-times with respect to the variable $x$ and using (1.12) with the (temporary) assumption $s(j,\ \ell)=0$ if

$j < \ell$, we have

$$\sum_{n=0}^{\infty} \frac{\partial^{\ell}}{\partial x^{\ell}} \mathfrak{D}_n^{(k)}(x;\lambda) \frac{t^n}{n!} = \left\{ \sum_{n=0}^{\infty} \mathfrak{D}_n^{(k)}(x;\lambda) \frac{t^n}{n!} \right\} \{\log(1+\lambda t)\}^{\ell}$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \mathfrak{D}_n^{(k)}(x;\lambda) \frac{t^n}{n!} \ell! \, s(j,\ell) \frac{\lambda^j t^j}{j!}$$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{j=0}^{n} \mathfrak{D}_{n-j}^{(k)}(x;\lambda) \frac{\ell! \, s(j,\ell)}{(n-j)!\, j!} \lambda^j \right\} t^n,$$

the first and last of which, upon equating the coefficients of $t^n$, yields the desired identity (2.9). □

**Theorem 4.** *The following addition formula for the* $\lambda$*-Apostol-Daehee polynomials* $\mathfrak{D}_n^{(k)}(x;\lambda)$ *of order k holds true*:

$$\mathfrak{D}_n^{(k)}(x+y;\lambda) = \sum_{j=0}^{n} \sum_{\ell=0}^{j} \binom{n}{j} \mathfrak{D}_{n-j}^{(k)}(x;\lambda) \lambda^j s(j,\ell) y^{\ell} \qquad (n \in \mathbb{N}_0), \quad (2.10)$$

*where* $s(j,\ell)$ *are Stirling numbers of the first kind given in* (1.11) *and* (1.12). □

**Proof.** A similar argument as in the above proofs can establish the result here. So its detailed account is omitted. □

**Theorem 5.** *The following integral formula holds true*:

$$\int_{\alpha}^{\alpha+1} \mathfrak{D}_n^{(k)}(x;\lambda)\,dx = \sum_{j=0}^{n} j! \binom{n}{j} \lambda^j \mathfrak{D}_{n-j}^{(k)}(\alpha;\lambda)\, p_j \qquad (j \in \mathbb{N}_0), \quad (2.11)$$

*where* $p_j$*'s are given in* (1.22).

**Proof.** Integrating both sides of (1.17) with respect to the variable *x* from $\alpha$ to $\alpha+1$, we have

$$\left(\frac{\log\lambda + \log(1+\lambda t)}{\lambda(1+\lambda t)-1}\right)^k (1+\lambda t)^{\alpha} \frac{\lambda t}{\log(1+\lambda t)} = \sum_{n=0}^{\infty}\int_{\alpha}^{\alpha+1}\mathfrak{D}_n^{(k)}(x;\lambda)dx\frac{t^n}{n!}.$$

Using (1.17) and (1.21), we obtain

$$\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\mathfrak{D}_{n-j}^{(k)}(\alpha;\lambda)\frac{\lambda^j p_j}{(n-j)!}\right)t^n = \sum_{n=0}^{\infty}\int_{\alpha}^{\alpha+1}\mathfrak{D}_n^{(k)}(x;\lambda)dx\frac{t^n}{n!},$$

which, upon equating the coefficients of $t^n$, yields the desired formula (2.11).

□

**Theorem 6.** *The following formula holds true*:

$$\sum_{\ell=0}^{m-1}\mathfrak{D}_n^{(k)}\left(x+\frac{\ell}{m};\lambda\right) = \sum_{j=0}^{n} j!\binom{n}{j}\mathfrak{D}_{n-j}^{(k)}(x;\lambda)q_j\left(\frac{1}{m}\right) \tag{2.12}$$

$$(m \in \mathbb{N};\ n \in \mathbb{N}_0),$$

*where* $q_j(\alpha)$ *are given in* (1.25).

**Proof.** We find from (1.17) that

$$\sum_{n=0}^{\infty}\left(\sum_{\ell=0}^{m-1}\mathfrak{D}_n^{(k)}\left(x+\frac{\ell}{m};\lambda\right)\right)\frac{t^n}{n!}$$

$$= \left(\frac{\log\lambda + \log(1+\lambda t)}{\lambda(1+\lambda t)-1}\right)^k (1+\lambda t)^x \cdot \frac{\lambda t}{(1+\lambda t)^{\frac{1}{m}}-1}.$$

Upon using (1.17) and (1.24) on the right-hand side, similarly as above, we obtain the desired identity (2.12). □

## 3. Further Remarks

It is seen that $G(t, x; \lambda)$ in (1.16) is analytic at $t = 0$ as a function of the variable $t$ and

$$\mathfrak{D}_n(x;\lambda)=\frac{\partial^n}{\partial t^n}G(t,\ x;\ \lambda)\bigg|_{t=0},$$

which, upon using the Maple, gives the following interesting identity:

$$\begin{aligned}\mathfrak{D}_n(x;\lambda)=\frac{1}{\lambda(\lambda-1)(n+1)\Gamma(1-x)}&\Bigg[(n+1)!\,\lambda\left(\frac{\lambda^2}{1-\lambda}\right)^n\\
&\times\left\{(\pi\cot(\pi x)+\log\lambda+\psi(x))\Gamma(1-x)-\left(\frac{1}{\lambda}\right)^x\Gamma(-x)\right.\\
&\left.-(\pi\cot(\pi x)+\psi(x))\Gamma(1-x)+\Gamma(-x)\right\}\\
&+(\lambda-1)(-\lambda)^n\Gamma(1-x+n)\{x(\psi(x)+\pi\cot(\pi x))+1+x\log\lambda\}\\
&\times{}_2F_1\left(1,\ n-x+1;\ n+2;\ \frac{\lambda-1}{\lambda}\right)+(n+1)!\,x(\lambda-1)\left(\frac{\lambda^2}{1-\lambda}\right)^n\\
&\times\sum_{k=0}^{n-1}\left(\frac{\lambda-1}{\lambda}\right)^k\Gamma(k-x+1)\psi(k-x+1)\Bigg],\end{aligned}$$

where $\Gamma$, $\psi$ and ${}_2F_1$ are Gamma function, Psi- (or Digamma) function, and hypergeometric function, respectively (see, e.g., [7, Chapter 1]).

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**References**

[1] T. M. Apostol, On the Lerch zeta-function, Pacific J. Math. 1 (1951), 161-167.

[2] Y. Simsek, Apostol type Daehee numbers and polynomials, Adv. Stud. Contemp. Math. 26(3) (2016), 555-566.

[3] Y. Simsek, Analysis of the *p*-adic *q*-Volkenborn integrals: an approach to generalized Apostol-type special numbers and polynomials and their applications, Cogent Math. 3(1) (2016), Article 1269393.

[4] Y. Simsek, Identities on the Changhee numbers and Apostol-Daehee polynomials, Adv. Stud. Contemp. Math. 27 (2017) (accepted).

[5] Y. Simsek and A. Yardimci, Applications on the Apostol-Daehee numbers and polynomials associated with special numbers, polynomials, and *p*-adic integrals, Adv. Diff. Equ. 2016 (2016), Article ID 308.

[6] H. M. Srivastava, Some formulas for the Bernoulli and Euler polynomials at rational arguments, Math. Proc. Cambridge Phil. Soc. 129 (2000), 77-84.

[7] H. M. Srivastava and J. Choi, Zeta and *q*-Zeta Functions and Associated Series and Integrals, Elsevier Science Publishers, Amsterdam, London, New York, 2012.