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THE HECKE ALGEBRA REPRESENTATION OF THE COMPLEX REFLECTION GROUP G_7 IS UNITARY

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Abstract

We consider a 2-dimensional representation of the Hecke algebra $\mathcal{H}(G_7, u)$, where G_7 is the complex reflection group and u is the set of indeterminates $u = (x_1, x_2, y_1, y_2, y_3, z_1, z_2, z_3)$. After specializing the indeterminates to non-zero complex numbers on the unit circle, we prove that the representation is unitary relative to a Hermitian positive definite matrix. We then determine a necessary and sufficient condition for an element of $\mathcal{H}(G_7, u)$ to belong to the kernel of the complex specialization of the representation of the Hecke algebra $\mathcal{H}(G_7, u)$.

1. Introduction

Let V be a complex vector space and W be a finite irreducible subgroup of $GL(V)$ generated by complex reflections. Let R be the set of reflections in

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W. For any element s of R, denote by H_s its pointwise fixed hyperplane. We define the set $V^{reg} = V - \bigcup_{s \in R} H_s$ and denote by $\overline{V}$ the quotient V^{reg}/W.

The braid group associated to (W, V) is the fundamental group $B(W) = \pi_1(\overline{V}, \bar{x}_0)$ of $\overline{V}$ with respect to any point $\bar{x}_0 \in \overline{V}$.

We choose the set of indeterminates, $u = (u_{s,j})_{s, 0 \leq j \leq o(s)-1}$, where s runs over the generators of W and $u_{s,j} = u_{t,j}$ if s and t are conjugate in W. Here $o(s)$ denotes the order of s. The cyclotomic Hecke algebra associated to W is the quotient of the group algebra $\mathbb{Z}[u, u^{-1}]B(W)$ by the ideal generated by the relations $\prod_{j=0}^{o(s)-1}(s - u_{s,j})$.

In [9], Malle and Michel constructed on the cyclotomic Hecke algebra $\mathcal{H}(G_7, u)$ of the complex reflection group, G_7, an irreducible representation $\phi : \mathcal{H}(G_7, u) \to M_2(\mathbb{C}(u^{\frac{1}{2}}, u^{-\frac{1}{2}}))$, where u is the set of indeterminates $u = (x_1, x_2, y_1, y_2, y_3, z_1, z_2, z_3)$.

In our work, we specialize the indeterminates $x_1, x_2, y_1, y_2, y_3, z_1, z_2$ and z_3 to nonzero complex numbers on the unit circle. We then get a representation $\varphi : \mathcal{H}(G_7, u) \to GL_2(\mathbb{C})$. In Section 2, we recall, from an earlier work, some results that determine necessary and sufficient conditions for the irreducibility of the representation φ [6]. In Section 3, we determine a necessary and sufficient condition that shows that φ is unitary relative to a non-zero invertible Hermitian matrix if and only if $y_1 y_2 = \pm 1$ and $x_1 \neq x_2$ (Theorem 4, Proposition 2). In Section 4, we find complex specializations under which the Hermitian matrices found in Section 3 are positive definite, and in which the representation φ remains to be irreducible (Theorem 5). In Section 5, we find a necessary and sufficient condition for an element of $\mathcal{H}(G_7, u)$ to belong to the kernel of the representation φ (Theorem 8).

2. Definitions, Notations and Theorems

Definition 1 [8]. Let V be a complex vector space of dimension n. A complex reflection of $GL(V)$ is a non-trivial element of $GL(V)$ which acts trivially on a hyperplane.

Definition 2 [8]. Let V be a complex vector space of dimension n. A complex reflection group is the subgroup of $GL(V)$ generated by complex reflections.

Examples of complex reflection groups include dihedral groups and symmetric groups. For $n \geq 3$, the dihedral group, D_n, is the group of the isometries of the plane preserving a regular polygon, with the operation being composition.

A classification of all irreducible reflection groups shows that there are 34 primitive irreducible reflection groups [10]. The starting point was with Cohen, who provided a data for those irreducible complex reflection groups of rank 2 [7].

Definition 3 [4]. The complex reflection group, G_7, is an abstract group defined by the presentation

$$G_7 = \langle t, u, s / t^2 = u^3 = s^3 = 1, tus = ust = stu \rangle.$$

Theorem 1 [2]. *The braid group of* G_7 *is isomorphic to the group*

$$B = \langle s_1, s_2, s_3 / s_1 s_2 s_3 = s_2 s_3 s_1 = s_3 s_1 s_2 \rangle.$$

Definitions and properties of braid groups are found in [3].

Definition 4 [9]. Let u be the set of indeterminates $u = (x_1, x_2, y_1, y_2, y_3, z_1, z_2, z_3)$. The cyclotomic Hecke algebra $\mathcal{H}(G_7, u)$ of G_7 is the quotient of the group algebra of B over $\mathbb{Z}[u, u^{-1}]$ by the relations

$$(s_1 - x_1)(s_1 - x_2) = 0, \ \prod_{i=1}^{3} (s_2 - y_i) = 0, \ \prod_{i=1}^{3} (s_3 - z_i) = 0.$$

For more details about the Hecke algebra of G_7, see [5].

Any 2-dimensional representation of B gives a representation of $\mathcal{H}(G_7, u)$ (see [9]).

Definition 5 [9]**.** Let $u = (x_1, x_2, y_1, y_2, y_3, z_1, z_2, z_3)$. The representation ϕ is defined as follows:

$$\phi : \mathcal{H}(G_7, u) \to M_2(\mathbb{C}(u^{\pm\frac{1}{2}})),$$

$$\phi(s_1) = \begin{pmatrix} x_1 & \frac{y_1 + y_2}{y_1 y_2} - \frac{(z_1 + z_2)x_2}{r} \\ 0 & x_2 \end{pmatrix}, \quad \phi(s_2) = \begin{pmatrix} y_1 + y_2 & \frac{1}{x_1} \\ -y_1 y_2 x_1 & 0 \end{pmatrix}$$

and

$$\phi(s_3) = \begin{pmatrix} 0 & \frac{-r}{y_1 y_2 x_1 x_2} \\ r & z_1 + z_2 \end{pmatrix},$$

where $r = \sqrt{x_1 x_2 y_1 y_2 z_1 z_2}$.

We specialize the indeterminates $x_1, x_2, y_1, y_2, y_3, z_1, z_2$ and z_3 to non-zero complex numbers on the unit circle. We then get a representation $\varphi : \mathcal{H}(G_7, u) \to GL_2(\mathbb{C})$.

Definition 6. Principal square root function is defined as follows: for $z = (1, \alpha)$, $\sqrt{z} = e^{i\frac{\alpha}{2}}$, where $-\pi < \alpha \leq \pi$.

Since $\alpha \in (-\pi, \pi]$, it follows that $\sqrt{z^2} = z$ for any complex number z.

We now recall two theorems that determine necessary and sufficient conditions that guarantee the irreducibility of φ.

Theorem 2 [6]**.** *Suppose that* $x_1 = x_2$. *Then the representation* φ *is irreducible if and only if* $z_1 \neq \frac{y_1 z_2}{y_2}$ *and* $z_1 \neq \frac{y_2 z_2}{y_1}$.

Theorem 3 [6]. *Suppose that* $x_1 \neq x_2$. *Then the representation* φ *is irreducible if and only if* $x_1y_2z_2 \neq x_2y_1z_1$, $x_1y_1z_2 \neq x_2y_2z_1$, $x_1y_2z_1 \neq x_2y_1z_2$ *and* $x_1y_1z_1 \neq x_2y_2z_2$.

3. $\varphi : \mathcal{H}(G_7, u) \to GL_2(\mathbb{C})$ is Unitary Relative to a Hermitian Matrix

We find a necessary and sufficient condition that proves that φ is unitary relative to a Hermitian invertible matrix.

Notation 1. Let $(*) : M_m(\mathbb{C}[t^{\pm 1}])$ be an involution defined as follows:

$$(f_{ij}(t))^* = f_{ji}(t^{-1}), \quad f_{ij}(t) \in \mathbb{C}[t^{\pm 1}].$$

Definition 7. Let U be an element of $GL_2(\mathbb{C})$. Then U is called *unitary* if $U^*U = UU^* = I_2$.

Definition 8. Let H and U be elements of $GL_2(\mathbb{C})$. Then U is called *unitary* relative to H if $UHU^* = H$.

Proposition 1 (Schur's Lemma). *Suppose that* L *is an* $n \times n$ *matrix such that* $L\alpha(g) = \alpha(g)L$ *for each* $g \in G$, *where* α *is an irreducible representation of the group* G. *Then* $L = \lambda I$ *for some* $\lambda \in \mathbb{C}$, *where* I *is the* $n \times n$ *identity matrix.*

Theorem 4. *The images of the generators of* $\mathcal{H}(G_7, u)$ *under the irreducible representation* φ *are unitary relative to some non-zero invertible matrix* K *if and only if* $y_1y_2 = \pm 1$ *and* $x_1 \neq x_2$.

Proof. Suppose that the images of the generators of $\mathcal{H}(G_7, u)$ under φ are unitary relative to a non-zero matrix $K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c and $d \in \mathbb{C}$. We consider the matrix of the image of s_i under the representation φ and still denoted by s_i. Here $i = 1, 2, 3$. Simple computations show that $s_3Ks_3^*(1, 1) = d$ and $s_2Ks_2^*(2, 2) = ay_1^2y_2^2$.

Since $s_3Ks_3^* = s_2Ks_2^* = K$, it follows that $d = a = ay_1^2y_2^2$. This implies that $a(y_1^2y_2^2 - 1) = 0$.

If we suppose that $a = 0$ then, by simple computations, we get that $K = 0$. Therefore, $y_1y_2 = \pm 1$.

Now, suppose to get a contradiction that $x_1 = x_2$. We have two cases to investigate:

Case 1. $y_1y_2 = 1$. Since $s_1Ks_1^*(1, 2) = \dfrac{bx_2 + a\left(\dfrac{1}{y_1} + y_1 - \dfrac{z_1 + z_2}{\sqrt{z_1z_2}}\right)}{x_2} = b$ $= K(1, 2)$, it follows that $y_1(z_1 + z_2) = \sqrt{z_1z_2}(1 + y_1^2)$.

Squaring both the sides, we obtain that $(y_1^2z_2 - z_1)(y_1^2z_1 - z_2) = 0$. This implies that $z_1 = y_1^2z_2$ or $z_2 = y_1^2z_1$, which contradicts the conditions of irreducibility of φ (Theorem 2).

Case 2. $y_1y_2 = -1$. Since $s_1Ks_1^*(1, 2) = \dfrac{bx_2 + a\left(\dfrac{1}{y_1} - y_1 - \dfrac{z_1 + z_2}{\sqrt{-z_1z_2}}\right)}{x_2} =$ $b = K(1, 2)$, it follows that $y_1(z_1 + z_2) = \sqrt{-z_1z_2}(1 - y_1^2)$.

Squaring both the sides, we obtain that $(y_1^2z_2 + z_1)(y_1^2z_1 + z_2) = 0$. This implies that $z_1 = -y_1^2z_2$ or $z_2 = -y_1^2z_1$, which contradicts the conditions of irreducibility of φ (Theorem 2).

On the other hand, suppose that $x_1 \neq x_2$ and $y_1y_2 = \pm 1$. We prove that the representation φ is unitary relative to a Hermitian matrix that will be determined explicitly. We have two cases to investigate:

Case 3. $x_1 \neq x_2$ and $y_1y_2 = 1$. Direct computations show that the images of the generators of $\mathcal{H}(G_7, u)$ under the representation φ are unitary relative to a matrix M given by

$$M = \begin{pmatrix} 1 & M_{1,2} \\ M_{2,1} & 1 \end{pmatrix},$$

where

$$M_{1,2} = \frac{-x_1(1+y_1^2)z_1z_2 + y_1\sqrt{x_1x_2z_1z_2}(z_1+z_2)}{x_1(x_1-x_2)y_1z_1z_2},$$

$$M_{2,1} = \frac{x_1(x_2(1+y_1^2)z_1z_2 - y_1\sqrt{x_1x_2z_1z_2}(z_1+z_2))}{(x_1-x_2)y_1z_1z_2}.$$

Case 4. $x_1 \neq x_2$ and $y_1y_2 = -1$. Direct computations show that the images of the generators of $\mathcal{H}(G_7, u)$ under the representation φ are unitary relative to a matrix N given by

$$N = \begin{pmatrix} 1 & N_{1,2} \\ N_{2,1} & 1 \end{pmatrix},$$

where

$$N_{1,2} = \frac{x_1(-1+y_1^2)z_1z_2 - y_1\sqrt{-x_1x_2z_1z_2}(z_1+z_2)}{x_1(x_1-x_2)y_1z_1z_2},$$

$$N_{2,1} = \frac{x_1(x_2(-1+y_1^2)z_1z_2 - y_1\sqrt{-x_1x_2z_1z_2}(z_1+z_2))}{(x_1-x_2)y_1z_1z_2}.$$

It is easy to show that the matrices M and N are invertible because of the irreducibility of the representation φ and the fact that $y_1y_2 = \pm 1$. □

Proposition 2. *The matrices M and N determined in Theorem* 4 *are Hermitian* $(M^* = M, N^* = N)$.

Proof. Since the conjugate of any complex number on the unit circle equals its inverse, it follows that

$$\overline{M(1, 2)} = \frac{\dfrac{-(1 + y_1^2)\sqrt{x_1x_2z_1z_2} + x_1y_1(z_1 + z_2)}{x_1y_1^2z_1z_2\sqrt{x_1x_2z_1z_2}}}{\dfrac{x_2 - x_1}{x_1^2x_2y_1z_1z_2}}$$

$$= \frac{[-(1 + y_1^2)\sqrt{x_1x_2z_1z_2} + x_1y_1(z_1 + z_2)]x_1x_2}{(x_2 - x_1)y_1\sqrt{x_1x_2z_1z_2}} = M(2, 1)$$

and

$$\overline{N(1, 2)} = \frac{\dfrac{(-y_1^2 + 1)\sqrt{-x_1x_2z_1z_2} - x_1y_1(z_1 + z_2)}{x_1y_1^2z_1z_2\sqrt{-x_1x_2z_1z_2}}}{\dfrac{x_2 - x_1}{x_1^2x_2y_1z_1z_2}}$$

$$= \frac{[(-y_1^2 + 1)\sqrt{-x_1x_2z_1z_2} - x_1y_1(z_1 + z_2)]x_1x_2}{(x_2 - x_1)y_1\sqrt{-x_1x_2z_1z_2}} = N(2, 1).$$

This implies that $M^* = M$ and $N^* = N$. □

We then show that the matrix M is unique up to scalar multiplication when $x_1 \neq x_2$ and $y_1y_2 = 1$. Similarly, for the matrix N when $x_1 \neq x_2$ and $y_1y_2 = -1$.

Proposition 3. *If the irreducible representation* φ *is unitary relative to an invertible matrix A*, *then A is unique up to scalar multiplication*.

Proof. Suppose that the irreducible representation φ is unitary relative to an invertible matrix A. This implies that $s_iAs_i^* = A$, $i = 1, 2, 3$.

If there exists another invertible matrix B such that $s_iBs_i^* = B$, then we get that

$$(s_iAs_i^*)(s_i^{*-1}B^{-1}s_i^{-1}) = AB^{-1}.$$

This implies that $s_i(AB^{-1}) = (AB^{-1})s_i$.

By Schur's Lemma, we have that $AB^{-1} = cI_2$ for some constant c. Therefore, $A = cB$. □

4. The Matrices *M* and *N* are Positive Definite for Some Complex Specializations

We determine complex specializations for the indeterminates under which the matrices M and N, computed in Section 3, are positive definite, and in which the representation φ remains to be irreducible. We consider two cases. In the case $(x_1 \neq x_2$ and $y_1 y_2 = 1)$, we consider the matrix M obtained in Case 3 of Theorem 4. Similarly, in the case $(x_1 \neq x_2$ and $y_1 y_2 = -1)$, we consider the matrix N obtained in Case 4 of Theorem 4. In both cases, we show that M and N are positive definite matrices.

Theorem 5. *If* $x_2 = -x_1$, $z_1 = 1$, $y_1 = e^{i\theta}$, $y_2 = e^{-i\theta}$ *and* $z_2 = e^{i\alpha}$, *where* $\frac{\pi}{4} < \theta < \frac{\pi}{2}$ *and* $\frac{\pi}{2} < \alpha < \pi$, *then the representation* φ *is irreducible and unitary relative to a positive definite Hermitian matrix M.*

Proof. Since $x_2 = -x_1$ and $z_1 = 1$, it follows that, by Theorem 3, φ is irreducible if and only if $y_1^2 z_2 \neq -1$ and $z_2 \neq -y_1^2$. We then substitute y_1 and z_2 by $e^{i\theta}$ and $e^{i\alpha}$, respectively. We get that φ is irreducible if and only if $2\theta + \alpha \neq (2k+1)\pi$ and $2\theta - \alpha \neq (2k+1)\pi$.

Under the hypothesis, we have that $\pi < 2\theta + \alpha < 2\pi$ and $-\frac{\pi}{2} < 2\theta - \alpha < \frac{\pi}{2}$. We now verify that φ is irreducible.

In contrary, suppose that the representation is reducible. So, either $2\theta + \alpha = (2k+1)\pi$ or $2\theta - \alpha = (2k+1)\pi$ for some $k \in \mathbb{Z}$.

If $2\theta + \alpha = (2k+1)\pi$ for some $k \in \mathbb{Z}$, then $\pi < (2k+1)\pi < 2\pi$. This implies that $1 < 2k+1 < 2$, a contradiction.

If $2\theta - \alpha = (2k+1)\pi$ for some $k \in \mathbb{Z}$, then $-\frac{\pi}{2} < (2k+1)\pi < \frac{\pi}{2}$. This implies that $-\frac{1}{2} < 2k+1 < \frac{1}{2}$, a contradiction.

In Theorem 4, we proved that the representation φ is unitary relative to the matrix *M*, when $x_1 \neq x_2$ and $y_1 y_2 = 1$. We now show that *M* is positive definite.

We denote the principal minors of *M* by d_i, where $i = 1, 2$. We have

$$d_1 = 1 \text{ and } d_2 = Det(M).$$

Under the hypothesis, we have

$$Det(M) = -\frac{(y_1^2 + z_2)(1 + y_1^2 z_2)}{4y_1^2 z_2} = -\frac{(e^{i2\theta} + e^{i\alpha})(e^{i0} + e^{i(2\theta+\alpha)})}{4e^{i(2\theta+\alpha)}}.$$

One can easily show that $e^{ia} + e^{ib} = 2e^{i\left(\frac{a+b}{2}\right)} \cos\left(\frac{a-b}{2}\right)$. This implies that

$$Det(M) = -\frac{2e^{i\left(\frac{2\theta+\alpha}{2}\right)} \cos\left(\frac{2\theta - \alpha}{2}\right) 2e^{i\left(\frac{2\theta+\alpha}{2}\right)} \cos\left(\frac{2\theta + \alpha}{2}\right)}{4e^{i(2\theta+\alpha)}}$$

$$= -\cos\left(\frac{2\theta - \alpha}{2}\right)\cos\left(\frac{2\theta + \alpha}{2}\right).$$

Since $\frac{\pi}{2} < \frac{2\theta + \alpha}{2} < \pi$ and $\frac{-\pi}{4} < \frac{2\theta - \alpha}{2} < \frac{\pi}{4}$, it follows that $\cos\left(\frac{2\theta + \alpha}{2}\right) < 0$ and $\cos\left(\frac{2\theta - \alpha}{2}\right) > 0$. Therefore, $d_2 > 0$. □

We might also want to consider another specialization of φ under which the matrix *N* is positive definite.

Theorem 6. *If* $x_2 = -x_1$, $z_1 = 1$, $y_1 = e^{i\theta}$, $y_2 = -e^{-i\theta}$ *and* $z_2 = e^{i\alpha}$, *where* $0 < \theta < \frac{\pi}{4}$ *and* $\frac{\pi}{2} < \alpha < \pi$, *then the representation* φ *is irreducible and unitary relative to a positive definite Hermitian matrix N.*

Proof. Since $x_2 = -x_1$ and $z_1 = 1$, it follows that φ is irreducible if and only if $y_1^2 z_2 \neq 1$ and $z_2 \neq y_1^2$. We then substitute y_1 and z_2 by $e^{i\theta}$ and $e^{i\alpha}$, respectively. We get that φ is irreducible if and only if $2\theta + \alpha \neq 2k\pi$ and $2\theta - \alpha \neq 2k\pi$.

Under the hypothesis, we have that $\frac{\pi}{2} < 2\theta + \alpha < \frac{3\pi}{2}$ and $-\pi < 2\theta - \alpha < 0$. We now verify that φ is irreducible.

In contrary, suppose that the representation is reducible. So, either $2\theta + \alpha = 2k\pi$ or $2\theta - \alpha = 2k\pi$ for some $k \in \mathbb{Z}$.

If $2\theta + \alpha = 2k\pi$ for some $k \in \mathbb{Z}$, then $\frac{\pi}{2} < 2k\pi < \frac{3\pi}{2}$. This implies that $\frac{1}{4} < k < \frac{3}{4}$, a contradiction.

If $2\theta - \alpha = 2k\pi$ for some $k \in \mathbb{Z}$, then $-\pi < 2k\pi < 0$. This implies that $-\frac{1}{2} < k < 0$, a contradiction.

We now show that *N* is positive definite.

We denote the principal minors of *N* by d_i, where $i = 1, 2$. We have

$$d_1 = 1 \text{ and } d_2 = Det(N).$$

Under the hypothesis, we have

$$Det(N) = \frac{(y_1^2 - z_2)(-1 + y_1^2 z_2)}{4 y_1^2 z_2} = \frac{(e^{i2\theta} - e^{i\alpha})(e^{i\pi} + e^{i(2\theta+\alpha)})}{4e^{i(2\theta+\alpha)}}.$$

One can also show that $e^{ia} - e^{ib} = 2e^{i\left(\frac{a+b+\pi}{2}\right)} \cos\left(\frac{a-b-\pi}{2}\right)$. This implies that

$$Det(N) = \frac{2e^{i\left(\frac{2\theta+\alpha+\pi}{2}\right)} \cos\left(\frac{2\theta-\alpha-\pi}{2}\right) 2e^{i\left(\frac{\pi+2\theta+\alpha}{2}\right)} \cos\left(\frac{\pi-2\theta-\alpha}{2}\right)}{4e^{i(2\theta+\alpha)}}$$

$$= -\sin\left(\frac{2\theta-\alpha}{2}\right)\sin\left(\frac{2\theta+\alpha}{2}\right).$$

Since $\frac{\pi}{4} < \frac{2\theta+\alpha}{2} < \frac{3\pi}{4}$ and $\frac{-\pi}{2} < \frac{2\theta-\alpha}{2} < 0$, it follows that $\sin\left(\frac{2\theta+\alpha}{2}\right) > 0$ and $\sin\left(\frac{2\theta-\alpha}{2}\right) < 0$. Therefore, $d_2 > 0$. □

5. Necessary and Sufficient Condition for an Element of $\mathcal{H}(G_7, u)$ to belong to the Kernel of φ

We show that under the hypothesis of Theorem 5 (or Theorem 6), the representation φ is conjugate to a unitary representation.

Theorem 7. *For some complex specialization of the indeterminates defined in Theorem* 5 *or Theorem* 6, *the representation* φ *is conjugate to a unitary representation*.

Proof. We have that $s_i M s_i^* = M$ or $s_i N s_i^* = N$ under the hypothesis of Theorem 5 or Theorem 6, respectively. Without loss of generality, we consider the matrix M.

Since M is positive definite, it follows that $M = VV^*$. This implies that

$$(V^{-1}s_iV)(V^{-1}s_iV)^* = V^{-1}s_i(VV^*)s_i^*(V^*)^{-1} = I.$$

Set $U_i = V^{-1}s_iV$. We have that $U_iU_i^* = I$. Similarly, $U_i^*U_i = I$. So, U_i is unitary. □

After we have shown that the representation φ is conjugate to a unitary representation, we find a necessary and sufficient condition for an element of $\mathcal{H}(G_7, u)$ to belong to the kernel of the representation. This argument is similar to that used in [1], where a criterion for the faithfulness of the Gassner representation of the pure braid group is given.

Theorem 8. *An element of* $\mathcal{H}(G_7, u)$ *belongs to the kernel of the representation if and only if the trace of its image is equal to* 2.

Proof. We show that if the trace of the image of an element s_i is 2, then s_i belongs to the kernel of the representation φ.

Since $trace(s_i) = 2$ and $U_i = V^{-1}s_iV$, it follows that $trace(U_i) = 2$.

U_i is unitary. This implies that $P_i^{-1}U_iP_i = D_i$ for some nonsingular matrix P_i. Here, D_i is a diagonal matrix which has its diagonal elements the eigenvalues of U_i.

Since $trace(U_i) = 2$, it follows that $\lambda_1 + \lambda_2 = 2$, where λ_i's are the eigenvalues of U_i. Being unitary, the eigenvalues of U_i are on the unit circle. This implies that $\lambda_1 = \lambda_2 = 1$. It follows that D_i is the identity matrix and so is U_i. This implies that $s_i = I$. □

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