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A NEAR CYCLIC $(m_1, m_2, \ldots, m_r)$-CYCLE SYSTEM OF COMPLETE MULTIGRAPH

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Abstract

Let v, λ be positive integers, λK_v denote a complete multigraph on v vertices in which each pair of distinct vertices joining with λ edges. In this article, difference method is used to introduce a new design that decomposes $4K_v$ into cycles, when $v \equiv 2, 10 (\mathrm{mod}\, 12)$. This design merging between cyclic $(m_1, \ldots, m_r)$-cycle system and near-four-factor is called a near cyclic $(m_1, \ldots, m_r)$-cycle system.

1. Introduction

In this paper, it is considered that all graphs are undirected with no loops and vertices set Z_v. We denote the complete graph on v vertices by K_v. An m-cycle (respectively, m-path), denoted by $(c_0, \ldots, c_{m-1})$ (respectively, $[c_0, \ldots, c_{m-1}]$), consists of m distinct vertices $\{c_0, c_1, \ldots, c_{m-1}\}$ and m edges

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$\{c_i c_{i+1}\}$, $0 \le i \le m-2$ and $c_0 c_{m-1}$ (respectively, $m-1$ edges $\{c_i c_{i+1}\}$, $0 \le i \le m-2$).

An $(m_1, ..., m_r)$-cycle is the union of all edges in each m_i-cycle, $1 \le i \le r$. A decomposition of a graph G is a set of subgraphs $\{H_1, ..., H_r\}$ of G whose edges set partitions the edge set of G. If K_v has a decomposition into r cycles of length $m_1, m_2, ..., m_r$, then it is said an $(m_1, ..., m_r)$-*cycle system of order v* that is defined as a pair (V, C) such that $V = V(K_v)$, and C is a collection of edge-disjoint m_i-cycles, for $1 \le i \le r$, which partitions the $E(K_v)$. In particular, if $m_1 = \cdots = m_r = m$, then it is called an *m-cycle system of order v* or (K_v, C_m)-design.

A complete multigraph of order v, denoted by λK_v, can be obtained by replacing each edge of K_v with λ edges. A $(m_1, ..., m_r)$-cycle system of λK_v is a pair (V, C), where $V = V(\lambda K_v)$ and C is a collection of edge-disjoint m_i-cycles for $1 \le i \le r$ which partitions the edge multiset of λK_v. An automorphism of $(m_1, ..., m_r)$-cycle system of λK_v is a bijection $\alpha : V(Z_v) \to V(Z_v)$ such that for any $(c_0, ..., c_{t-1}) \in C$ if and only if $(\alpha(c_0), ..., \alpha(c_{t-1})) \in C$, $(m_1, ..., m_r)$-cycle system of λK_v is called *cyclic* if it has automorphism that is a permutation consisting of a single cycle of order v, for instance, $\alpha = (0, 1, ..., v-1)$ and is said to be *simple* if all its cycles are distinct.

Given an m-cycle $C_m = (c_0, c_1, ..., c_{m-1})$, by $C_m + i$ we mean $(c_0 + i, c_1 + i, ..., c_{m-1} + i)$, where $i \in Z_v$. Analogously, if $C = \{C_{m_1}, C_{m_2}, ..., C_{m_r}\}$ is an $(m_1, ..., m_r)$-cycle, then we use $C + i$ instead of $\{C_{m_1} + i, C_{m_2} + i, ..., C_{m_r} + i\}$. A set of cycles that generates the cyclic $(m_1, ..., m_r)$-cycle system of λK_v by repeated addition of 1 modular v which is called a *starter set* (briefly δ).

The study of $(m_1, ..., m_r)$-cycle system of λK_v has been considered the most important problems in graph decomposition. The important is case $\lambda = 1,\ m_1 = \cdots = m_r = m.$ The existence question for a $(K_v,\ C_m)$-design has been solved by Alspach and Gavlas [2] in the case of m odd and by Šajna [11] for m even. While the existence question for a cyclic m-cycle has been settled when $m = 3$ [8], 5 and 7 [10]. For m even and $v \equiv 1(\mathrm{mod}\ 2m)$, a cyclic m-cycle system of order v was proved for $m \equiv 0,\ 2(\mathrm{mod}\ 4)$ in [6, 9]. Recently, Bryant et al. [3] showed the necessary and sufficient conditions for decomposing K_v into r cycles of lengths $m_1, m_2, ..., m_r$ or into r cycles of lengths $m_1, m_2, ..., m_r$ and perfect matching. Thus, the Alspach's problem has been settled which was posed in 1981 [1]. More recently, it has been extended to this decomposition for the complete multigraph λK_v in [4].

A k-factor of a graph G is a spanning subgraph whose vertices have a degree k. While a near-k-factor is a subgraph in which all vertices have a degree k with exception of one vertex (isolated vertex) which has a degree zero.

Moreover, in [7], Matarneh and Ibrahim introduced the decomposition of a complete multigraph $2K_v$, when $v \equiv 0(\mathrm{mod}\ 12)$, by combination of cyclic $(m_1,\ m_2,\ ...,\ m_r)$-cycle system and near-two-factor. In our paper, we propose a new design for decomposing a complete multigraph $4K_v$ when $v \equiv 2, 10(\mathrm{mod}\ 12)$. This is obtained by merging a cyclic $(m_1,\ ...,\ m_r)$-cycle system and near-four-factors that is called a *near cyclic* $(m_1,\ ...,\ m_r)$*-cycle system* denoted by $NCCS(4K_v,\ \delta)$. Thus, we present $NCCS(4K_v,\ \delta)$ as a $(v \times |\ \delta\ |)$ array satisfying the following conditions:

- the cycles in row r and column i form a near-four-factor with focus r,
- the cycles associated with rows contain no repetitions.

The main result of this paper is the following:

Theorem 1.1. *There exists a full simple cyclic* $(m_1,\ ...,\ m_r)$*-cycle system of* $4K_v$, $NCCS(4K_v,\ \delta)$, *when* $v \equiv 2, 10(\mathrm{mod}\ 12)$.

2. Preliminaries

Throughout this paper, we use difference set method that will be clarified in this section to obtain the main results.

Let $G = K_v$, for $a, b \in V(K_v)$ and $a \neq b$, the difference d of pair $\{a, b\}$ is $|a - b|$ or $v - |a - b|$, whichever is smaller. We define the difference d of any edge $ab \in E(K_v)$ as $\min\{|a - b|, v - |a - b|\}$. So, the difference of any edge in $E(K_v)$ is not exceeding $\frac{v}{2}$, $(1 \leq d \leq \lfloor v/2 \rfloor)$. Let $C_n = (a_0, a_1, ..., a_{n-1})$ (respectively, $P_n = [a_0, a_1, ..., a_{n-1}]$) be an n-cycle (respectively, n-path) of K_v, the list of differences from C_n is a multiset $D(C_n) = \{\min\{|a_i - a_{i-1}|, v - |a_i - a_{i-1}|\} | i = 1, 2, ..., n\}$, where $a_0 = a_n$ (respectively, $D(P_n) = \{\min\{|a_i - a_{i-1}|, v - |a_i - a_{i-1}| i = 1, 2, ..., n-1\}\}$). The list difference from $\delta = \{C_{m_1}, ..., C_{m_t}\}$ is the multiset $D(C) = \bigcup_{i=1}^{t} D(C_{m_i})$.

Definition 2.1. Given a complete multigraph λK_v, when v even. A set $\delta = \{C_{m_1}, ..., C_{m_t}\}$ of cycles of λK_v is $(\lambda K_v, \delta)$-difference system if $D(\delta) = \bigcup_{i=1}^{t} D(C_i)$ covers each element of $Z^*_{\frac{v}{2}} = Z_{\frac{v}{2}} - \{0\}$ exactly λ times and the middle difference $\left(\frac{v}{2}\right)$ appears $\left\{\frac{\lambda}{2}\right\}$ times.

As a particular result of the theory developed in [5], we have:

Proposition 2.1. *A set* $\delta = \{C_1, ..., C_t\}$ *of* m_i*-cycles, where* $i = 1, 2, ..., t$ *is a starter set of a cyclic* $(m_1, ..., m_t)$*-cycle system of* $4K_v$, *if and only if* δ *is a* $(4K_v, \delta)$*-difference system.*

The orbit of cycle C_n, denoted by $orb(C_n)$, is the set of all distinct n-cycles in the collection $\{C_n + i | i \in Z_v\}$. The length of $orb(C_n)$ is its cardinality, i.e., $orb(C_n) = k$, where k is the minimum positive integer such that $C_n + k = C_n$. A cycle orbit of length v on λK_v is said to be *full* and otherwise *short*.

3. A Near Cyclic $(m_1, m_2, ..., m_r)$-cycle System

In this section, we present new definitions and results of a near cyclic $(m_1, m_2, ..., m_r)$-cycle system, that are useful for our proof.

Definition 3.1. A near cyclic $(m_1, ..., m_r)$-cycle system of $4K_v$, $NCCS(4K_v, \delta)$, combining a near-four-factor and cyclic $(m_1, ..., m_r)$-cycle system that is generated by the starter set δ. In addition, $NCCS(4K_v, \delta)$ is a $(v \times |\delta|)$ array that satisfies the following conditions:

- the cycles in row r and column i form a near-four-factor with focus r,
- the cycles associated with rows contain no repetitions.

Undoubtedly, for presenting the $NCCS(4K_v, \delta)$, it is sufficient to provide a starter set δ that satisfied a near-four-factor.

We present here some of new definitions which will be needed in the sequel.

Definition 3.2. Two m-cycles H and F of a graph G of order v are said to be *parallel* if they have the same difference set.

Definition 3.3. Let H and F be two m-cycles of a graph G of order v. If the sum of each two corresponding vertices of them is v, then it is called *adjoined m-cycles*, i.e., for $H = (h_1, h_2, ..., h_m)$ and $F = (f_1, f_2, ..., f_m)$ if $h_i + f_i = v, i = 1, ..., m,$ then H and F are adjoined cycles.

Corollary 3.1. *Any two adjoined cycles are parallel cycles.*

Throughout the paper, we shall sometimes use superscripts to identify the number of the cycles in a set. So, let us consider $\delta = \{C_{m_1}^{n_1}, C_{m_2}^{n_2}, ..., C_{m_r}^{n_r}\}$ to be the set comprised of n_i cycles of length m_i, for $i = 1, 2, ..., r$. In addition, we consider that C_{m_i} is the ith m-cycle in starter set δ. Therefore, it is convenient to provide an example here to clarify the above discussion.

Example 3.1. Let $G = 4K_{22}$ and $\delta = \{C_4^5, C_{11}^2\}$ be a set of cycles of G such that

$$C_{4_1} = (1, 21, 12, 10),\ C_{4_2} = (2, 20, 13, 9),\ C_{4_3} = (3, 19, 14, 8),$$

$$C_{4_4} = (4, 18, 7, 15),\ \ C_{4_5} = (5, 17, 16, 6),$$

$$C_{11_1} = (2, 11, 3, 10, 4, 9, 6, 8, 7, 17, 21),$$

$$C_{11_2} = (20, 11, 19, 12, 18, 13, 16, 14, 15, 5, 1).$$

Firstly, we note that each nonzero element in Z_{22} occurs twice in the cycles of δ. So every vertex has a degree 4 except zero element (isolated vertex) has degree zero. So, it satisfies the near-four-factor. Secondly, the difference sets for the cycles in δ are listed in Table 3.1 and Table 3.2 for 4-cycles and 11-cycles, respectively.

Table 3.1

4-cycle	(1, 21, 12, 10)	(2, 20, 13, 9)	(3, 19, 14, 8)	(4, 18, 7, 15)	(5, 17, 16, 6)
Difference set	{2, 9, 2, 9}	{4, 7, 4, 7}	{6, 5, 6, 5}	{8, 11, 8, 11}	{10, 1, 10, 1}

Table 3.2

11-cycle	(2, 11, 3, 10, 4, 9, 6, 8, 7, 17, 21)	(20, 11, 19, 12, 18, 13, 16, 14, 15, 5, 1)
Difference set	{9, 8, 7, 6, 5, 3, 2, 1, 10, 4, 3}	{9, 8, 7, 6, 5, 3, 2, 1, 10, 4, 3}

As clearly shown, we observe that $D(\delta) = D\left(\bigcup_{i=1}^{5} C_{4_i}\right) \cup D\left(\bigcup_{i=1}^{2} C_{11_i}\right)$ covers each element of Z_{11}^* four times while the middle difference $\frac{22}{2} = 11$ appears exactly twice. Therefore, the set $\delta = \{C_4^5, C_{11}^2\}$ is a $(4K_{22}, \delta)$-difference system. Then an $NCCS(4K_{22}, \delta)$ is (22×7) array and the starter set $\delta = \{C_4^5, C_{11}^2\}$ generates all the cycles in (22×7) array by repeated addition of 1 (mod 22) as shown in Table 3.3.

Table 3.3

Focus	$NCCS(4K_v, \delta)$				
0	1 21 12 10	2 20 13 9	3 19 14 8	…	20 11 19 12 18 13 16 14 15 5 1
1	2 0 13 11	3 21 14 10	4 20 15 9	…	21 12 20 13 19 14 17 15 16 6 2
⋮	⋮	⋮	⋮	…	⋮
20	21 19 10 8	0 18 11 7	1 17 12 6	…	18 9 17 10 16 11 14 12 13 3 21
21	0 20 11 9	1 19 12 8	2 18 13 7	…	19 10 18 11 17 12 15 13 14 4 0

As usual, any m-cycle has been written as a permutation

$$(a_{1,1}, ..., a_{1,n}, a_{2,1}, ..., a_{2,r}, a_{3,1}, ..., a_{3,l}),$$

where $n + r + l = m.$ For the sake of simplicity, it can be represented as connected paths, we mean that $C_m = (P_{1,n}, P_{2,r}, P_{3,l})$ such that $P_{1,n} = [a_{1,1}, ..., a_{1,n}],$ $P_{2,r} = [a_{2,1}, ..., a_{2,r}],$ $P_{3,l} = [a_{3,1}, ..., a_{3,l}].$

We will define the difference between any two paths H and K, denoted by $D(H, K),$ as the difference between the last vertex in the path H and the first vertex in the path K. Thus, for the cycle $C_m = (P_{1,n}, P_{2,r}, P_{3,l}),$ we find that $D(P_{1,n}, P_{2,r}) = D([a_{1,n}, a_{2,1}]),$ $D(P_{2,r}, P_{3,l}) = D([a_{2,r}, a_{3,1}])$ and $D(P_{3,l}, P_{1,n}) = D([a_{3,l}, a_{1,1}]).$ Subsequently,

$$D(C_m) = D(P_{1,n}) \cup D(P_{2,r}) \cup D(P_{3,l}) \cup D(P_{1,n}, P_{2,r})$$
$$\cup D(P_{2,r}, P_{3,l}) \cup D(P_{3,l}, P_{1,n})$$

and $V(C_m) = V(P_{1,n}) \cup V(P_{2,r}) \cup V(P_{3,l}).$

Now we are ready to present the proof for Theorem 1.1, the main aim of our paper. We distinguish two cases according to the congruence class of $v \equiv (\text{mod}\, 12).$

Case 1. There exists a full near cyclic $(m_1, ..., m_r)$-cycle system of $4K_{12n+10},$ $NCCS(4K_{12n+10}, \delta).$

Proof. We have two subcases:

Subcase 1. n is odd.

Suppose $\delta = \{C_4^{3n+2}, C_{6n+5}^2\}$ is the starter set of $4K_{12n+10}$ such that the list of 4-cycles is:

$$C_{4_i} = \bigcup_{\substack{i=1 \\ i \neq \frac{5n+3}{2}}}^{3n+2} (c_{1,i}, c_{2,i}, c_{3,i}, c_{4,i})$$

$$= \bigcup_{\substack{i=1 \\ i \neq \frac{5n+3}{2}}}^{3n+2} (i, 12n + 10 - i, 6n + 5 + i, 6n + 5 - i),$$

when $i = \dfrac{5n+3}{2}$, let

$$C_{4_i} = \left(\frac{5n+3}{2}, 12n + 10 - \frac{5n+3}{2}, 6n + 5 - \frac{5n+3}{2}, 6n + 5 + \frac{5n+3}{2}\right).$$

While we consider C_{6n+5}^{*} and C_{6n+5}^{**} that are adjoined $(6n + 5)$-cycle such that $C_{6n+5}^{*} = (P_1^{*}, P_2^{*}, P_3^{*})$, $C_{6n+5}^{**} = (P_1^{**}, P_2^{**}, P_3^{**})$, where $\{P_i^{*}, P_i^{**} \mid 1 \leq i \leq 3\}$ are paths as follows:

$P_1^{*} = [2, 6n + 5, 3, 6n + 4, ..., 2n + 2, 4n + 5]$, $P_2^{*} = [3n + 3, 3n + 5, 3n + 4]$,

$P_3^{*} = [9n + 8, 9n + 4, 9n + 9, 9n + 3, ..., 8n + 6, 10n + 7, 12n + 9]$,

$P_1^{**} = [12n + 8, 6n + 5, 12n + 7, 6n + 6, ..., 10n + 8, 8n + 5]$,

$P_2^{**} = [9n + 7, 9n + 5, 9n + 6]$,

$P_3^{**} = [3n + 2, 3n + 6, 3n + 1, 3n + 7, ..., 4n + 4, 2n + 3, 1]$.

We will divide the proof into two parts as follows:

Part 1. In this part, we prove that δ is a near-four-factor. To do this, we need to calculate the vertices

$$V\left(\bigcup_{i=1}^{3n+2} C_{4_i}\right) = c_{1,i} \cup c_{2,i} \cup c_{3,i} \cup c_{4,i},\ 1 \le i \le 3n+2$$

such that $c_{1,i} = i,\ c_{2,i} = 12n + 10 - i,\ c_{3,i} = 6n + 5 + i,\ c_{4,i} = 6n + 5 - i,$ $1 \le i \le 3n + 2,\ i \ne \frac{5n+3}{2}$. Then

$$c_{1,i} = \{1, 2, 3, ..., 3n + 2\} - \left\{\frac{5n+3}{2}\right\},$$

$$c_{2,i} = \{12n + 9, 12n + 8, ..., 9n + 8\} - \left\{\frac{19n+17}{2}\right\},$$

$$c_{3,i} = \{6n + 6, 6n + 7, ..., 9n + 7\} - \left\{\frac{17n+13}{2}\right\},$$

$$c_{4,i} = \{6n + 4, 6n + 3, ..., 3n + 3\} - \left\{\frac{7n+7}{2}\right\}.$$

While, if $i = \frac{5n+3}{2}$, then

$$V(C_{4_i}) = \left\{\frac{5n+3}{2}, \frac{19n+17}{2}, \frac{7n+7}{2}, \frac{17n+13}{2}\right\}.$$

Observe that the vertices of all 4-cycles cover every nonzero elements of $\{Z_{12n+10} - \{6n+5\}\}$ exactly once, whereas we provide the vertices of $(6n+5)$-cycles as $V(P_i^*) \cup V(P_i^{**}),\ i = 1, 2, 3$ as follows:

$$V(P_1^*) = \{2, 3, 4, ..., 2n + 2\} \cup \{6n + 5, 6n + 4, ..., 4n + 5\},$$

$$V(P_2^*) = \{3n + 3, 3n + 5, 3n + 4\},$$

$$V(P_3^*) = \{9n + 8, 9n + 9, ..., 10n + 7\}$$

$$\cup \{9n + 4, 9n + 3, ..., 8n + 6\} \cup \{12n + 9\},$$

$$V(P_1^{**}) = \{12n + 8, 12n + 7, ..., 10n + 8\} \cup \{6n + 5, 6n + 6, ..., 8n + 5\},$$

$V(P_2^{**}) = \{9n + 7,\ 9n + 5,\ 9n + 6\},$

$V(P_3^{**}) = \{3n + 2,\ 3n + 1,\ \ldots,\ 2n + 3\} \cup \{3n + 6,\ 3n + 7,\ \ldots,\ 4n + 4\} \cup \{1\}.$

Then the vertices of $(6n + 5)$-cycles cover each nonzero element of Z_{12n+10} exactly once except $\{6n + 5\}$ twice. Then the vertex set of the cycles in δ, $V(\delta)$, covers each element of Z^*_{12n+10} twice. Consequently, it satisfies near-four-factor (with isolated zero element).

Part 2. In this part, we prove that $\delta = \{C_4^{3n+2},\ C_{6n+5}^2\}$ is the $(4K_{12n+10},\ \delta)$-difference system. So, we will check the difference as follows:

$$\bigcup_{i=1}^{3n+2} D(c_{1,i},\ c_{2,i},\ c_{3,i},\ c_{4,i}) = \bigcup_{i=1}^{3n+2} D(c_{j,i},\ c_{j+1,i}),\ 1 \leq j \leq 4,$$

where $c_{5,i} = c_{1,i}$,

$$\bigcup_{\substack{i=1 \\ i \neq \frac{5n+3}{2}}}^{3n+2} D(c_{1,i},\ c_{2,i}) = \bigcup_{\substack{i=1 \\ i \neq \frac{5n+3}{2}}}^{3n+2} (2i) = \{2,\ 4,\ \ldots,\ 6n + 4\} - \{5n + 3\},$$

$$\bigcup_{\substack{i=1 \\ i \neq \frac{5n+3}{2}}}^{3n+2} D(c_{2,i},\ c_{3,i}) = \bigcup_{\substack{i=1 \\ i \neq \frac{5n+3}{2}}}^{3n+2} (6n + 5 - 2i)$$

$$= \{6n + 3,\ 6n + 1,\ \ldots,\ 3,\ 1\} - \{n + 2\},$$

$$\bigcup_{\substack{i=1 \\ i \neq \frac{5n+3}{2}}}^{3n+2} D(c_{3,i},\ c_{4,i}) = \bigcup_{\substack{i=1 \\ i \neq \frac{5n+3}{2}}}^{3n+2} (2i) = \{2,\ 4,\ \ldots,\ 6n + 4\} - \{5n + 3\},$$

$$\bigcup_{\substack{i=1 \\ i \neq \frac{5n+3}{2}}}^{3n+2} D(c_{4,i},\ c_{1,i}) = \bigcup_{\substack{i=1 \\ i \neq \frac{5n+3}{2}}}^{3n+2} (6n + 5 - 2i)$$

$$= \{6n + 3,\ 6n + 1,\ \ldots,\ 3,\ 1\} - \{n + 2\}.$$

When $i = \frac{5n+3}{2}$, then $D(C_{4_i}) = \{5n + 3, 6n + 5, 5n + 3, 6n + 5\}$.

Then the list of difference set of 4-cycles covers every element of $\{Z^*_{6n+5} - (n + 2)\} \cup \{6n + 5\}$ exactly twice. Similarly, we compute $D(C^*_{6n+5}) \cup D(C^{**}_{6n+5})$ as follows:

$D(C^*_{6n+5}) = D(P_1^*) \cup D(P_2^*) \cup D(P_3^*) \cup D(P_1^*, P_2^*) \cup D(P_2^*, P_3^*) \cup D(P_3^*, P_1^*)$,

$D(P_1^*) = \{6n + 3, 6n + 2, ..., 2n + 4, 2n + 3\}$, $D(P_2^*) = \{2, 1\}$,

$D(P_3^*) = \{4, 5, ..., 2n + 1, 2n + 2\}$,

$D(P_1^*, P_2^*) = D(4n + 5, 3n + 3) = \{n + 2\}$,

$D(P_2^*, P_3^*) = D(3n + 4, 9n + 8) = \{6n + 4\}$,

$D(P_3^*, P_1^*) = D(12n + 9, 2) = \{3\}$.

Relying on adjoined cycles C^{**}_{6n+5} and C^*_{6n+5}, we find the same difference set by Corollary 3.1. Then $D(C^*_{6n+5}) \cup D(C^{**}_{6n+5})$ covers each element of Z^*_{6n+5} exactly twice except $\{n + 2\}$ four times. From the above discussion, we deduce that $D(\delta)$ covers each element in Z^*_{6n+5} four times and the middle difference $\{6n + 5\}$ twice.

This assures that $\delta = \{C_4^{3n+2}, C^2_{6n+5}\}$ is $(4K_{12n+10}, \delta)$-difference system, n is odd. Therefore, $\delta = \{C_4^{3n+2}, C^2_{6n+1}\}$ is starter set for the $NCCS(4K_{12v+10}, \delta)$ when n is odd. □

Subcase 2. *n* is even.

Suppose $\delta = \{C_4^{3n+2}, C^2_{6n+5}\}$ is the starter set of $4K_{12n+10}$ such that the list of 4-cycles is:

$$C_{4_i} = \bigcup_{\substack{i=1 \\ i \neq \frac{n}{2}}}^{3n+2} (c_{1,i}, c_{2,i}, c_{3,i}, c_{4,i})$$

$$= \bigcup_{\substack{i=1 \\ i \neq \frac{n}{2}}}^{3n+2} (i, 12n + 10 - i, 6n + 5 + i, 6n + 5 - i).$$

When $i = \frac{n}{2}$, then $C_{4_i} = \left(\frac{n}{2}, 6n + 5 - \frac{n}{2}, 12n + 10 - \frac{n}{2}, 6n + 5 + \frac{n}{2}\right)$ whereas C^*_{6n+5} and C^{**}_{6n+5} are adjoined $(6n + 5)$-cycles such that $C^*_{6n+5} = (P^*_1, P^*_2, P^*_3)$, $C^{**}_{6n+5} = (P^{**}_1, P^{**}_2, P^{**}_3)$, where $\{P^*_i, P^{**}_i \mid 1 \leq i \leq 3\}$ are paths as follows:

$$P^*_1 = [2, 6n + 5, 3, 6n + 4, ..., 2n + 2, 4n + 5],$$

$$P^*_2 = [3n + 5, 3n + 3, 3n + 4],$$

$$P^*_3 = [9n + 8, 9n + 4, 9n + 9, 9n + 3, ..., 8n + 6, 10n + 7, 12n + 9],$$

$$P^{**}_1 = [12n + 8, 6n + 5, 12n + 7, 6n + 6, ..., 10n + 8, 8n + 5],$$

$$P^{**}_2 = [9n + 5, 9n + 7, 9n + 6],$$

$$P^{**}_3 = [3n + 2, 3n + 6, 3n + 1, 3n + 7, ..., 4n + 4, 2n + 3, 1].$$

In similar way for the Subcase 1, one may easily verify that $V(\delta) = \left(V\left(\bigcup_{i=1}^{3n+2} C_{4_i}\right) \cup V(C^*_{6n+5}) \cup V(C^{**}_{6n+5})\right)$ covers each element in Z^*_{12n+10} exactly twice. Now, we are going to calculate the difference set of 4-cycles as follows:

$$\bigcup_{\substack{i=1 \\ i \neq \frac{n}{2}}}^{3n+2} D(c_{1,i}, c_{2,i}, c_{3,i}, c_{4,i}) = \bigcup_{\substack{i=1 \\ i \neq \frac{n}{2}}}^{3n+2} D(c_{j,i}, c_{j+1,i}), 1 \leq j \leq 4,$$

where $c_{5,i} = c_{1,i}$,

$$\bigcup_{\substack{i=1\\ i\neq\frac{n}{2}}}^{3n+2} D(c_{1,i}, c_{2,i}) = \bigcup_{\substack{i=1\\ i\neq\frac{n}{2}}}^{3n+2} (2i) = \{2, 4, ..., 6n+4\} - \{n\},$$

$$\bigcup_{\substack{i=1\\ i\neq\frac{n}{2}}}^{3n+2} D(c_{2,i}, c_{3,i}) = \bigcup_{\substack{i=1\\ i\neq\frac{n}{2}}}^{3n+2} (6n+5-2i)$$

$$= \{6n+3, 6n+1, ..., 3, 1\} - \{5n+5\},$$

$$\bigcup_{\substack{i=1\\ i\neq\frac{n}{2}}}^{3n+2} D(c_{3,i}, c_{4,i}) = \bigcup_{\substack{i=1\\ i\neq\frac{n}{2}}}^{3n+2} (2i) = \{2, 4, ..., 6n+4\} - \{n\},$$

$$\bigcup_{\substack{i=1\\ i\neq\frac{n}{2}}}^{3n+2} D(c_{4,i}, c_{1,i}) = \bigcup_{\substack{i=1\\ i\neq\frac{n}{2}}}^{3n+2} (2i6n+5-2i)$$

$$= \{6n+3, 6n+1, ..., 3, 1\} - \{5n+5\}.$$

When $i = \frac{n}{2}$, $D(C_{4_i}) = \{5n+5, 6n+5, 5n+5, 6n+5\}$.

Then the list of difference set of 4-cycles covers each element of $\{Z_{6n+5}^{*} - (n)\} \cup \{6n+5\}$ exactly twice. Correspondingly, the list of difference set of $(6n+5)$-cycles calculates as follows:

$$D(C_{6n+5}^{*}) = D(P_1^{*}) \cup D(P_2^{*}) \cup D(P_3^{*}) \cup D(P_1^{*}, P_2^{*})$$
$$\cup D(P_2^{*}, P_3^{*}) \cup D(P_3^{*}, P_1^{*}),$$

$$D(P_1^{*}) = \{6n+3, 6n+2, ..., 2n+4, 2n+3\}, D(P_2^{*}) = \{2, 1\},$$

$$D(P_3^{*}) = \{4, 5, ..., 2n+1, 2n+2\}, D(P_1^{*}, P_2^{*}) = D(4n+5, 3n+5) = \{n\},$$

$$D(P_2^{*}, P_3^{*}) = D(3n+4, 9n+8) = \{6n+4\}, D(P_3^{*}, P_1^{*}) = D(12n+9, 2) = \{3\}.$$

As clearly shown, in the previous equation, the vertices of $6n+5$-cycles cover every element of Z_{6n+5}^{*} exactly twice except $\{n\}$ four times. Thus, we realize now that $\delta = \{C_4^{3n+2}, C_{6n+5}^2\}$ is $(4K_{12n+10}, \delta)$-difference system, n is even. Then $\delta = \{C_4^{3n+2}, C_{6n+5}^2\}$ is starter set for the $NCCS(4K_{12v+10}, \delta)$ when n is even. □

Case 2. There exists a full cyclic $(m_1, ..., m_r)$-cycle system of $4K_{12n+2}$, $NCCS(4K_{12n+2}, \delta)$.

Proof. We also have two subcases:

Subcase 1. n is even.

When $n = 2$, $v = 26$, let $\delta = \{C_4^6, C_7^2, C_6^2\}$ be the starter set of $NCCS(4K_{26}, \delta)$ as follows:

$C_{4_1} = (1, 25, 14, 12)$, $C_{4_2} = (2, 24, 15, 11)$, $C_{4_3} = (3, 23, 16, 10)$,

$C_{4_4} = (4, 22, 17, 9)$, $C_{4_5} = (5, 21, 18, 8)$, $C_{4_6} = (6, 19, 7, 20)$,

$C_7^* = (13, 2, 12, 3, 11, 4, 10)$, $C_7^{**} = (13, 24, 14, 23, 15, 22, 16)$,

$C_6^* = (6, 1, 5, 17, 19, 18)$, $C_6^{**} = (20, 25, 21, 9, 7, 8)$.

It is straightforward to check that δ is actually a starter set of $NCCS(4K_{26}, \delta)$.

When $n \geq 4$, suppose $\delta = \{C_4^{3n}, C_{2n+2}^2, C_{4n-1}^2\}$ is the starter set of $NCCS(4K_{12n+2}, \delta)$ such that the list of 4-cycles is:

$$C_{4_i} = \bigcup_{\substack{i=1 \\ i \neq \frac{5n+4}{2}}}^{3n} (c_{1,i}, c_{2,i}, c_{3,i}, c_{4,i})$$

$$= \bigcup_{\substack{i=1 \\ i \neq \frac{5n+4}{2}}}^{3n} (i, 12n + 2 - i, 6n + 1 + i, 6n + 1 - i),$$

when $i = \dfrac{5n+4}{2}$ let

$$C_{4_i} = \left(\frac{5n+4}{2},\ 6n+1-\frac{5n+4}{2},\ 12n+2-\frac{5n+4}{2},\ 6n+1+\frac{5n+4}{2}\right).$$

While we consider C^{*}_{4n-1} and C^{**}_{4n-1} that are adjoined $(4n-1)$-cycles such that

$$C^{*}_{4n-1} = (6n+1, 2, 6n, 3, 6n-1, 4, ..., 2n-1, 4n+3, 2n, 4n+2),$$

$$C^{**}_{4n-1} = (6n+1, 12n, 6n+2, 12n-1, 6n+3, ..., 10n+3, 8n-1, 10n+2, 8n).$$

As well, we consider that C^{*}_{2n+2} and C^{**}_{2n+2} are adjoined $(2n+2)$-cycles such that

$$C^{*}_{2n+2}$$
$$= (2n+2, 1, 2n+1, 8n+1, 10n-1, 8n+2, 10n-2, ..., 9n+2, 9n-1, 9n+1, 9n),$$

$$C^{**}_{2n+2}$$
$$= (10n, 12n+1, 10n+1, 4n+1, 2n+3, 4n, 2n+4, ..., 3n, 3n+3, 3n+1, 3n+2).$$

Similarly, it will be following the same manner of the previous case to prove that the set δ is the starter set of $4K_{12n+2}$. We will divide the proof into two parts as follows:

Part 1. In this part, we prove a near-four-factor. So, we need to calculate the vertices $V\left(\bigcup_{i=1}^{3n} C_{4_i}\right) = c_{1,i} \cup c_{2,i} \cup c_{3,i} \cup c_{4,i}, 1 \le i \le 3n$ such that

$$c_{1,i} = i,\ c_{2,i} = 12n+2-i,\ c_{3,i} = 6n+1+i,$$

$$c_{4,i} = 6n+1-i,\ 1 \le i \le 3n+2,\ i \ne \frac{5n+4}{2}.$$

$$c_{1,i} = \{1, 2, 3, ..., 3n\} - \left\{\frac{5n+4}{2}\right\},\ c_{2,i} = \{12n+1, 12n, ..., 9n+2\} - \left\{\frac{19n}{2}\right\},$$

$$c_{3,i} = \{6n+2, 6n+3, ..., 9n+1\} - \left\{\frac{17n+6}{2}\right\},$$

$c_{4,i} = \{6n, 6n-1, ..., 3n+1\} - \left\{\frac{7n-2}{2}\right\}.$

And when $i = \frac{5n+4}{2}$, then $V(C_{4_i}) = \left\{\frac{5n+4}{2}, \frac{7n-2}{2}, \frac{19n}{2}, \frac{17n+6}{2}\right\}.$

At the same time, the vertex set of remaining cycles can be written as follows:

$$V(C^{*}_{4n-1}) = \{2, 3, 4, ..., 2n\} \cup \{4n+2, 4n+3, ..., 6n+1\},$$

$$V(C^{**}_{4n-1}) = \{6n+1, 6n+2, ..., 8n\} \cup \{10n+2, 10n+3, ..., 12n\},$$

$$V(C^{*}_{2n+2}) = \{1, 2n+1, 2n+2\} \cup \{8n+1, 8n+2, 8n+3, ..., 10n-2, 10n-1\},$$

$$V(C^{**}_{2n+2}) = \{12n+1, 10n, 10n+1\} \cup \{2n+3, 2n+4, 2n+5, ..., 4n, 4n+1\}.$$

Simply we can note that $V(\delta)$ covers $\{Z^{*}_{12n+2}\}$ exactly twice.

Part 2. In this part, we prove that $\delta = \{C_4^{3n}, C_{4n-1}^{2}, C_{2n+2}^{2}\}$ is the $(4K_{12n+2}, \delta)$-difference system. So, we check the difference as follows:

The list of difference set of all 4-cycles $\left(\bigcup_{i=1}^{3n} D(C_{4_i})\right)$ is determined as follows:

$$\bigcup_{i=1}^{3n} D(C_{4_i}) = \bigcup_{i=1}^{3n} D(c_{j,i}, c_{j+1,i}),\ 1 \le j \le 4, \text{ where } c_{5,i} = c_{1,i},$$

$$\bigcup_{\substack{i=1 \\ i \neq \frac{5n+4}{2}}}^{3n} D(c_{1,i}, c_{2,i}) = \bigcup_{\substack{i=1 \\ i \neq \frac{5n+4}{2}}}^{3n} (2i) = \{2, 4, ..., 6n\} - \{5n+4\},$$

$$\bigcup_{\substack{i=1 \\ i \neq \frac{5n+4}{2}}}^{3n} D(c_{2,i}, c_{3,i}) = \bigcup_{\substack{i=1 \\ i \neq \frac{5n+4}{2}}}^{3n} (6n+1-2i)$$

$$= \{6n+3, 6n+1, ..., 3, 1\} - \{n-3\},$$

$$\bigcup_{\substack{i=1 \\ i \neq \frac{5n+4}{2}}}^{3n} D(c_{3,i}, c_{4,i}) = \bigcup_{\substack{i=1 \\ i \neq \frac{5n+4}{2}}}^{3n} (2i) = \{2, 4, ..., 6n\} - \{5n+4\},$$

$$\bigcup_{\substack{i=1 \\ i \neq \frac{5n+4}{2}}}^{3n} D(c_{4,i}, c_{1,i}) = \bigcup_{\substack{i=1 \\ i \neq \frac{5n+4}{2}}}^{3n} (6n+1-2i)$$

$$= \{6n+3, 6n+1, ..., 3, 1\} - \{n-3\}.$$

Also, when $i = \dfrac{5n+4}{2}$, $D(C_{4_i}) = \{n-3, 6n+1, n-3, 6n+1\}$.

Then the list of difference set of all 4-cycles $(D(C_4^{3n}))$ covers each element of $\{Z_{6n+1}^* - (5n+4)\} \cup \{6n+1\}$ precisely twice. Correspondingly, the list of difference set of remaining cycles $\{C_{2n+2}^*, C_{2n+2}^{**}, C_{4n-1}^*, C_{4n-1}^{**}\}$ is computed as below:

$$D(C_{4n-1}^*) = D\{(6n+1, 2, 6n, 3, 6n-1, 4, ..., 2n-1, 4n+3, 2n, 4n+2)\},$$

$$D(C_{4n-1}^{**}) = \{6n-1, 6n-2, 6n-3, ..., 2n+3, 2n+2\} \cup \{2n-1\}.$$

Since C_{4n-1}^* and C_{4n-1}^{**} are adjoined cycles in $4K_{12n+2}$, $D(C_{4n-1}^{**}) = D(C_{4n-1}^*)$.

We also have:

$$D(C_{2n+2}^*) = D\{(2n+2, 1, 2n+1, 8n+1, 10n-1, 8n+2,$$

$$10n-2, ..., 9n+2, 9n-1, 9n+1, 9n)\}$$

$$= \{2n+1, 2n, 6n, 2n-2, 2n-3, 2n-4, ..., 3, 2, 1\} \cup \{5n+4\}.$$

Since C_{2n+2}^* and C_{2n+2}^{**} are adjoined cycles in $4K_{12n+2}$, $D(C_{2n+2}^{**}) = D(C_{2n+2}^*)$.

Thus, each element in the multiset Z_{6n+1}^{*} is covered by $D(C_{4n-1}^{*}) \cup D(C_{4n-1}^{**}) \cup D(C_{2n+2}^{*}) \cup D(C_{2n+2}^{**})$ twice except $\{5n+4\}$ four times. In view of previous observation, we conclude that $\delta = \{C_4^{3n}, C_{2n+2}^{2}, C_{4n-1}^{2}\}$ is $(4K_{12n+2}, \delta)$-difference system, n is even. □

Subcase 2. n is odd.

Suppose $\delta = \{C_4^{3n}, C_{2n+2}^{2}, C_{4n-1}^{2}\}$ is the starter set of cycles of $NCCS(4K_{12n+2}, \delta)$ such that the list of 4-cycles is:

$$C_{4_i} = \bigcup_{\substack{i=1 \\ i \neq \frac{5n+1}{2}}}^{3n} (c_{1,i}, c_{2,i}, c_{3,i}, c_{4,i})$$

$$= \bigcup_{\substack{i=1 \\ i \neq \frac{5n+1}{2}}}^{3n} (i, 12n+2-i, 6n+1+i, 6n+1-i),$$

when $i = \dfrac{5n+1}{2}$, let

$$C_{4_i} = \left(\frac{5n+1}{2}, 12n+2-\frac{5n+1}{2}, 6n+1-\frac{5n+1}{2}, 6n+1+\frac{5n+1}{2}\right)$$

whereas that C_{4n-1}^{*} and C_{4n-1}^{**} are adjoined $(4n-1)$-cycles such that

$$C_{4n-1}^{*} = (6n+1, 2, 6n, 3, 6n-1, 4, \ldots, 2n-1, 4n+3, 2n, 4n+2),$$

$$C_{4n-1}^{**} = (6n+1, 12n, 6n+2, 12n-1, 6n+3, \ldots, 10n+3, 8n-1, 10n+2, 8n).$$

Also, we consider that C_{2n+2}^{*} and C_{2n+2}^{**} are adjoined $(2n+2)$-cycles such that $C_{2n+2}^{*} = (P_1^{*}, P_2^{*})$, $C_{2n+2}^{**} = (P_1^{**}, P_2^{**})$, where $\{P_i^{*}, P_i^{**} \mid 1 \leq i \leq 2\}$ are paths as follows:

$$P_1^{*} = [2n+2, 1, 10n+1],$$

$P_2^* = [4n+1, 2n+3, 4n, 2n+4, ..., 3n, 3n+3, 3n+1, 3n+2],$

$P_1^{**} = [10n, 12n+1, 2n+1],$

$P_2^{**} = [8n+1, 10n-1, 8n+2, 10n-2, ..., 9n+2, 9n-1, 9n+1, 9n].$

Obviously, as the Subcase 1, it can be found that $V(\delta)$ covers each element of Z_{12n+2}^* exactly twice and the list of difference set of all 4-cycles $(D(C_4^{3n}))$ covers each element of $\{Z_{6n+1}^* - n\}$ precisely twice, whereas the difference set of $(4n-1)$-cycles $(D(C_{4n-1}^*) \cup D(C_{4n-1}^{**}))$ contains elements $\{6n-1, 6n-2, 6n-3, ..., 2n+3, 2n+2\} \cup \{2n-1\}$ twice. Now, we calculate the difference set of $(2n+2)$-cycles as follows:

$D(C_{2n+2}^*) = D(P_1^*) \cup D(P_2^*) \cup D(P_1^*, P_2^*) \cup D(P_2^*, P_1^*),$

$D(P_1^*) = \{2n+1, 2n\}, D(P_2^*) = \{2n-2, 2n-3, 2n-4, ..., 3, 2, 1\},$

$D(P_1^*, P_2^*) = D(10n+1, 4n+1) = \{6n\}, D(P_2^*, P_1^*) = D(2n+2, 3n+2) = \{n\}.$

Then all elements in the set $\{1, 2, 3, ..., 2n-3, 2n-2, 2n, 2n+1, 6n\}$ appear in $D(C_{2n+2}^*)$ exactly once except $\{n\}$ twice. Therefore, the multiset of $D(C_{4n-1}^*) \cup D(C_{4n-1}^{**}) \cup D(C_{2n+2}^*) \cup D(C_{2n+2}^{**})$ covers each element of $\{Z_{6n+1}^*\}$ exactly twice except $\{n\}$ four times.

Hence, $\delta = \{C_4^{3n}, C_{2n+2}^2, C_{4n-1}^2\}$ is $(4K_{12n+2}, \delta)$-difference system, n is odd. Then $\delta = \{C_4^{3n}, C_{2n+2}^2, C_{4n-1}^2\}$ is starter set of $NCCS(4K_{12n+2}, \delta)$.

□

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