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PERTURBATIONS OF NONLINEAR PERTURBED DIFFERENTIAL SYSTEMS

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Abstract

In this paper, we show that the solutions to the nonlinear perturbed differential system

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s), T_1 y(s))\, ds + h(t, y(t), T_2 y(t)),$$

have boundedness, uniformly Lipschitz stability, and asymptotic behavior by imposing conditions on the perturbed part $\int_{t_0}^{t} g(s, y(s), T_1 y(s))\, ds,\ h(t, y(t), T_2 y(t)),$ and on the fundamental

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matrix of the unperturbed system $y' = f(t, y)$ using the notion of h-stability.

1. Introduction and Preliminaries

We consider the unperturbed nonlinear nonautonomous differential system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{1.1}$$

and the perturbed differential system of (1.1) including an operator T such that

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s), T_1 y(s))ds + h(t, y(t), T_2 y(t)), \; y(t_0) = y_0, \tag{1.2}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $g, h \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$, $f(t, 0) = 0$, $g(t, 0, 0) = h(t, 0, 0) = 0$, and $T_1, T_2 : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ are continuous operators and $\mathbb{R}^n$ is an n-dimensional Euclidean space. We always assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$. The symbol $|\cdot|$ will be used to denote any convenient vector norm in $\mathbb{R}^n$.

Let $x(t, t_0, x_0)$ denote the unique solution of (1.1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (1.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t), \quad v(t_0) = v_0 \tag{1.3}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \quad z(t_0) = z_0. \tag{1.4}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (1.4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (1.3).

We give some of the main definitions that we need in the sequel [10].

Definition 1.1. The system (1.1) (the zero solution $x = 0$ of (1.1)) is called (S) *stable* if for any $\varepsilon > 0$ and $t_0 \geq 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that if $|x_0| < \delta$, then $|x(t)| < \varepsilon$ for all $t \geq t_0 \geq 0$,

(US) *uniformly stable* if the δ in (S) is independent of the time t_0,

(ULS) *uniformly Lipschitz stable* if there exist $M > 0$ and $\delta > 0$ such that $|x(t)| \leq M|x_0|$ whenever $|x_0| \leq \delta$ and $t \geq t_0 \geq 0$,

(ULSV) *uniformly Lipschitz stable in variation* if there exist $M > 0$ and $\delta > 0$ such that $|\Phi(t, t_0, x_0)| \leq M$ for $|x_0| \leq \delta$ and $t \geq t_0 \geq 0$,

(AS) *asymptotically stable* if it is stable and if there exists $\delta = \delta(t_0) > 0$ such that if $|x_0| < \delta$, then $|x(t)| \to 0$ as $t \to \infty$,

(EAS) *exponentially asymptotically stable* if there exist constants $K > 0$, $c > 0$, and $\delta > 0$ such that

$$|x(t)| \leq K|x_0|e^{-c(t-t_0)}, \quad 0 \leq t_0 \leq t$$

provided that $|x_0| < \delta$,

(EASV) *exponentially asymptotically stable in variation* if there exist constants $K > 0$ and $c > 0$ such that

$$|\Phi(t, t_0, x_0)| \leq Ke^{-c(t-t_0)}, \quad 0 \leq t_0 \leq t$$

provided that $|x_0| < \infty$.

Remark 1.2 [12]**.** The last definition implies that for $|x_0| \leq \delta$

$$|x(t)| \leq K|x_0|e^{-c(t-t_0)}, \quad 0 \leq t_0 \leq t.$$

We recall some notions of *h*-stability [21].

Definition 1.3. The system (1.1) (the zero solution $x = 0$ of (1.1)) is called an *h-system* if there exist a constant $c \geq 1$, and a positive continuous function *h* on $\mathbb{R}^+$ such that

$$|x(t)| \le c|x_0|h(t)h(t_0)^{-1}$$

for $t \ge t_0 \ge 0$ and $|x_0|$ small enough $\left(\text{here } h(t)^{-1} = \dfrac{1}{h(t)}\right)$.

Definition 1.4. The system (1.1) (the zero solution $x = 0$ of (1.1)) is called (hS) *h-stable* if there exists $\delta > 0$ such that (1.1) is an h-system for $|x_0| \le \delta$ and h is bounded.

Let $\mathcal{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞-similarity in $\mathcal{M}$ was introduced by Conti [9].

Definition 1.5. A matrix $A(t) \in \mathcal{M}$ is t_∞-similar to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)|dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \tag{1.5}$$

for some $S(t) \in \mathcal{N}$.

The notion of t_∞-similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on $\mathbb{R}^+$, and it preserves some stability concepts [9, 15].

Pinto [21, 22] introduced the notion of h-stability (hS) with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. That is, Pinto extended the study of exponential asymptotic stability to a variety of reasonable systems called *h-systems*. The new notion of uniformly Lipschitz stability (ULS) was introduced by Dannan and Elaydi [10]. This notion of ULS lies somewhere between uniformly stability on one side and the notions of asymptotic stability in variation of Brauer [4] and uniformly stability in variation of Brauer and Strauss [3] on the other side. An important feature of ULS is that for linear systems, the notion of uniformly Lipschitz stability and that of uniformly stability are equivalent. However, for nonlinear systems, the two notions are quite distinct and Choi and Ryu [7] and Choi et al. [8] investigated bounds of solutions for nonlinear perturbed systems. Also, Goo [13] and Choi and Goo [5] studied the boundedness of solutions for the perturbed differential systems. Goo [14] and Goo et al. [6, 16, 17] investigated uniform Lipschity stability and asymptotic property of perturbed nonlinear systems. Elaydi and Farran [11] introduced the notion of exponential asymptotic stability (EAS) which is a stronger notion than that of ULS. They investigated some analytic criteria for an autonomous differential system and its perturbed systems to be EAS. Pachpatte [19, 20] investigated the stability, boundedness, and the asymptotic behavior of the solutions of perturbed nonlinear systems under some suitable conditions on the perturbation term g and on the operator T.

In this paper, we investigate bounds, ULS, and asymptotic behavior for solutions of the nonlinear differential systems further allowing more general perturbations that were previously allowed using the notion of h-stability.

We give some related properties that we need in the sequel.

Lemma 1.6 [22]**.** *The linear system*

$$x' = A(t)x, \quad x(t_0) = x_0, \tag{1.6}$$

where $A(t)$ *is an* $n \times n$ *continuous matrix, is an* h*-system* (*respectively,* h*-stable*) *if and only if there exist* $c \geq 1$ *and a positive and continuous* (*respectively bounded*) *function* h *defined on* $\mathbb{R}^+$ *such that*

$$|\phi(t, t_0)| \leq ch(t)h(t_0)^{-1} \tag{1.7}$$

for $t \geq t_0 \geq 0,$ *where* $\phi(t, t_0)$ *is a fundamental matrix of* (1.6).

We need Alekseev formula to compare between the solutions of (1.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y), \quad y(t_0) = y_0, \tag{1.8}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (1.8) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 1.7 [2]**.** *Let x and y be a solution of* (1.1) *and* (1.8), *respectively*. *If* $y_0 \in \mathbb{R}^n$, *then for all* $t \geq t_0$ *such that* $x(t, t_0, y_0) \in \mathbb{R}^n$, $y(t, t_0, y_0) \in \mathbb{R}^n$,

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s)) g(s, y(s)) ds.$$

Theorem 1.8 [7]**.** *If the zero solution of* (1.1) *is hS, then the zero solution of* (1.3) *is hS*.

Theorem 1.9 [8]**.** *Suppose that* $f_x(t, 0)$ *is* t_∞*-similar to* $f_x(t, x(t, t_0, x_0))$ *for* $t \geq t_0 \geq 0$ *and* $|x_0| \leq \delta$ *for some constant* $\delta > 0$. *If the solution* $v = 0$ *of* (1.3) *is hS, then the solution* $z = 0$ *of* (1.4) *is hS.*

Lemma 1.10 (*Bihari-type inequality*)**.** *Let* $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ *and* $w(u)$ *be nondecreasing in u. Suppose that, for some* $c > 0$,

$$u(t) \leq c + \int_{t_0}^{t} \lambda(s) w(u(s)) ds, \quad t \geq t_0 \geq 0.$$

Then

$$u(t) \leq W^{-1}\left[W(c) + \int_{t_0}^{t} \lambda(s) ds\right], \quad t_0 \leq t < b_1,$$

where $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $W^{-1}(u)$ *is the inverse of* $W(u)$ *and*

$$b_1 = \sup\left\{t \geq t_0 : W(c) + \int_{t_0}^{t} \lambda(s)ds \in \operatorname{dom} W^{-1}\right\}.$$

Lemma 1.11 [5]. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ *and* $w(u)$ *be nondecreasing in* u, $u \leq w(u)$. *Suppose that for some* $c > 0$,

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s)w(u(s))ds$$

$$+ \int_{t_0}^{t} \lambda_3(s)\int_{t_0}^{s} \lambda_4(\tau)u(\tau)d\tau ds$$

$$+ \int_{t_0}^{t} \lambda_5(s)\int_{t_0}^{s} \lambda_6(\tau)w(u(\tau))d\tau ds, \quad 0 \leq t_0 \leq t.$$

Then

$$u(t) \leq W^{-1}\Bigg[W(c) + \int_{t_0}^{t} \Bigg(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s} \lambda_4(\tau)d\tau$$

$$+ \lambda_5(s)\int_{t_0}^{s} \lambda_6(\tau)d\tau\Bigg)ds\Bigg],$$

where $t_0 \leq t < b_1$, W, W^{-1} *are the same functions as in Lemma* 1.10, *and*

$$b_1 = \sup\Bigg\{t \geq t_0 : W(c) + \int_{t_0}^{t} \Bigg(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s} \lambda_4(\tau)d\tau$$

$$+ \lambda_5(s)\int_{t_0}^{s} \lambda_6(\tau)d\tau\Bigg)ds \in \operatorname{dom} W^{-1}\Bigg\}.$$

Lemma 1.12 [6]. *Let* u, λ_1, λ_2, λ_3, λ_4, λ_5, λ_6, λ_7, λ_8, λ_9, $\lambda_{10} \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, *and* $w(u)$ *be nondecreasing in* u, $u \leq w(u)$. *Suppose that for some* $c > 0$ *and* $0 \leq t_0 \leq t$,

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s)w(u(s))ds$$

$$+ \int_{t_0}^{t} \lambda_3(s)\int_{t_0}^{s}\Big(\lambda_4(\tau)u(\tau) + \lambda_5(\tau)\int_{t_0}^{\tau} \lambda_6(r)u(r)dr$$

$$+ \lambda_7(\tau)\int_{t_0}^{\tau} \lambda_8(r)w(u(r))dr\Big)d\tau ds + \int_{t_0}^{t} \lambda_9(s)\int_{t_0}^{s} \lambda_{10}(\tau)w(u(\tau))d\tau ds.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t}\Big(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s}\Big(\lambda_4(\tau) + \lambda_5(\tau)\int_{t_0}^{\tau}\lambda_6(r)dr$$

$$+ \lambda_7(\tau)\int_{t_0}^{\tau}\lambda_8(r)dr\Big)d\tau + \lambda_9(s)\int_{t_0}^{s}\lambda_{10}(\tau)d\tau\Big)ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma* 1.10, *and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t}\Big(\lambda_1(s) + \lambda_2(s)$$

$$+ \lambda_3(s)\int_{t_0}^{s}\Big(\lambda_4(\tau) + \lambda_5(\tau)\int_{t_0}^{\tau}\lambda_6(r)dr + \lambda_7(\tau)\int_{t_0}^{\tau}\lambda_8(r)dr\Big)d\tau$$

$$+ \lambda_9(s)\int_{t_0}^{s}\lambda_{10}(\tau)d\tau\Big)ds \in \operatorname{dom} W^{-1}\Big\}.$$

Corollary 1.13. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, *and* $w(u)$ *be nondecreasing in* $u, u \le w(u)$. *Suppose that for some* $c > 0$ *and* $0 \le t_0 \le t$,

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s)w(u(s))ds + \int_{t_0}^{t} \lambda_2(s)\int_{t_0}^{s}\Big(\lambda_3(\tau)u(\tau)$$

$$+ \lambda_4(\tau)\int_{t_0}^{\tau}\lambda_5(r)u(r)dr + \lambda_6(\tau)\int_{t_0}^{\tau}\lambda_7(r)w(u(r))dr\Big)d\tau ds$$

$$+ \int_{t_0}^{t}\lambda_8(s)\int_{t_0}^{s}\lambda_9(\tau)w(u(\tau))d\tau ds.$$

Then

$$u(t) \le W^{-1}\Bigg[W(c) + \int_{t_0}^{t}\Bigg(\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s}\Bigg(\lambda_3(\tau) + \lambda_4(\tau)\int_{t_0}^{\tau}\lambda_5(r)dr$$

$$+ \lambda_6(\tau)\int_{t_0}^{\tau}\lambda_7(r)dr\Bigg)d\tau + \lambda_8(s)\int_{t_0}^{s}\lambda_9(\tau)d\tau\Bigg)ds\Bigg],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma* 1.10, *and*

$$b_1 = \sup\Bigg\{t \ge t_0 : W(c) + \int_{t_0}^{t}\Bigg(\lambda_1(s)$$

$$+ \lambda_2(s)\int_{t_0}^{s}\Bigg(\lambda_3(\tau) + \lambda_4(\tau)\int_{t_0}^{\tau}\lambda_5(r)dr + \lambda_6(\tau)\int_{t_0}^{\tau}\lambda_7(r)dr\Bigg)d\tau$$

$$+ \lambda_8(s)\int_{t_0}^{s}\lambda_9(\tau)d\tau\Bigg)ds \in \operatorname{dom} W^{-1}\Bigg\}.$$

2. Main Results

In this section, we investigate boundedness, ULS, and asymptotic behavior for solutions of perturbed functional differential systems using the notion of t_∞-similarity.

To obtain these properties, the following assumptions are needed:

(H1) $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \ge t_0 \ge 0$ and $|x_0| \le \delta$ for some constant $\delta > 0$.

(H2) The solution $x = 0$ of (1.1) is hS with the increasing function *h*.

(H3) $w(u)$ be nondecreasing in u such that $u \le w(u)$ and $\frac{1}{v}w(u) \le w\left(\frac{u}{v}\right)$ for some $v > 0$.

(H4) The solution $x = 0$ of (2.1) is ULSV.

(H5) The solution $x = 0$ of (2.1) is EASV.

Lemma 2.1. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{10} \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$ *and* $0 \le t_0 \le t$,

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s)\int_{t_0}^{s}\Big(\lambda_3(\tau)u(\tau) + \lambda_4(\tau)w(u(\tau))$$

$$+ \lambda_5(\tau)\int_{t_0}^{\tau} \lambda_6(r)u(r)dr + \lambda_7(\tau)\int_{t_0}^{\tau} \lambda_8(r)w(u(r))dr\Big)d\tau ds$$

$$+ \int_{t_0}^{t} \lambda_9(s)\int_{t_0}^{s} \lambda_{10}(\tau)w(u(\tau))d\tau ds. \tag{2.1}$$

Then, we have

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t}\Big(\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s}\Big(\lambda_3(\tau) + \lambda_4(\tau) + \lambda_5(\tau)\int_{t_0}^{\tau}\lambda_6(r)dr$$

$$+ \lambda_7(\tau)\int_{t_0}^{\tau}\lambda_8(r)dr\Big)d\tau + \lambda_9(s)\int_{t_0}^{s}\lambda_{10}(\tau)d\tau\Big)ds\Big], \tag{2.2}$$

where $t_0 \le t \le b_1$, W, W^{-1} *are the same functions as in Lemma* 1.10, *and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t}\Big(\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s}\Big(\lambda_3(\tau) + \lambda_4(\tau)$$

$$+ \lambda_5(\tau)\int_{t_0}^{\tau}\lambda_6(r)dr + \lambda_7(\tau)\int_{t_0}^{\tau}\lambda_8(r)dr\Big)d\tau$$

$$+ \lambda_9(s)\int_{t_0}^{s}\lambda_{10}(\tau)d\tau\Big)ds \in \operatorname{dom} W^{-1}\Big\}.$$

Proof. Define a function $v(t)$ by the right member of (2.1) and let us differentiate $v(t)$. Then, we have

$$v'(t) = \lambda_1(t)u(t) + \lambda_2(t)\int_{t_0}^{t}\Big(\lambda_3(s)u(s) + \lambda_4(s)w(u(s))$$

$$+ \lambda_5(s)\int_{t_0}^{s} \lambda_6(\tau)u(\tau)d\tau + \lambda_7(s)\int_{t_0}^{s} \lambda_8(\tau)w(u(\tau))d\tau\Bigg)ds$$

$$+ \lambda_9(t)\int_{t_0}^{t} \lambda_{10}(s)w(u(s))ds.$$

This reduces to

$$v'(t) \le \Bigg[\lambda_1(t) + \lambda_2(t)\int_{t_0}^{t}\Bigg(\lambda_3(s) + \lambda_4(s) + \lambda_5(s)\int_{t_0}^{s} \lambda_6(\tau)d\tau$$

$$+ \lambda_7(s)\int_{t_0}^{s} \lambda_8(\tau)d\tau\Bigg)ds + \lambda_9(t)\int_{t_0}^{t} \lambda_{10}(s)ds\Bigg]w(v(t)),$$

$t \ge t_0$, since $v(t)$ is nondecreasing, $u \le w(u)$, and $u(t) \le v(t)$. Now, by integrating the above inequality on $[t_0, t]$ and using $v(t_0) = c$, we obtain

$$v(t) \le c + \int_{t_0}^{t}\Bigg(\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s}\Bigg(\lambda_3(\tau) + \lambda_4(\tau) + \lambda_5(\tau)\int_{t_0}^{\tau} \lambda_6(r)dr$$

$$+ \lambda_7(\tau)\int_{t_0}^{\tau} \lambda_8(r)dr\Bigg)d\tau + \lambda_9(s)\int_{t_0}^{s} \lambda_{10}(\tau)d\tau\Bigg)w(v(s))ds. \tag{2.3}$$

It follows from Lemma 1.10 that (2.3) yields the estimate (2.2). □

Theorem 2.2. *Let* $a, b, c, d, k, p, q \in C(\mathbb{R}^+)$. *Suppose that* (H1), (H2), (H3), *and* g *in* (1.2) *satisfy*

$$|g(t, y, T_1y)| \le a(t)|y(t)| + b(t)w(|y(t)|) + |T_1y(t)|, \tag{2.4}$$

$$|T_1y(t)| \le b(t)\int_{t_0}^{t} k(s)|y(s)|ds + d(t)\int_{t_0}^{t} p(s)w(|y(s)|)ds, \tag{2.5}$$

and

$$|h(t, y(t), T_2y(t))|$$

$$\le c(t)(|y(t)| + |T_2y(t)|), \; |T_2y(t)| \le \int_{t_0}^{t} q(s)w(|y(s)|)ds, \tag{2.6}$$

where $a, b, c, d, k, p, q \in L^1(\mathbb{R}^+)$, $w \in C((0, \infty))$, T_1, T_2 *are continuous operators. Then, any solution* $y(t) = y(t, t_0, y_0)$ *of* (1.2) *is bounded on* $[t_0, \infty)$ *and it satisfies*

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}\Big(c(s) + \int_{t_0}^{s}\Big(a(\tau) + b(\tau) + b(\tau)\int_{t_0}^{\tau}k(r)dr + d(\tau)\int_{t_0}^{\tau}p(r)dr\Big)d\tau + c(s)\int_{t_0}^{\tau}q(r)dr\Big)ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma* 1.10, *and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2\int_{t_0}^{t}\Big(c(s) + \int_{t_0}^{s}\Big(a(\tau) + b(\tau) + b(\tau)\int_{t_0}^{\tau}k(r)dr + d(\tau)\int_{t_0}^{\tau}p(r)dr\Big)d\tau + c(s)\int_{t_0}^{\tau}q(r)dr\Big)ds \in \operatorname{dom} W^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By Theorem 1.8, since the solution $x = 0$ of (1.1) is hS, the solution $v = 0$ of (1.3) is hS. Therefore, from (H1), by Theorem 1.9, the solution $z = 0$ of (1.4) is hS. Using the nonlinear variation of constants formula due to Lemma 1.7, together with (2.4), (2.5), and (2.6), we have

$$\begin{aligned}
|y(t)| &\le |x(t)| + \int_{t_0}^{t}|\Phi(t, s, y(s))|\Big(\int_{t_0}^{s}|g(\tau, y(\tau), T_1y(\tau))|d\tau + |h(s, y(s), T_2y(s))|\Big)ds \\
&\le c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^{t}c_2h(t)h(s)^{-1}\Big(\int_{t_0}^{s}\Big(a(\tau)|y(\tau)| + b(\tau)w(|y(\tau)|) \\
&\quad + b(\tau)\int_{t_0}^{\tau}k(r)|y(r)|dr + d(\tau)\int_{t_0}^{\tau}p(r)w(|y(r)|)dr\Big)d\tau \\
&\quad + c(s)\Big(|y(s)| + \int_{t_0}^{s}q(\tau)w(|y(\tau)|)d\tau\Big)\Big)ds.
\end{aligned}$$

It follows from (H2) and (H3) that

$$|y(t)| \le c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t) \left(c(s) \frac{|y(s)|}{h(s)} \right.$$

$$+ \int_{t_0}^{s} \left(a(\tau) \frac{|y(\tau)|}{h(\tau)} + b(\tau) w\left(\frac{|y(\tau)|}{h(\tau)} \right) + b(\tau) \int_{t_0}^{\tau} k(r) \frac{|y(r)|}{h(r)} dr \right.$$

$$\left. \left. + d(\tau) \int_{t_0}^{\tau} p(r) w\left(\frac{|y(r)|}{h(r)} \right) dr \right) d\tau + c(s) \int_{t_0}^{s} q(\tau) w\left(\frac{|y(\tau)|}{h(\tau)} \right) d\tau \right) ds.$$

Define $u(t) = |y(t)| |h(t)|^{-1}$. Then, by Lemma 2.1, we have

$$|y(t)| \le h(t) W^{-1} \left[W(c) + c_2 \int_{t_0}^{t} \left(c(s) + \int_{t_0}^{s} \left(a(\tau) + b(\tau) + b(\tau) \int_{t_0}^{\tau} k(r) dr \right. \right. \right.$$

$$\left. \left. \left. + d(\tau) \int_{t_0}^{\tau} p(r) dr \right) d\tau + c(s) \int_{t_0}^{\tau} q(\tau) d\tau \right) ds \right],$$

where $c = c_1 |y_0| h(t_0)^{-1}$. The above estimation yields the desired result since the function h is bounded. Thus, the proof is complete. □

Remark 2.3. Letting $c(t) = d(t) = 0$ in Theorem 2.2, we obtain the same result as that of Theorem 3.1 in [13].

Theorem 2.4. *Let* $a, b, c, d, k, p, q \in C(\mathbb{R}^+)$. *Suppose that* (H1), (H2), (H3), *and* g *in* (1.2) *satisfy*

$$\int_{t_0}^{t} |g(s, y(s), T_1 y(s))| ds \le a(t) |y(t)| + b(t) w(|y(t)|) + |T_1 y(t)|, \tag{2.7}$$

$$|T_1 y(t)| \le b(t) \int_{t_0}^{t} k(s) w(|y(s)|) ds + d(t) \int_{t_0}^{t} p(s) |y(s)| ds, \tag{2.8}$$

and

$$|h(t, y(t), T_2 y(t))| \le (c(t) w(|y(t)|) + |T_2 y(t)|), \ |T_2 y(t)|$$

$$\le d(t) \int_{t_0}^{t} q(s) |y(s)| ds, \tag{2.9}$$

where $a, b, c, d, k, p, q \in L^1(\mathbb{R}^+)$, $w \in C((0, \infty))$, T_1, T_2 *are continuous operators. Then, any solution* $y(t) = y(t, t_0, y_0)$ *of* (1.2) *is bounded on* $[t_0, \infty)$ *and it satisfies*

$$|y(t)| \le h(t)W^{-1}\bigg[W(c) + c_2\int_{t_0}^{t}\bigg(a(s) + b(s) + c(s)$$

$$+ b(s)\int_{t_0}^{s} k(\tau)d\tau + d(s)\int_{t_0}^{s}(p(\tau) + q(\tau))d\tau\bigg)ds\bigg],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma* 1.10, *and*

$$b_1 = \sup\bigg\{t \ge t_0 : W(c) + c_2\int_{t_0}^{t}\bigg(a(s) + b(s) + c(s)$$

$$+ b(s)\int_{t_0}^{s} k(\tau)d\tau + d(s)\int_{t_0}^{s}(p(\tau) + q(\tau))d\tau\bigg)ds \in \operatorname{dom} W^{-1}\bigg\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the same argument as in the proof in Theorem 2.2, the solution $z = 0$ of (1.4) is hS. Applying the nonlinear variation of constants formula due to Lemma 1.7, together with (2.7), (2.8), and (2.9), we have

$$|y(t)| \le c_1|y_0|h(t)h(t_0)^{-1}$$

$$+ \int_{t_0}^{t} c_2 h(t)h(s)^{-1}\bigg(a(s)|y(s)| + (b(s) + c(s))w(|y(s)|)$$

$$+ b(s)\int_{t_0}^{s} k(\tau)w(|y(\tau)|)d\tau + d(s)\int_{t_0}^{s}(p(\tau) + q(\tau))|y(\tau)|d\tau\bigg)ds.$$

Using (H2) and (H3), we obtain

$$|y(t)| \le c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)\bigg(a(s)\frac{|y(s)|}{h(s)} + (b(s) + c(s))w\bigg(\frac{|y(s)|}{h(s)}\bigg)$$

$$+ b(s)\int_{t_0}^{s} k(\tau)w\bigg(\frac{|y(\tau)|}{h(\tau)}\bigg)d\tau + d(s)\int_{t_0}^{s}(p(\tau) + q(\tau))\frac{|y(\tau)|}{h(\tau)}d\tau\bigg)ds.$$

Let $u(t) = |y(t)||h(t)|^{-1}$. Then, by Lemma 1.11, we have

$$|y(t)| \le h(t)W^{-1}\left[W(c) + c_2\int_{t_0}^{t}\left(a(s) + b(s) + c(s) + b(s)\int_{t_0}^{s}k(\tau)d\tau\right.\right.$$

$$\left.\left. + d(s)\int_{t_0}^{s}(p(\tau) + q(\tau))d\tau\right)ds\right],$$

where $c = c_1|y_0|h(t_0)^{-1}$. Thus, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$, and so the proof is complete. □

Remark 2.5. Letting $c(t) = d(t) = 0$ in Theorem 2.4, we obtain the same result as that of Theorem 3.7 in [13].

Theorem 2.6. *Suppose that* (H3), (H4), *and that the perturbing term g in* (1.2) *satisfy*

$$|g(t, y, T_1y)| \le a(t)|y(t)| + b(t)w(|y(t)|) + |T_1y(t)|, \tag{2.10}$$

$$|T_1y(t)| \le b(t)\int_{t_0}^{t}k(s)|y(s)|ds + p(t)\int_{t_0}^{t}q(s)w(|y(s)|)ds, \tag{2.11}$$

and

$$|h(t, y(t), T_2y(t))| \le c(t)w(|y(t)|) + |T_2y(t)|, |T_2y(t)| \le d(t)|y(t)|, \tag{2.12}$$

where $a, b, c, d, k, p, q \in C(\mathbb{R}^+)$, $a, b, c, d, k, p, q \in L^1(\mathbb{R}^+)$, $w \in C((0, \infty))$, T_1, T_2 *are continuous operators, and*

$$M(t_0) = W^{-1}\left[W(M) + M\int_{t_0}^{\infty}\left(c(s) + d(s) + \int_{t_0}^{s}\left(a(\tau) + b(\tau) + b(\tau)\int_{t_0}^{\tau}k(r)dr\right.\right.\right.$$

$$\left.\left.\left. + p(\tau)\int_{t_0}^{\tau}q(r)dr\right)d\tau\right)ds\right], \tag{2.13}$$

where $M(t_0) < \infty$ *and* $b_1 = \infty$. *Then the zero solution of* (1.2) *is ULS.*

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the assumption (H4), it is ULS ([10], Theorem 3.3). Using (H3), together with (2.10), (2.11), and (2.12), we have

$$|y(t)| \le M|y_0| + \int_{t_0}^{t} M|y_0|\left(\int_{t_0}^{s}\left(a(\tau)\frac{|y(\tau)|}{|y_0|} + b(\tau)w\left(\frac{|y(\tau)|}{|y_0|}\right)\right.\right.$$
$$\left. + b(\tau)\int_{t_0}^{\tau} k(r)\frac{|y(r)|}{|y_0|}dr + p(\tau)\int_{t_0}^{\tau} q(r)w\left(\frac{|y(r)|}{|y_0|}\right)dr\right)d\tau$$
$$\left. + c(s)w\left(\frac{|y(s)|}{|y_0|}\right) + d(s)\frac{|y(s)|}{|y_0|}\right)ds.$$

Let $u(t) = |y(t)||y_0|^{-1}$. Now an application of Lemma 1.12 yields

$$|y(t)| \le |y_0|W^{-1}\left[W(M) + M\int_{t_0}^{t}\left(c(s) + d(s) + \int_{t_0}^{s}\left(a(\tau) + b(\tau)\right.\right.\right.$$
$$\left.\left.\left. + b(\tau)\int_{t_0}^{\tau} k(r)dr + p(\tau)\int_{t_0}^{\tau} q(r)dr\right)d\tau\right)ds\right].$$

Thus, by (2.13), we have $|y(t)| \le M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$. This completes the proof. □

Remark 2.7. Letting $c(t) = d(t) = p(t) = 0$ in Theorem 2.6, we obtain the same result as that of Theorem 3.1 in [16].

Theorem 2.8. *Suppose that* (H3), (H4), *and that the perturbing term g in* (1.2) *satisfy*

$$\int_{t_0}^{t} |g(s, y(s), T_1y(s))|ds \le a(t)|y(t)| + b(t)w(|y(t)|) + |T_1y(t)|, \quad (2.14)$$

$$|T_1y(t)| \le b(t)\int_{t_0}^{t} k(s)|y(s)|ds + d(t)\int_{t_0}^{t} p(s)w(|y(s)|)ds, \quad (2.15)$$

and

$$| h(t,\ y(t),\ T_2 y(t)) |$$

$$\leq c(t) | y(t) | + | T_2 y(t) |, \ | T_2 y(t) | \leq d(t) \int_{t_0}^{t} q(s) w(| y(s) |) ds, \qquad (2.16)$$

where $a, b, c, d, k, p, q \in C(\mathbb{R}^+)$, $a, b, c, d, k, p, q \in L^1(\mathbb{R}^+)$, $w \in C((0, \infty))$, T_1, T_2 *are continuous operators, and*

$$M(t_0) = W^{-1}\left[W(M) + M\int_{t_0}^{\infty}\left(a(s) + b(s) + c(s) + b(s)\int_{t_0}^{s} k(\tau) d\tau\right.\right.$$

$$\left.\left. + d(s)\int_{t_0}^{s}(p(\tau) + q(\tau)) d\tau\right) ds\right], \qquad (2.17)$$

where $M(t_0) < \infty$ *and* $b_1 = \infty$. *Then the zero solution of* (1.2) *is ULS.*

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the assumption (H4), it is ULS. Applying (H3), together with (2.14), (2.15), and (2.16), we have

$$| y(t) | \leq M | y_0 | + \int_{t_0}^{t} M | y_0 | \left((a(s) + c(s)) \frac{| y(s) |}{| y_0 |} + b(s) w\left(\frac{| y(s) |}{| y_0 |} \right) \right) ds$$

$$+ \int_{t_0}^{t} M | y_0 | \left(b(s) \int_{t_0}^{s} k(\tau) \frac{| y(\tau) |}{| y_0 |} d\tau \right.$$

$$\left. + d(s) \int_{t_0}^{s} (p(\tau) + q(\tau)) w\left(\frac{| y(\tau) |}{| y_0 |} \right) d\tau \right) ds.$$

Defining $u(t) = | y(t) | | y_0 |^{-1}$, then it follows from Lemma 1.11 that

$$| y(t) | \leq | y_0 | W^{-1}\left[W(M) + M \int_{t_0}^{t} \left(a(s) + b(s) + c(s) + b(s) \int_{t_0}^{s} k(\tau) d\tau \right.\right.$$

$$\left.\left. + d(s) \int_{t_0}^{s} (p(\tau) + q(\tau)) d\tau \right) ds \right].$$

Hence, by (2.17), we have $|y(t)| \le M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$. Thus, the proof is complete. $\square$

Remark 2.9. Letting $c(t) = d(t) = q(t) = 0$ in Theorem 2.8, we obtain the same result as that of Theorem 3.3 in [16].

Theorem 2.10. *Suppose that* (H3), (H5), *and that the perturbing term* $g(t, y, T_1 y)$ *satisfy*

$$|g(t, y(t), T_1 y(t))| \le e^{-\alpha t}(a(t)|y(t)| + b(t)w(|y(t)|) + |T_1 y(t)|), \quad (2.18)$$

$$|T_1 y(t)| \le c(t)\int_{t_0}^{t} k(s)|y(s)|ds + m(t)\int_{t_0}^{t} p(s)w(|y(s)|)ds, \quad (2.19)$$

and

$$|h(t, y(t), T_2 y(t))|$$

$$\le (e^{-\alpha t}d(t)w(|y(t)|) + |T_2 y(t)|), \; |T_2 y(t)| \le \int_{t_0}^{t} e^{-\alpha s}q(s)|y(s)|ds, \quad (2.20)$$

where $\alpha > 0$, $a, b, c, d, k, m, p, q, w \in C(\mathbb{R}^+)$, $a, b, c, d, k, m, p, q \in L^1(\mathbb{R}^+)$. *If*

$$M(t_0) = W^{-1}\Bigg[W(c) + M\int_{t_0}^{\infty} e^{\alpha s}\Bigg(d(s) + \int_{t_0}^{s}\Bigg(a(\tau) + b(\tau) + q(\tau)$$

$$+ c(\tau)\int_{t_0}^{\tau} k(r)dr + m(\tau)\int_{t_0}^{\tau} p(r)dr\Bigg)d\tau\Bigg)ds\Bigg], \quad (2.21)$$

where $M(t_0) < \infty$, $b_1 = \infty$, $c = |y_0|Me^{\alpha t_0}$, W *and* W^{-1} *are the same functions as in Lemma* 1.10, *then all solutions of* (1.2) *approach zero as* $t \to \infty$.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. Using (H5), it is EAS by Remark 1.2. Therefore, by view of Lemma 1.7, together with (2.18), (2.19), and (2.20), we have

$$
\begin{aligned}
|y(t)| \le M|y_0|e^{-\alpha(t-t_0)}
&+ \int_{t_0}^{t} Me^{-\alpha(t-s)}\left(\int_{t_0}^{s} e^{-\alpha\tau}\left(a(\tau)|y(\tau)| + b(\tau)w(|y(\tau)|)\right.\right. \\
&+ c(\tau)\int_{t_0}^{\tau} k(r)|y(r)|dr + m(\tau)\int_{t_0}^{\tau} p(r)w(|y(r)|)dr\bigg)d\tau \\
&+ e^{-\alpha s}d(s)w(|y(s)|) + \int_{t_0}^{s} e^{-\alpha\tau}q(\tau)|y(\tau)|d\tau\bigg)ds.
\end{aligned}
$$

By (H3), it follows that

$$
\begin{aligned}
|y(t)| \le M|y_0|e^{-\alpha(t-t_0)} &+ \int_{t_0}^{t} Me^{-\alpha(t-s)}\Big(d(s)w(|y(s)|e^{\alpha s}) \\
&+ \int_{t_0}^{s}\Big((a(\tau)+q(\tau))|y(\tau)|e^{\alpha\tau} + b(\tau)w(|y(\tau)|e^{\alpha\tau}) \\
&+ c(\tau)\int_{t_0}^{\tau} k(r)|y(r)|e^{\alpha r}dr + m(\tau)\int_{t_0}^{\tau} p(r)w(|y(r)|e^{\alpha r})dr\Big)d\tau\Big)ds.
\end{aligned}
$$

Let $u(t) = |y(t)|e^{\alpha t}$. Then, by Corollary 1.13 and (2.21), we obtain

$$
\begin{aligned}
|y(t)| \le e^{-\alpha t}W^{-1}\Big[W(c) &+ M\int_{t_0}^{t} e^{\alpha s}\Big(d(s) + \int_{t_0}^{s}\Big(a(\tau)+b(\tau)+q(\tau) \\
&+ c(\tau)\int_{t_0}^{\tau} k(r)dr + m(\tau)\int_{t_0}^{\tau} p(r)dr\Big)d\tau\Big)ds\Big] \le ce^{-\alpha t}M(t_0),
\end{aligned}
$$

where $t \ge t_0$ and $c = M|y_0|e^{\alpha t_0}$. Hence, all solutions of (1.2) approach zero as $t \to \infty$, and so the theorem is complete. □

Remark 2.11. Letting $c(t) = d(t) = q(t) = 0$ in Theorem 2.10, we obtain the same result as that of Theorem 3.2 in [17].

Theorem 2.12. *Suppose that* (H3), (H5), *and that the perturbed term* $g(t, y, T_1y)$ *satisfy*

$$\int_{t_0}^{t} |g(s, y(s), T_1 y(s))| ds \le e^{-\alpha t}(a(t)|y(t)| + b(t)w(|y(t)|) + |T_1 y(t)|), \quad (2.22)$$

and

$$|T_1 y(t)| \le b(t)\int_{t_0}^{t} k(s)w(|y(s)|)ds + d(t)\int_{t_0}^{t} p(s)|y(s)|ds, \quad (2.23)$$

$$|h(t, y(t), T_2 y(t))|$$

$$\le e^{-\alpha t}\left(d(t)\int_{t_0}^{t} c(s)|y(s)|ds + |T_2 y(t)|\right), |T_2 y(t)| \le q(t)w(|y(t)|), \quad (2.24)$$

where $\alpha > 0$, $a, b, c, d, k, p, q, w \in C(\mathbb{R}^+)$, $a, b, c, d, k, p, q \in L^1(\mathbb{R}^+)$. *If*

$$M(t_0) = W^{-1}\left[W(c) + M\int_{t_0}^{\infty}\left(a(s) + b(s) + q(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau\right.\right.$$

$$\left.\left. + d(s)\int_{t_0}^{s}(c(\tau) + p(\tau))d\tau\right)ds\right], \quad (2.25)$$

where $b_1 = \infty$, $M(t_0) < \infty$, $c = M|y_0|e^{\alpha t_0}$, W *and* W^{-1} *are the same functions as in Lemma* 1.10, *then all solutions of* (1.2) *approach zero as* $t \to \infty$.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. Using (H5), it is EAS. Therefore, by Lemma 1.7, together with (2.22), (2.23), and (2.24), we have

$$|y(t)| \le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)}\left(e^{-\alpha s}\left(a(s)|y(s)| + b(s)w(|y(s)|)\right.\right.$$

$$+ b(s)\int_{t_0}^{s} k(\tau)w(|y(\tau)|)d\tau$$

$$\left.\left. + d(s)\int_{t_0}^{s}(c(\tau) + p(\tau))|y(\tau)|d\tau + q(s)w(|y(s)|)\right)\right)ds.$$

By (H3), we obtain

$$|y(t)| \le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha t}\Big(a(s)|y(s)|e^{\alpha s}$$

$$+ (b(s) + q(s))w(|y(s)|e^{\alpha s}) + b(s)\int_{t_0}^{s} k(\tau)w(|y(\tau)|e^{\alpha\tau})d\tau$$

$$+ d(s)\int_{t_0}^{s}(c(\tau) + p(\tau))|y(\tau)|e^{\alpha\tau}d\tau\Big)ds.$$

Define $u(t) = |y(t)|e^{\alpha t}$. Then, an application of Lemma 1.11 and (2.25) yields

$$|y(t)| \le e^{-\alpha t}W^{-1}\Big[W(c) + M\int_{t_0}^{t}\Big(a(s) + b(s) + q(s)$$

$$+ b(s)\int_{t_0}^{s} k(\tau)d\tau + d(s)\int_{t_0}^{s}(c(\tau) + p(\tau))d\tau\Big)ds\Big] \le e^{-\alpha t}M(t_0),$$

where $t \ge t_0$ and $c = M|y_0|e^{\alpha t_0}$. Hence, all solutions of (1.2) approach zero as $t \to \infty$. This completes the proof. □

Remark 2.13. Letting $d(t) = q(t) = 0$ in Theorem 2.12, we obtain the same result as that of Theorem 3.4 in [17].

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