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AN ANALOGOUS EULER SUM

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Abstract

Since Euler discovered the Euler sum, the sum has been redeveloped in various ways and a large number of its variants have been presented. The object of this note is to evaluate an analogue of the original Euler sum.

1. Introduction and Preliminaries

The following well-known Euler sum

$$\sum_{n=1}^{\infty} \frac{H_n}{n^p} = \left(1 + \frac{p}{2}\right)\zeta(p+1) - \frac{1}{2}\sum_{j=2}^{p-1} \zeta(j)\zeta(p-j+1) \qquad (p \in \mathbb{N}\setminus\{1\}) \quad (1.1)$$

was first discovered by Euler. Since then, the formula (1.1) has been redeveloped in various ways and a large number of its variants have been presented (see, e.g., [1, 2, 3, 4, 5, 6] and the references therein). Here and in the following, an empty sum is assumed to be zero. The H_n denote the

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harmonic numbers defined by

$$H_n := \sum_{j=1}^{n} \frac{1}{j} \qquad (n \in \mathbb{N}), \tag{1.2}$$

whose generalized harmonic numbers are defined by

$$H_n^{(s)} := \sum_{j=1}^{n} \frac{1}{j^s} \qquad (n \in \mathbb{N};\ s \in \mathbb{C}). \tag{1.3}$$

Also $\zeta(s)$ is the Riemann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (\Re(s) > 1), \tag{1.4}$$

one of whose simplest generalizations is called *generalized* (*Hurwitz*) *zeta function* defined by

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \qquad (\Re(s) > 1;\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-), \tag{1.5}$$

Here and in the following, let $\mathbb{N}$, $\mathbb{C}$ and $\mathbb{Z}_0^-$ be the sets of positive integers, complex numbers, and non-positive integers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Among many properties of $\zeta(s)$ and $\zeta(s, a)$, the following formulas are recalled (see, e.g., [6, Sections 2.2 and 2.3]):

$$\zeta(s, a) = \zeta(s, n+a) + \sum_{k=0}^{n-1} \frac{1}{(k+a)^s} \qquad (n \in \mathbb{N}), \tag{1.6}$$

whose special case when $a = 1$ is given by

$$\zeta(s) = \zeta(s, n+1) + \sum_{k=1}^{n} \frac{1}{k^s} \qquad (n \in \mathbb{N}). \tag{1.7}$$

We also recall the following formula (see, e.g., [6, Eq. (6), p. 270]):

$$\sum_{k=2}^{\infty}(-1)^{k}\zeta(k,\,a)t^{k-1} = \psi(a+t)-\psi(a) \qquad (|\,t\,| < |\,a\,|), \tag{1.8}$$

where $\psi(s)$ is the Psi (or Digamma) function defined by $\psi(s) = \Gamma'(s)/\Gamma(s)$ ($\Gamma(s)$ being the familiar Gamma function) one of whose properties is given (see, e.g., [6, Eq. (7), p. 25]):

$$\psi(s+n)-\psi(s) = \sum_{k=1}^{n}\frac{1}{s+k-1} \qquad (n \in \mathbb{N}). \tag{1.9}$$

Here, in this paper, like the Euler sum in (1.1), we aim to express the following analogous Euler sum:

$$\sum_{n=1}^{\infty}\frac{H_{n+m}}{n^{p}} \qquad (p \in \mathbb{N}\setminus\{1\};\ m \in \mathbb{N}_0) \tag{1.10}$$

in terms of the Riemann zeta functions $\zeta(s)$.

2. Main Result

For our purpose, we first give an evaluation asserted by the following lemma.

Lemma 1. *Let*

$$\mathcal{S}_p(j) := \sum_{n=1}^{\infty}\frac{1}{n^{p}(n+j)} \qquad (j,\ p \in \mathbb{N}). \tag{2.1}$$

The following formula holds true:

$$\mathcal{S}_p(j) = \sum_{n=1}^{j}\frac{1}{n^{p}(n+j)} + \frac{(-1)^{p+1}}{j^{p}}\left\{H_{2j} - H_j + \sum_{k=2}^{p}(-1)^{k}\left(H_j^{(k)} - \zeta(k)\right)j^{k-1}\right\}, \tag{2.2}$$

where, throughout this paper, an empty sum is assumed to be zero.

Proof. We begin by

$$\mathcal{S}_p(j) = \sum_{n=1}^{j} \frac{1}{n^p (n+j)} + \sum_{n=j+1}^{\infty} \frac{1}{n^{p+1}} \cdot \frac{1}{1+j/n}.$$

Applying the geometric expansion in the last term and changing the order of summations, we have

$$\begin{aligned}
\mathcal{S}_p(j) &= \sum_{n=1}^{j} \frac{1}{n^p (n+j)} + \sum_{k=0}^{\infty} (-1)^k j^k \sum_{n=j+1}^{\infty} \frac{1}{n^{p+k+1}} \\
&= \sum_{n=1}^{j} \frac{1}{n^p (n+j)} + \sum_{k=0}^{\infty} (-1)^k j^k \sum_{n=0}^{\infty} \frac{1}{(n+j+1)^{p+k+1}} \\
&= \sum_{n=1}^{j} \frac{1}{n^p (n+j)} + \sum_{k=0}^{\infty} (-1)^k j^k \zeta(p+k+1,\ j+1).
\end{aligned}$$

We find

$$\begin{aligned}
\mathcal{S}_p(j) &= \sum_{n=1}^{j} \frac{1}{n^p (n+j)} + \frac{(-1)^{p+1}}{j^p} \sum_{k=p+1}^{\infty} (-1)^k \zeta(k,\ j+1)\, j^{k-1} \\
&= \sum_{n=1}^{j} \frac{1}{n^p (n+j)} + \frac{(-1)^{p+1}}{j^p} \left\{ -\sum_{k=2}^{p} (-1)^k \zeta(k,\ j+1)\, j^{k-1} \right. \\
&\qquad \left. + \sum_{k=2}^{\infty} (-1)^k \zeta(k,\ j+1)\, j^{k-1} \right\}.
\end{aligned}$$

Using (1.8) and (1.9), we obtain

$$\begin{aligned}
\mathcal{S}_p(j) &= \sum_{n=1}^{j} \frac{1}{n^p (n+j)} \\
&\quad + \frac{(-1)^{p+1}}{j^p} \left\{ \psi(2j+1) - \psi(j+1) - \sum_{k=2}^{p} (-1)^k \zeta(k,\ j+1)\, j^{k-1} \right\}
\end{aligned}$$

$$= \sum_{n=1}^{j} \frac{1}{n^p (n+j)} + \frac{(-1)^{p+1}}{j^p} \left\{ \sum_{k=1}^{j} \frac{1}{k+j} - \sum_{k=2}^{p} (-1)^k \zeta(k,\ j+1)\, j^{k-1} \right\}$$

$$= \sum_{n=1}^{j} \frac{1}{n^p (n+j)} + \frac{(-1)^{p+1}}{j^p} \left\{ H_{2j} - H_j - \sum_{k=2}^{p} (-1)^k \zeta(k,\ j+1)\, j^{k-1} \right\}. \tag{2.3}$$

We find from (1.3) and (1.7) that

$$\zeta(s) = \zeta(s,\ n+1) + H_n^{(s)} \qquad (n \in \mathbb{N};\ s \in \mathbb{C}). \tag{2.4}$$

Finally, applying (2.4) to the $\zeta(k,\ j+1)$ in (2.3), we obtain the desired formula (2.2). □

Now we are ready to give the main identity asserted by the following theorem.

Theorem 1. *The following analogous Euler sum holds true*:

$$\sum_{n=1}^{\infty} \frac{H_{n+m}}{n^p} = \left(1 + \frac{p}{2}\right) \zeta(p+1) - \frac{1}{2} \sum_{j=2}^{p-1} \zeta(j) \zeta(p-j+1)$$

$$+ \frac{1}{2} H_m^{(p+1)} + \sum_{n=1}^{m} \frac{1}{n^p} \{((-1)^{p+1} - 1) H_{2n} + (-1)^p H_n + H_{n+m}\}$$

$$+ (-1)^{p+1} \sum_{k=2}^{p} (-1)^k \left\{ \sum_{j=1}^{m} \frac{H_j^{(k)}}{j^{p+1-k}} - \zeta(k) H_m^{(p+1-k)} \right\} \tag{2.5}$$

$$(p \in \mathbb{N} \setminus \{1\};\ m \in \mathbb{N}_0).$$

Proof. We see that

$$\sum_{n=1}^{\infty} \frac{H_{n+m}}{n^p} = \sum_{n=1}^{\infty} \frac{H_n}{n^p} + \sum_{n=1}^{\infty} \frac{1}{n^p} \sum_{j=1}^{m} \frac{1}{n+j}$$

$$= \sum_{n=1}^{\infty} \frac{H_n}{n^p} + \sum_{j=1}^{m} \sum_{n=1}^{\infty} \frac{1}{n^p (n+j)}. \tag{2.6}$$

Now applying (1.1) and (2.2) to the last equality in (2.6) and noting (see [2, Eq. (1.23)])

$$\sum_{j=1}^{m}\sum_{n=1}^{j}\frac{1}{n^p(n+j)} = \sum_{n=1}^{m}\sum_{j=n}^{m}\frac{1}{n^p(n+j)}$$

$$= \sum_{n=1}^{m}\frac{1}{n^p}\left(H_{n+m} - H_{2n} + \frac{1}{2n}\right)$$

$$= \sum_{n=1}^{m}\frac{1}{n^p}(H_{n+m} - H_{2n}) + \frac{1}{2}H_m^{(p+1)},$$

after some simplification, we obtain the desired identity (2.5). □

References

[1] D. Borwein, J. M. Borwein and R. Girgensohn, Explicit evaluation of Euler sums, Proc. Edinburgh Math. Soc. 38 (1995), 277-294.

[2] J. Choi, Notes on formal manipulations of double series, Commun. Korean Math. Soc. 18(4) (2003), 781-789.

[3] J. Choi and H. M. Srivastava, Explicit evaluation of Euler and related sums, Ramanujan J. 10 (2005), 51-70.

[4] N. Nielsen, Die Gammafunktion, Chelsea Publishing Company, Bronx, New York, 1965.

[5] L.-C. Shen, Remarks on some integrals and series involving the Stirling numbers $\zeta(n)$, Trans. Amer. Math. Soc. 347 (1995), 1391-1399.

[6] H. M. Srivastava and J. Choi, Zeta and *q*-zeta Functions and Associated Series and Integrals, Elsevier Science Publishers, Amsterdam, London, New York, 2012.