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SOME NOTES ON THE REGULAR GRAPH DEFINED BY SCHMIDT AND SUMMERER AND UNIFORM APPROXIMATION

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Abstract

Within the study of parametric geometry of numbers, Schmidt and Summerer introduced so-called regular graphs. Roughly speaking, the successive minima functions for the classical simultaneous Diophantine approximation problem have a very special pattern if the vector $\underline{\zeta}$ induces a regular graph. The regular graph is, in particular, of interest due to a conjecture by Schmidt and Summerer concerning classic approximation constants. This paper aims to provide several new results on the behavior of the successive minima functions for the regular graph. Moreover, we improve the best known upper bounds for the classic approximation constants $\hat{w}_n(\zeta)$, provided that the Schmidt-Summerer conjecture is true.

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1. Introduction

1.1. Outline

This paper aims, on one hand, to give a better understanding of the regular graph defined by Schmidt and Summerer, and on the other hand, to establish a connection to the uniform approximation constants $\hat{w}_n$. Theorems 2.4 and 2.6 in this paper are the main results. In Subsection 1.3, we will define the regular graph and explain its significance for simultaneous Diophantine approximation. We recommend the reader to look at the illustrations of combined graphs and, in particular, the regular graph in [23, p. 90], another sketch adopted from [19, p. 72] is visible in Subsection 1.3. See also [19] for Matlab plots of the combined graph for special choices of real vectors. Finally, in Section 4, we discuss the consequences of another reasonable conjecture to uniform approximation.

1.2. Geometry of numbers

We start with a classical problem of simultaneous approximation. Assume $\underline{\zeta} = (\zeta_1, \dots, \zeta_n)$ in $\mathbb{R}^n$ is given. For $1 \le j \le n+1$, let $\lambda_{n,j} = \lambda_{n,j}(\underline{\zeta})$ be the supremum of real ν for which there are arbitrarily large X such that the system

$$|x| \le X, \quad \max_{1 \le j \le k} |\zeta_j x - y_j| \le X^{-\nu} \tag{1}$$

has j linearly independent solution vectors $(x, y_1, \dots, y_n)$ in $\mathbb{Z}^{n+1}$. Moreover, let $\hat{\lambda}_{n,j} = \hat{\lambda}_{n,j}(\underline{\zeta})$ be the supremum of ν such that the system (1) has j linearly independent integer vector solutions $(x, y_1, \dots, y_n)$ for all large X. For $\lambda_{n,1}$, we will also simply write λ_n, and similarly $\hat{\lambda}_n$ for $\hat{\lambda}_{n,1}$. For all $\underline{\zeta} \in \mathbb{R}^n$, Minkowski's first lattice point theorem (or Dirichlet's theorem) implies the estimates

$$\lambda_n \geq \hat{\lambda}_n \geq \frac{1}{n}. \tag{2}$$

More generally, it can be shown that

$$\frac{1}{n} \leq \lambda_n \leq \infty, \quad \frac{1}{n} \leq \lambda_{n,2} \leq 1, \quad 0 \leq \lambda_{n,j} \leq \frac{1}{j-1} \ \ (1 \leq j \leq n+1), \tag{3}$$

and similarly

$$\frac{1}{n} \leq \hat{\lambda}_n \leq 1, \quad 0 \leq \hat{\lambda}_{n,j} \leq \frac{1}{j} \ \ (2 \leq j \leq n), \quad 0 \leq \hat{\lambda}_{n,n+1} \leq \frac{1}{n}. \tag{4}$$

See [19, (14)-(18)]. Moreover, $\lambda_{n,j} \geq \hat{\lambda}_{n,j-1}$ holds for $2 \leq j \leq n+1$ as pointed out in [21].

Schmidt and Summerer investigated a parametric version of the simultaneous approximation problem above [21, 22]. We will now introduce some concepts and results of the evolved parametric geometry of numbers from [21]. Our notation will partially deviate from [21] for technical reasons. Keep $n \geq 1$ an integer and $\underline{\zeta} = (\zeta_1, \ldots, \zeta_n)$ a fixed vector in $\mathbb{R}^n$. For any parameter $Q > 1$ and any $1 \leq j \leq n+1$, consider the largest number ν such that

$$|x| \leq Q^{1+\nu}, \quad \max_{1 \leq j \leq k} |\zeta_j x - y_j| \leq Q^{-1/n+\nu}$$

has j linearly independent integral solution vectors $(x, y_1, \ldots, y_n) \in \mathbb{Z}^{n+1}$. Denote by $\psi_{n,j}(Q)$ this value. Dirichlet's theorem yields $\psi_{n,1}(Q) < 0$ for all $Q > 1$. Further, let

$$\underline{\psi}_{n,j} = \liminf_{Q \to \infty} \psi_j(Q), \quad \overline{\psi}_{n,j} = \limsup_{Q \to \infty} \psi_j(Q).$$

It is not hard to see that

$$-1 \leq \psi_{n,1}(Q) \leq \psi_{n,2}(Q) \leq \cdots \leq \psi_{n,n+1}(Q) \leq \frac{1}{n}, \quad Q > 1,$$

and, in particular,

$$-1 \leq \underline{\psi}_{n,j} \leq \overline{\psi}_{n,j} \leq \frac{1}{n}, \quad 1 \leq j \leq n+1.$$

For $q = \log Q$, consider the derived functions

$$L_{n,j}(q) = q\psi_{n,j}(Q), \quad 1 \leq j \leq n+1. \tag{5}$$

They have the nice property of being piecewise linear with slope among $\{-1, 1/n\}$. The functions $\psi_{n,j}$ and the derived $L_{n,j}$ can alternatively be defined via a classical successive minima problem of a parametrized family of convex bodies with respect to a lattice. For the details, see [21]. A crucial observation from this point of view is that Minkowski's second theorem yields pointed out in [21] is that the sum of $L_{n,j}$ over j is uniformly bounded by absolute value for $q > 0$. The connection between the constants $\lambda_{n,j}$ and the functions $\psi_{n,j}$ is given by the formula

$$(1 + \lambda_{n,j})(1 + \underline{\psi}_{n,j}) = (1 + \hat{\lambda}_{n,j})(1 + \overline{\psi}_{n,j}) = \frac{n+1}{n}, \quad 1 \leq j \leq n+1.$$

This was pointed out in [19, (13)], which generalized [21, Theorem 1.4]. In particular, for $1 \leq j \leq n+1$, we have the equivalences

$$\underline{\psi}_{n,j} < 0 \Leftrightarrow \lambda_{n,j} > \frac{1}{n}, \quad \overline{\psi}_{n,j} < 0 \Leftrightarrow \hat{\lambda}_{n,j} > \frac{1}{n}. \tag{6}$$

We now briefly introduce the dual problem studied in [21] as well. Define the classic approximation constants $w_{n,j}$ and $\hat{w}_{n,j}$, respectively, as the supremum of ν such that the system

$$\max\{|x|, |y_1|, \ldots, |y_n|\} \leq X, \quad |x + \zeta_1 y_1 + \cdots + \zeta_n y_n| \leq X^{-\nu}$$

has j linearly independent integer vector solutions for arbitrarily large X and all large X, respectively. Again, we also write w_n instead of $w_{n,1}$ and $\hat{w}_n$ instead of $\hat{w}_{n,1}$. In this context, Minkowski's first lattice point theorem (or Dirichlet's theorem) implies

$$w_n \geq \hat{w}_n \geq n. \tag{7}$$

As already mentioned in [18, (1.24)], it can be shown that

$$w_{n,j} = \frac{1}{\hat{\lambda}_{n,n+2-j}}, \quad \lambda_{n,j} = \frac{1}{\hat{w}_{n,n+2-j}}, \quad 1 \leq j \leq n+1. \tag{8}$$

Together with the bounds in (3) and (4), for the spectra of the exponents we obtain

$$n \leq w_n \leq \infty, \quad n+2-j \leq w_{n,j} \leq \infty, \quad 2 \leq j \leq n, \quad 1 \leq w_{n,n+1} \leq n \tag{9}$$

such as

$$n+1-j \leq \hat{w}_{n,j} \leq \infty,\ 1 \leq j \leq n-1,\ 1 \leq \hat{w}_{n,n} \leq n,\ 0 \leq \hat{w}_{n,n+1} \leq n. \tag{10}$$

Schmidt and Summerer studied a parametric version of the linear form problem as well in [21], however, the above classic exponents will suffice for our purposes.

1.3. The regular graph and the Schmidt-Summerer conjecture

For fixed $n \geq 1$ and a parameter $\rho \in [1, \infty]$, in [23] Schmidt and Summerer defined what is called the regular graph. This geometrically describes a special pattern of the combined graph of the successive minima functions $L_{n,j}(q) = L_{n,j}(\log Q)$ from Subsection 1.2. We refer to [23, p. 90] for an idealized illustration of the functions $L_{n,j}(q)$ for the regular graph connected to approximation of three numbers, i.e., $n = 3$ in our notation. Figure 1 depicts a sketch for $n = 2$, which was already presented in [19, p. 72]. The solid lines depict the graphs of the functions $L_{2,1}$, $L_{2,2}$, $L_{2,3}$ whereas the dotted lines correspond to the quantities $\underline{\psi}_{2,j}$, $\overline{\psi}_{2,j}$ for $1 \leq j \leq 3$. Notice that $\underline{\psi}_{2,j+1} = \overline{\psi}_{2,j}$ in the regular graph.

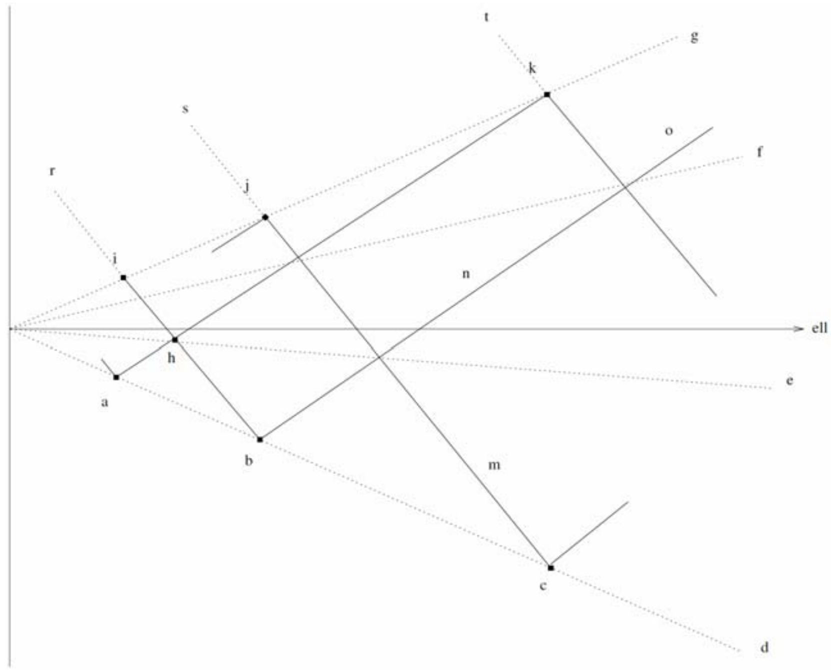

Figure 1. Sketch of the regular graph for $n = 2$.

Roughly speaking, the integers $(x_k)_{k \geq 1}$ that induce a not too short falling period of all $L_{n,j}(q)$, coincide for all $1 \leq j \leq n+1$ and have the additional property that the logarithmic quotients $\log x_{k+1}/\log x_k$ tend to $\lambda_n/\hat{\lambda}_n$. An immediate consequence already mentioned in [19, Section 3] is that all quotients $\lambda_{n,j}/\lambda_{n,j+1} = \lambda_{n,j}/\hat{\lambda}_{n,j}$ coincide for $1 \leq j \leq n+1$. That is,

$$\frac{\lambda_n}{\lambda_{n,2}} = \frac{\lambda_{n,2}}{\lambda_{n,3}} = \cdots = \frac{\lambda_{n,n+1}}{\lambda_{n,n+2}} = \frac{\hat{\lambda}_n}{\hat{\lambda}_{n,2}} = \cdots = \frac{\hat{\lambda}_{n,n}}{\hat{\lambda}_{n,n+1}}, \tag{11}$$

where we have put $\lambda_{n,n+2} := \hat{\lambda}_{n,n+1}$, which shall remain for the sequel. Moreover, it is obvious from its definition that the regular graph satisfies

$$\lambda_{n,j} = \hat{\lambda}_{n,j-1}, \quad 2 \leq j \leq n+2. \tag{12}$$

In view of (11) and (12), all $\lambda_{n,j}, \hat{\lambda}_{n,j}$ are determined by one real parameter $\lambda \geq 1/n$. According to (8), this applies to all exponents $w_{n,j}$ and $\hat{w}_{n,j}$ as well. The parameter $\rho \in [1, \infty]$ in Schmidt-Summerer notation coincides with the value $\lambda_n/\hat{\lambda}_n$ in (11). We will use a different parametrization. We consider the equivalent situation that the constant λ_n is prescribed in the interval $[1/n, \infty]$. Any such choice again uniquely determines a regular graph in dimension n and vice versa. Thus, we have the assignment

$$(n, \lambda) \to (\lambda_n, \lambda_{n,2}, \ldots, \lambda_{n,n+1}, \lambda_{n,n+2}), \quad \lambda \in [1/n, \infty], \tag{13}$$

where $\lambda_n = \lambda$. We call the graph arising from (13) *the regular graph in dimension n with parameter* λ. For $n = 2$, the graphs of the functions $\lambda_{2,j}$ are illustrated in Figure 2:

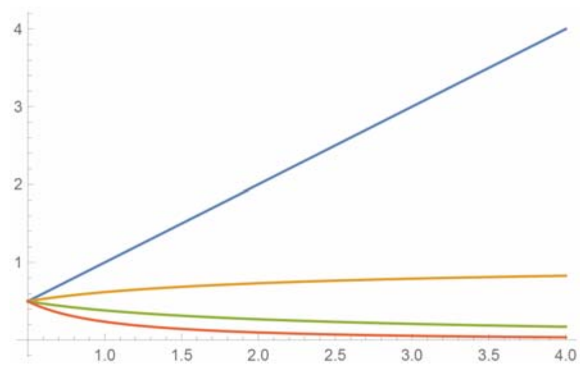

Figure 2. The functions $\lambda_{2,1}(\lambda), \lambda_{2,2}(\lambda), \lambda_{2,3}(\lambda), \lambda_{2,4}(\lambda)$ in the interval $\lambda \in [1/2, 4]$.

It is rather obvious and will follow from (28) in Subsection 5.1 that the right hand side in (13) depends continuously on λ. In view of (12), the assignment (13) contains the entire information on all exponents $\lambda_{n,j}, \hat{\lambda}_{n,j}$. We will also write $\lambda_{n,j}(\lambda)$ and $\hat{\lambda}_{n,j}(\lambda)$ for the quantities $\lambda_{n,j}$ and $\hat{\lambda}_{n,j}$, respectively, in the regular graph in dimension n and parameter λ. It is worth noting that for $\lambda_n = \lambda = 1/n$ all constants in (13) take the value $1/n$, which is a very general elementary consequence of Minkowski's second theorem. Moreover, in the other degenerate case of the regular graph $\lambda = \infty$, it is not hard to see that $\lambda_{n,2}(\infty) = 1$ and $\lambda_{n,j}(\infty) = 0$ hold for $3 \leq j \leq n+2$, see also Proposition 2.3. Roy [16] proved that for any pair (n, λ) as in (13), there exist $\mathbb{Q}$-linearly independent vectors $\underline{\zeta}$ (together with $\{1\}$) that induce the corresponding regular graph. The existence of the regular graph for the special "degenerate" case $\lambda = \infty$ had already been constructively proved before by the author [19, Theorem 4].

The importance of the regular graph stems, in particular, from a conjecture by Schmidt and Summerer [23]. It suggests that the regular graph with assignment (13) maximizes the value $\hat{\lambda}_n$ among all $\underline{\zeta}$ that are $\mathbb{Q}$-linearly independent with 1 and share the prescribed value $\lambda_n(\underline{\zeta}) = \lambda$. A dual version of the conjecture states that $\hat{w}_n$ is *maximized* for given value of w_n in the regular graph as well. For convenience, we introduce some notation.

Definition 1. Let ϕ_n be the function that expresses $\hat{w}_n$ in terms of $w_n \in [n, \infty]$ and ϑ_n the function that expresses the value $\hat{\lambda}_n$ in terms of $\lambda_n \in [1/n, \infty]$ in the regular graph.

Note that $\vartheta_n(\lambda)$ coincides with $\hat{\lambda}_n(\lambda) = \lambda_{n,2}(\lambda)$ defined above. The Schmidt-Summerer conjecture can now be stated in the following way.

Conjecture 1.1 (Schmidt and Summerer)**.** For any positive integer n and every $\underline{\zeta} \in \mathbb{R}^n$ which is $\mathbb{Q}$-linearly independent together with $\{1\}$, we have $\hat{w}_n(\underline{\zeta}) \leq \phi_n(w_n(\underline{\zeta}))$ and $\hat{\lambda}_n(\underline{\zeta}) \leq \vartheta_n(\lambda_n(\underline{\zeta}))$. In particular, for any real transcendental ζ and $n \geq 1$, we have $\hat{w}_n(\zeta) \leq \phi_n(w_n(\zeta))$ and $\hat{\lambda}_n(\zeta) \leq \vartheta_n(\lambda_n(\zeta))$.

For $n \in \{2, 3\}$, Schmidt and Summerer settled Conjecture 1.1 in [22] and [23], see also German and Moshchevitin [9]. For $n \geq 4$, it is open. As mentioned above, equality holds for suitable $\underline{\zeta}$, so Conjecture 1.1 would lead to sharp bounds.

2. Structural Study of the Regular Graph

2.1. Fixed λ

In this short subsection, let $\lambda > 0$ be given. We investigate constants $\lambda_{n,j}$ in the regular graph for prescribed value $\lambda_n = \lambda$ in dependence of *n*, for which, obviously, it is necessary and sufficient to assume $n \geq \lceil \lambda^{-1} \rceil$. Recall the notations $\lambda_{n,j}(\lambda)$ and $\hat{\lambda}_{n,j}(\lambda)$ for the constants $\lambda_{n,j}$, $\hat{\lambda}_{n,j}$ obtained in the regular graph in dimension n and the parameter $\lambda_{n,1} = \lambda_n = \lambda$. Our first result shows, roughly speaking, that for fixed $\lambda_n = \lambda$, the remaining constants $\lambda_{n,j}(\lambda)$ for fixed $j \geq 2$ are decreasing as the dimension *n* increases.

Proposition 2.1. *Let* $\lambda > 0$ *be fixed and* $n_1 > n_2 \geq j - 1 \geq 1$ *be integers such that* $n_2 \geq \lceil \lambda^{-1} \rceil$. *Then the constants* $\lambda_{n_i,j}(\lambda)$, $i \in \{1, 2\}$ *in the regular graph in dimensions* n_1 *and* n_2, *respectively*, *and parameter* λ *are well-defined and satisfy* $\lambda_{n_1,j}(\lambda) < \lambda_{n_2,j}(\lambda)$.

Remark 1. The proposition can be used to obtain the following. Consider the regular graphs in some fixed dimension $n \geq 2$ and let the parameter λ tend to infinity. Then we have the asymptotic behaviour

$$\lim_{\lambda \to \infty} \lambda + 1 - \frac{\lambda}{\hat{\lambda}_n(\lambda)} = 0, \tag{14}$$

with $\lambda_n = \lambda$ and $\hat{\lambda}_n(\lambda)$ as in Subsection 1.3. The formula (14) was remarked but not proved in [19]. Observe that (14), in particular, yields

$$\lim_{\lambda\to\infty} \hat{\lambda}_n(\lambda) = \lim_{\lambda\to\infty} \lambda_{n,2}(\lambda) = 1.$$

This property can be roughly seen in Figure 1.

Corollary 2.2. *Let* $j \geq 2$ *be an integer and* $\lambda > 0$ *be a fixed parameter. Consider the regular graphs in all dimensions* $n \geq \lceil \lambda^{-1} \rceil$ *with* $\lambda_n = \lambda$ *as in* (13), *which are well-defined. Then we have*

$$\lambda_{n,j}(\lambda) \geq \frac{\lambda}{(1+\lambda)^{j-1}}$$

and the asymptotic behavior

$$\lim_{n\to\infty} \hat{\lambda}_{n,j-1}(\lambda) = \lim_{n\to\infty} \lambda_{n,j}(\lambda) = \frac{\lambda}{(1+\lambda)^{j-1}}.$$

2.2. Fixed n and Schmidt's conjecture

First we state a (by now settled) conjecture of Schmidt. Recall the simultaneous approximation problem from Subsection 1.2 can be interpreted as a successive minima problem of a parametrized family of convex bodies with respect to a lattice. Schmidt conjectured that for any integers $1 \leq T \leq n-1$ there exist vectors $\underline{\zeta}$ that are $\mathbb{Q}$-linearly independent together with $\{1\}$, and for which the corresponding Tth successive minimum tends to 0 whereas the $(T+2)$nd tends to infinity. In the language of Subsection 1.2, this means precisely that the function $L_{n,T}(q)$ tends to $-\infty$ whereas $L_{n,T+2}(q)$ tends to $+\infty$ as $q \to \infty$. For convenience, we introduce some notation.

Definition 2. Let n, T be integers with $1 \leq T \leq n-1$. We say $\underline{\zeta} \in \mathbb{R}^n$ satisfies Schmidt's property for (n, T) if $\underline{\zeta}$ is $\mathbb{Q}$-linearly independent together with $\{1\}$ and the induced functions $L_{n,j}$ from Subsection 1.2 satisfy $\lim_{q\to\infty} L_{n,T}(q) = -\infty$ and $\lim_{q\to\infty} L_{n,T+2}(q) = \infty$.

So Schmidt's conjecture claims that for any reasonable pair (n, T), the set of vectors that satisfy Schmidt's property is non-empty. The conjecture was proved in a complicated non-constructive way by Moshchevitin [14]. In case of T not too close to n, where the condition $T < n/\log n$ is sufficient, it was reproved constructively in [19]. We should remark that the modified Schmidt property for $L_{n,T}$ and $L_{n,T+1}$ instead of $L_{n,T}$ and $L_{n,T+2}$ cannot be satisfied if $\underline{\zeta}$ is $\mathbb{Q}$-linearly independent together with $\{1\}$. Indeed, it must fail since then $L_{n,j}(q) = L_{n,j+1}(q)$ has arbitrarily large solutions q for any $1 \leq j \leq n$, see [21, Theorem 1.1]. On the other hand, if one drops the linear independence condition, then the conjecture would be true as well by a rather easy argument, as carried out in [14].

By (5), a sufficient condition for a vector to satisfy Schmidt's conjecture is given by $\overline{\psi}_{n,T} < 0 < \underline{\psi}_{n,T+2}$. In view of (6), that is in turn equivalent to $\lambda_{n,T+2} < 1/n < \hat{\lambda}_{n,T}$. In this context, recall that for the regular graph we have $\hat{\lambda}_{n,T} = \lambda_{n,T+1}$. We will investigate below how the quantities $\lambda_{n,j}$ for the regular graph in fixed dimension n depends on the parameter $\lambda \geq 1/n$. Concretely, when we ask for the largest index j such that $\lambda_{n,j}$ is larger than $1/n$ in such intervals, then the above correspondence indicates the close connection to Schmidt's conjecture. Indeed, Theorem 2.6 will provide the link. We start with an easy but important preparatory observation.

Proposition 2.3. *Let* $n \geq 2$ *and* $1 \leq j \leq n+2$. *Then the quantities* $\lambda_{n,j}(\lambda) = \hat{\lambda}_{n,j-1}(\lambda)$ *for the regular graph in dimension n with parameter* λ *satisfy*

$$\frac{\lambda}{(1+\lambda)^{j-1}} \leq \lambda_{n,j}(\lambda) \leq \lambda^{2-j}, \quad \lambda \in [1/n, \infty].$$

In particular, if $j \geq 3$, *then* $\lambda_{n,j}(\lambda)$ *tends to* 0 *as* λ *tends to infinity.*

Proof. The left inequality was already established in Corollary 2.2. For the right estimate, observe $\hat{\lambda}_n(\lambda) = \lambda_{n,2}(\lambda) \leq 1$ always holds by (3). Together with the constant quotients property (11), we have $\lambda_{n,j}(\lambda) = \hat{\lambda}_n(\hat{\lambda}_n/\lambda)^{j-2} \leq \lambda^{2-j}$, which clearly tends to 0 for $j \geq 3$ as $\lambda \to \infty$. □

In particular, $\lambda_{n,j}(\lambda) \sim \lambda^{2-j}$ for $1 \leq j \leq n+2$ as $\lambda \to \infty$. Dually, if we denote by $w_{n,j}(w)$ the constants $w_{n,j}$ for the regular graph for the parameter $w_{n,1} = w$, then with (8) we deduce $w_{n,j}(w) \sim w^{(n-j+1)/n}$ as $w \to \infty$ for $1 \leq j \leq n+2$. The next theorem provides more detailed information on the functions $\lambda_{n,j}(\lambda)$ in (13).

Theorem 2.4. *Let $j \geq 3$ and $n \geq j-2$ be integers. If $n \geq 2j-2$, then there exist $\widetilde{\lambda} \in (1/n, n)$ with the following properties. The regular graph in dimension n with parameter λ satisfies $\lambda_{n,j}(\lambda) > 1/n$ for $\lambda \in (1/n, \widetilde{\lambda})$, $\lambda_{n,j}(\lambda) = 1/n$ for $\lambda \in \{1/n, \widetilde{\lambda}\}$ and $\lambda_{n,j}(\lambda) < 1/n$ for $\lambda \in (\widetilde{\lambda}, \infty]$. If on the other hand $n \leq 2j-3$, then for all $\lambda \in (1/n, \infty]$ the regular graph in dimension n with parameter λ satisfies $\lambda_{n,j}(\lambda) < 1/n$.*

It is easy to check the following consequence of Theorem 2.4.

Corollary 2.5. *Precisely in case of $n \leq 3$ none of the functions $\lambda_{n,j}(\lambda) - 1/n$ changes sign on $\lambda \in (1/n, \infty)$.*

The claims of Theorem 2.4 and Corollary 2.5 are (to some degree) visible in Figure 3 for $n = 8$.

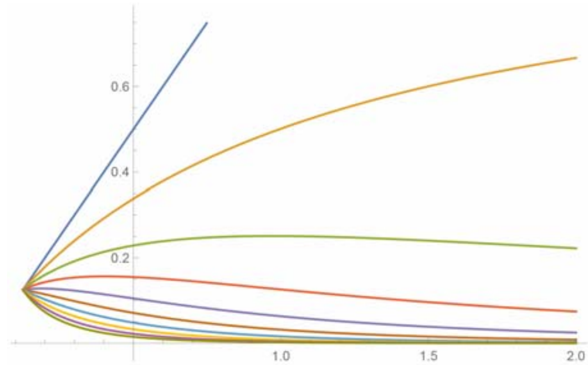

Figure 3. The functions $\lambda_{8,1}(\lambda), \ldots, \lambda_{8,10}(\lambda)$ in the interval $\lambda \in [1/8, 2]$.

Remark 2. For $j \in \{1, 2\}$ and $n \geq 2$, clearly we have $\lambda_{n,j}(\lambda) > 1/n$ for all $\lambda \in (1/n, \infty]$ by (3) and (4), with equality in both inequalities only for $\lambda = 1/n$. See also Proposition 2.1. A similar dual argument shows $\lambda_{n,j}(\lambda) < 1/n$ for $j \in \{n+1, n+2\}$, as we will carry out in the proof. In particular, for $n = 2$ it is clear that $\lambda_{2,1}(\lambda) > \lambda_{2,2}(\lambda) > 1/2 > \lambda_{n,3}(\lambda) > \lambda_{n,4}(\lambda)$ for all $\lambda > 1/2$, and it can be shown easily that all functions $\lambda_{2,i}(\lambda)$ are monotonic on $[1/n, \infty]$, see also Figure 2. On the other hand, for $n = 3$ the above argument is already too weak to imply $\lambda_{3,3}(\lambda) < 1/3$ for all $\lambda > 1/3$, as Theorem 2.4 does.

Moreover, it should be true that the derivative of $\lambda_{n,j}(\lambda)$ with respect to the parameter λ changes sign at most once, and precisely for $3 \leq j < \frac{n+3}{2}$, somewhere in the interval $(1/n, \tilde{\lambda})$ with $\tilde{\lambda}$ from Theorem 2.4. However, we omit a most likely cumbersome proof.

From Theorem 2.4, it is not hard to deduce explicit examples for Schmidt's property if T does not exceed roughly $n/2$.

Theorem 2.6. *Let* $n \geq 2$ *be an integer. Then for any* $1 \leq T \leq \lfloor n/2 \rfloor$ *there exists a non-empty subinterval* $I = I(T)$ *of* $(1/n, n)$ *such that for all*

$\lambda \in I$ the regular graph in dimension n with parameter λ satisfies

$$\hat{\lambda}_{n,T}(\lambda) > \frac{1}{n}, \quad \lambda_{n,T+2}(\lambda) < \frac{1}{n}.$$

In other words, for any pair (n, T) with $1 \leq T \leq \lfloor n/2 \rfloor$ there exist $\underline{\zeta}$ that induce the regular graph and satisfy Schmidt's property for (n, T). For $T > \lfloor n/2 \rfloor$ such $\underline{\zeta}$ does not exist.

Proof. First let $3 \leq j \leq \lfloor n/2 \rfloor + 1$. Then the first case of Theorem 2.4 applies and yields $\lambda_{n,j}(\widetilde{\lambda}) = 1/n$ and $\lambda_{n,j}(t) > 1/n$ for some $\widetilde{\lambda} > 1/n$ and $t \in (1/n, \widetilde{\lambda})$. Since $\lambda_{n,j+1} < \lambda_{n,j}$ unless both are equal to $\lambda = 1/n$, we have $\lambda_{n,j+1}(\widetilde{\lambda}) < 1/n$. Hence, by continuity of the function $\lambda_{n,j+1}(\lambda)$ in the parameter λ, there exists some non-empty interval $J = J(j) = (\delta - \varepsilon, \delta)$ such that for $t_0 \in J$ the inequalities $\lambda_{n,j+1}(t_0) < 1/n < \lambda_{n,j}(t_0)$ are satisfied. Since in the regular graph $\hat{\lambda}_{n,j-1} = \lambda_{n,j}$ holds by (12), the claim follows for $T \geq 2$ with $T = j - 1$, and the fact that $\underline{\zeta}$ inducing the corresponding regular graphs exist as mentioned above. For $T = 1$, a very similar argument applies with $j = 2$. We may take any value λ sufficiently large that $\lambda_{n,3}(\lambda) - 1/n < 0$, observing $\lambda_{n,2}(\lambda) > 1/n$ for $\lambda > 1/n$ but $\lambda_{n,3}(\lambda) - 1/n$ changes sign somewhere in $(1/n, n)$. Finally, concerning the claim for $T > \lfloor n/2 \rfloor$, suitable $\underline{\zeta}$ cannot exist, since $\lambda_{n,T+1}(\lambda) = \hat{\lambda}_{n,T}(\lambda) < 1/n$ for all $\lambda > 1/n$ by the last claim of Theorem 2.4. □

Remark 3. For $T < n/e$, the claim concerning Schmidt's property could be derived directly from Proposition 2.3 instead of the deeper Theorem 2.4, where e is Euler's number.

3. Implications of Conjecture 1.1 for Uniform Approximation

In this section, we restrict to the case of successive powers $(\zeta, \zeta^2, \ldots, \zeta^n)$. We will write $w_{n,j}(\zeta)$ for $w_{n,j}(\zeta, \zeta^2, \ldots, \zeta^n)$ and similarly for $\hat{w}_{n,j}(\zeta), \lambda_{n,j}(\zeta), \hat{\lambda}_{n,j}(\zeta)$. We will also consider related constants connected to approximation by algebraic numbers. For a given real number ζ, let $w_n^*(\zeta)$ be the supremum of ν such that

$$0 < |\zeta - \alpha| \leq H(\alpha)^{-\nu-1}$$

has infinitely many real algebraic solutions α of degree at most n. Here $H(\alpha)$ is the height of the irreducible minimal polynomial P of α over $\mathbb{Z}[X]$, which is the maximum modulus among its coefficients. Similarly, let the uniform constant $\hat{w}_n^*(\zeta)$ be the supremum of real ν for which the system

$$H(\alpha) \leq X, \quad 0 < |\zeta - \alpha| \leq H(\alpha)^{-1} X^{-\nu}$$

has a solution as above for all large values of X. For all $n \geq 1$ and all real ζ, the estimates

$$w_n^*(\zeta) \leq w_n(\zeta) \leq w_n^*(\zeta) + n - 1, \quad \hat{w}_n^*(\zeta) \leq \hat{w}_n(\zeta) \leq \hat{w}_n^*(\zeta) + n - 1 \tag{15}$$

are well-known, see [2, Lemma A8]. We aim to establish a conditional improvement of the known upper bound for the exponents $\hat{w}_n(\zeta), \hat{w}_n^*(\zeta)$ valid for all transcendental real ζ, under the assumption of Conjecture 1.1. The bound $\hat{w}_n(\zeta) \leq 2n - 1$ was given by Davenport and Schmidt [7]. This has recently been refined in [6, Theorem 2.1] to

$$\hat{w}_n(\zeta) \leq n - \frac{1}{2} + \sqrt{n^2 - 2n + \frac{5}{4}}. \tag{16}$$

For large n, the right hand side in (16) is of order $2n - 3/2 + o(1)$. For $n = 3$, the stronger estimate

$$\hat{w}_3(\zeta) \leq 3 + \sqrt{2} \approx 4.4142... \tag{17}$$

was established in [6, Theorem 2.1]. For $n = 2$, the bound in (16) is best possible as proved by Roy, see [17]. Our main result of this section is the following asymptotic estimation, conditioned on Conjecture 1.1.

Theorem 3.1. *Suppose Conjecture* 1.1 *holds for every* $n \geq 2$. *Let* $\tau \approx 0.5693$ *be the solution* $y \in (0, 1)$ *of* $ye^{1/y} = 2\sqrt{e}$, *where e is Euler's number*, *and put* $\Delta := \log(2/\tau) + 1 \approx 2.2564$. *Then for any* $\varepsilon > 0$ *there exists* $n_0 = n_0(\varepsilon)$ *such that for all real transcendental numbers* ζ *we have*

$$\hat{w}_n^*(\zeta) \leq 2n - \Delta + \varepsilon, \quad n \geq n_0. \tag{18}$$

The same bound holds for $\hat{w}_n(\zeta)$ *unless* $w_{n-2}(\zeta) < w_{n-1}(\zeta) = w_n(\zeta)$. *In any case*, *we have*

$$\hat{w}_n(\zeta) \leq 2n - 2, \quad n \geq 10. \tag{19}$$

Furthermore, in Subsection 5.3, we will derive conditioned concrete upper bounds for $\hat{w}_n(\zeta)$, $\hat{w}_n^*(\zeta)$ for certain values of *n*, see (52). We close this section with another related result, whose proof will be omitted as it is very similar to that of Theorem 3.1. Assume that the estimate

$$\hat{w}_n \leq n^{\frac{1}{n+1}} w_n^{\frac{n}{n+1}} \tag{20}$$

is satisfied. Then for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that

$$\hat{w}_n(\zeta) \leq 2n - 1 - \log 2 + \varepsilon, \quad n \geq n_0. \tag{21}$$

Observe that (21) is still stronger than (16), although it is weaker than (18). On the other hand, we will see in Subsection 5.1 that the involved assumption (20) is reasonably weaker than the assumption of Conjecture 1.1 in Theorem 3.1.

4. Conditioned Results under Assumption of Another Conjecture

4.1. Uniform approximation

Let $n \geq 1$ be an integer and ζ be a real number. We call $P \in \mathbb{Z}[X]$ of degree at most n a *best approximation* for (n, ζ) if there is no $Q \in \mathbb{Z}[X]$ of degree at most n with strictly smaller height $H(Q) < H(P)$ that satisfies $|Q(\zeta)| < |P(\zeta)|$. Obviously, every real transcendental ζ induces a sequence of best approximation polynomials $P_1, P_2, \ldots$ with $|P_1(\zeta)| > |P_2(\zeta)| > \cdots$ and $H(P_1) \leq H(P_2) \leq \cdots$. Similarly, for $\underline{\zeta} = (\zeta_1, \ldots, \zeta_n)$ define the best approximations for $(n, \underline{\zeta})$ for the linear forms in $\underline{\zeta}$.

Conjecture 4.1. For any $n \geq 1$ and any real transcendental ζ, there exist infinitely many k such that $n + 1$ successive best approximations $P_k, P_{k+1}, \ldots, P_{k+n}$ for (n, ζ) are linearly independent (i.e., the coefficient vectors span the entire space $\mathbb{R}^{n+1}$).

Remark 4. The claim is known to hold for $n = 2$. More generally, for any n there are three linearly independent consecutive best approximations infinitely often, see [21]. On the other hand, Moshchevitin [13] proved the existence of counterexamples for the analogous claim for vectors $\underline{\zeta} \in \mathbb{R}^n$ that are $\mathbb{Q}$-linearly independent together with $\{1\}$, for $n > 2$. Vectors can even be chosen such that the $(n+1) \times (n+1)$-matrix whose columns are formed by $n + 1$ successive best approximation vectors has rank at most 3 for all large k. However, it seems plausible that such vectors cannot lie on the Veronese curve.

Theorem 4.2. *For any $n \geq 2$ and any real vector $\underline{\zeta}$ linearly independent over $\mathbb{Q}$ together with $\{1\}$, we have*

$$w_{n,3}(\underline{\zeta}) \geq \frac{\hat{w}_n(\underline{\zeta})^2}{w_n(\underline{\zeta})}. \tag{22}$$

If $(n, \underline{\zeta})$ *satisfies the assumption of Conjecture* 4.1, *then*

$$w_{n,i}(\underline{\zeta}) \geq \frac{\hat{w}_n(\underline{\zeta})^{i-1}}{w_n(\underline{\zeta})^{i-2}}, \quad 1 \leq i \leq n+1 \tag{23}$$

and

$$w_n(\underline{\zeta}) \geq \hat{w}_n(\underline{\zeta})\left(\frac{\hat{w}_n(\underline{\zeta}) - 1}{n-1}\right)^{\frac{1}{n-1}}. \tag{24}$$

Analogous claims of (22) *and* (23) *hold for* $\lambda_{n,j}, \hat{\lambda}_{n,j}$ *with respect to the obvious dual definition of the best approximations and Conjecture* 4.1, *and* (24) *has to be replaced by*

$$\lambda_n(\underline{\zeta}) \geq \hat{\lambda}_n(\underline{\zeta}) \cdot \left(\frac{(n-1)\hat{\lambda}_n(\underline{\zeta})}{1 - \hat{\lambda}_n(\underline{\zeta})}\right)^{\frac{1}{n-1}}. \tag{25}$$

For $n = 2$, the estimate (24) is unconditioned by Remark 4 and yields the inequality $w_2(\underline{\zeta}) \geq \hat{w}_2(\underline{\zeta})(\hat{w}_2(\underline{\zeta}) - 1)$ known by Laurent [11]. There is equality in all inequalities of Theorem 4.2 for (ζ, ζ^2) when ζ is an extremal number defined by Roy, see for example [17]. See also Moshchevitin [15, Section 3] for results related to (24) and (25). For us, the main purpose of Theorem 4.2 is the connection to uniform approximation, portrayed in the following theorem.

Theorem 4.3. *Assume Conjecture* 4.1 *is true*. *Then* (21) *holds*.

5. Proofs

5.1. Preliminary results

In this section, we establish several identities involving the exponents $\lambda_{n,j}, \hat{\lambda}_{n,j}, w_{n,j}, \hat{w}_{n,j}$ in the regular graph, to prepare the proofs of the main results. They are essentially derived by algebraic rearrangements of the identity

$$\frac{(\lambda_n + 1)^{n+1}}{\lambda_n} = \frac{(\hat{\lambda}_{n,n+1} + 1)^{n+1}}{\hat{\lambda}_{n,n+1}}, \tag{26}$$

which was proved in [19, (95) in Section 3]. In view of (26), we define the auxiliary functions

$$f_n(x) := \frac{(1+x)^{n+1}}{x}. \tag{27}$$

It is easily verified that f_n decays on $(0, 1/n)$ and increases on $(1/n, \infty)$. Hence, we see that for given $\lambda_n \in [1/n, \infty]$, the constant $\hat{\lambda}_{n,n+1}$ is the unique solution of (26) in the interval $[0, 1/n]$. Observe that by (26) and the constant quotients (11), the constants $\lambda_n = \lambda$ and $\lambda_{n,j}(\lambda)$ satisfy the implicit equation

$$\frac{(1+\lambda)^{n+1}}{\lambda} = \frac{\left(1 + \lambda^{1-\frac{n+1}{j-1}}\lambda_{n,j}(\lambda)^{\frac{n+1}{j-1}}\right)^{n+1}}{\lambda^{1-\frac{n+1}{j-1}}\lambda_{n,j}(\lambda)^{\frac{n+1}{j-1}}}. \tag{28}$$

Moreover, from (26) and (11) we infer

$$\hat{\lambda}_n = \lambda_n^{\frac{n}{n+1}}\hat{\lambda}_{n,n+1}^{\frac{1}{n+1}} = \lambda_n\left(\frac{\hat{\lambda}_{n,n+1}}{\lambda_n}\right)^{\frac{1}{n+1}}. \tag{29}$$

By combining (26) with (29), after some rearrangements, we derive an implicit polynomial equation involving λ_n and $\hat{\lambda}_n$ of the form

$$(\hat{\lambda}_n - 1)\lambda_n^n + \hat{\lambda}_n\lambda_n^{n-1} - \hat{\lambda}_n^{n+1} = 0, \tag{30}$$

where in the special case $\lambda_n = \infty$ we have to put $\hat{\lambda}_n = 1$. Noticing that $\hat{\lambda}_n = \lambda_n$ is a solution of (30) not of interest, we can decrease the degree by one

$$\lambda_n^{n-1}(\hat{\lambda}_n - 1) + \hat{\lambda}_n^2 \frac{\lambda_n^{n-1} - \hat{\lambda}_n^{n-1}}{\lambda_n - \hat{\lambda}_n}$$

$$= \lambda_n^{n-1}(\hat{\lambda}_n - 1) + \hat{\lambda}_n^2(\lambda_n^{n-2} + \lambda_n^{n-3}\hat{\lambda}_n + \cdots + \hat{\lambda}_n^{n-2}) = 0.$$

Now we want to establish the dual results. One can either proceed similarly as in [19] for (26), or immediately apply (8) to (26), to derive

$$\frac{(1 + w_n)^{n+1}}{w_n^n} = \frac{(1 + \hat{w}_{n,n+1})^{n+1}}{\hat{w}_{n,n+1}^n} \tag{31}$$

for the regular graph. Observe that $\hat{w}_{n,n+1} = 1/\lambda_n \in [0, n]$ by (8) and (2), whereas $w_n \in [n, \infty]$ by (7). In particular, it is not hard to see that for given $w_n \in [n, \infty]$ the approximation constant $\hat{w}_{n,n+1}$ is the unique real solution of (31) in the interval $[0, n]$. Moreover, again for the regular graph all quotients $w_{n,j}/w_{n,j+1} = w_{n,j}/\hat{w}_{n,j}$ coincide for $1 \leq j \leq n+1$, where we put $w_{n,n+2} := \hat{w}_{n,n+1}$. This yields

$$\hat{w}_n = w_n^{\frac{n}{n+1}} \hat{w}_{n,n+1}^{\frac{1}{n+1}} = w_n \left(\frac{\hat{w}_{n,n+1}}{w_n} \right)^{\frac{1}{n+1}}. \tag{32}$$

From (32) and the most right inequality of (10), we obtain (20), where equality holds only in case of $\hat{w}_{n,n+1} = n$ or equivalently $w_n = n$. Expressing $\hat{w}_n$ in terms of w_n, $\hat{w}_n$ by rearranging (32) and inserting in (31), some further rearrangements lead to the nice implicit equation

$$w_n - \hat{w}_n + 1 = \left(\frac{w_n}{\hat{w}_n} \right)^n. \tag{33}$$

We summarize the above observations in a proposition.

Proposition 5.1. *The function* ϕ_n *coincides with the unique solution of* $\hat{w}_n$ *in* (33) *in terms of* w_n *in the interval* $[n, w_n)$, *unless* $w_n = \phi_n(w_n) =$

$\hat{w}_n = n.$ *The function* ϑ_n *coincides with the unique solution of* $\hat{\lambda}_n$ *in* (30) *in terms of* λ_n *in the interval* $[1/n, \lambda_n)$, *unless* $\lambda_n = \vartheta_n(\lambda_n) = \hat{\lambda}_n = 1/n.$

Proof. The asserted uniqueness can be easily proved. It has been established that (33) and (30) are satisfied and the claim on the intervals follows from (2) and (7). □

We remark that similarly to (28), one can obtain an implicit equation involving w_n and $w_{n,j} = \hat{w}_{n,j-1}$ for $2 \leq j \leq n+2,$ and dual interpretations of Theorem 2.4 and Remark 2 provide some information on the monotonicity of the functions $w_{n,j}$ in dependence of w_n. We do not carry this out.

5.2. Proofs of Section 2

For the first proof, recall the functions f_n from (27) and their properties.

Proof of Proposition 2.1. By the assumptions, the regular graphs with parameter λ in dimensions n_1, n_2 are well-defined (and exist due to Roy [16]). Since in the regular graph the quotients (11) coincide, it suffices to prove that $\lambda_{n,2}(\lambda) = \hat{\lambda}_n(\lambda)$ decreases for fixed λ as n increases.

Recall the functions f_n defined in Section 2. We have $f_{n+1}(\lambda)/f_n(\lambda) = 1 + \lambda$ and hence in view of (26) also

$$\frac{f_{n+1}(\hat{\lambda}_{n+1,n+2}(\lambda))}{f_n(\hat{\lambda}_{n,n+1}(\lambda))} = 1 + \lambda. \tag{34}$$

On the other hand, we claim that

$$\hat{\lambda}_{n+1,n+2}(\lambda) < \hat{\lambda}_{n,n+1}(\lambda). \tag{35}$$

In case of $\hat{\lambda}_{n,n+1}(\lambda) > 1/(n+1),$ this is trivial since $\hat{\lambda}_{n+1,n+2}(\lambda) \leq 1/(n+1).$ If otherwise $\hat{\lambda}_{n,n+1}(\lambda) \leq 1/(n+1),$ then (35) follows from the decay of the function f_{n+1} on $(0, 1/(n+1))$ and

$$f_{n+1}(\hat{\lambda}_{n,n+1}(\lambda))/f_n(\hat{\lambda}_{n,n+1}(\lambda)) = 1 + \hat{\lambda}_{n,n+1}(\lambda) \leq 1 + \lambda,$$

in combination with (34). From (35), we deduce

$$\begin{aligned}(1 + \hat{\lambda}_{n+1,n+2}(\lambda))^{n+2} &\leq (1 + \hat{\lambda}_{n,n+1}(\lambda))^{n+2} \\ &= (1 + \hat{\lambda}_{n,n+1}(\lambda))^{n+1} \frac{f_{n+1}(\hat{\lambda}_{n,n+1}(\lambda))}{f_n(\hat{\lambda}_{n,n+1}(\lambda))}.\end{aligned}$$

Observe the left and middle quantities are the nominators of $f_{n+1}(\hat{\lambda}_{n,n+1}(\lambda))$ and $f_{n+1}(\hat{\lambda}_{n+1,n+2}(\lambda))$, respectively. Together with (34), we infer

$$\frac{\hat{\lambda}_{n,n+1}(\lambda)}{\hat{\lambda}_{n+1,n+2}(\lambda)} > \frac{1+\lambda}{\hat{\lambda}_{n,n+1}(\lambda)}. \tag{36}$$

The identities (29) for $n, n+1$ yield

$$\hat{\lambda}_n(\lambda) = \lambda^{n/(n+1)} \hat{\lambda}_{n,n+1}(\lambda)^{1/(n+1)},$$

$$\hat{\lambda}_{n+1}(\lambda) = \lambda^{(n+1)/(n+2)} \hat{\lambda}_{n+1,n+2}(\lambda)^{1/(n+2)}.$$

Taking quotients, with

$$(n+1)/(n+2) - n/(n+1) = 1/(n+1) - 1/(n+2) = (n+1)^{-1}(n+2)^{-1},$$

we get

$$\frac{\hat{\lambda}_n(\lambda)}{\hat{\lambda}_{n+1}(\lambda)} \geq \lambda^{-\frac{1}{(n+1)(n+2)}} \hat{\lambda}_{n,n+1}(\lambda)^{\frac{1}{(n+1)(n+2)}} \left(\frac{\hat{\lambda}_{n,n+1}(\lambda)}{\hat{\lambda}_{n+1,n+2}(\lambda)} \right)^{\frac{1}{n+2}}.$$

Inserting the bound (36), for the last expression we obtain

$$\frac{\hat{\lambda}_n(\lambda)}{\hat{\lambda}_{n+1}(\lambda)} \geq \lambda^{-\frac{1}{(n+1)(n+2)}} \hat{\lambda}_{n,n+1}(\lambda)^{\frac{1}{(n+1)(n+2)}} \left(\frac{1+\lambda}{\hat{\lambda}_{n,n+1}(\lambda)} \right)^{\frac{1}{n+2}}. \tag{37}$$

One readily checks that the right hand side in (37) equals 1, since this is equivalent to $f_k(\lambda) = f_k(\hat{\lambda}_{n,n+1}(\lambda))$, which is (26). This finishes the proof.

□

Proof of Corollary 2.2. It was shown in [21, Proposition 5] that we have $\hat{\lambda}_n(\lambda)/\lambda > (\lambda + 1)^{-1}$ in the regular graph with parameter $\lambda_n = \lambda$. On the other hand, the quotients $\lambda_{n,j}/\lambda_{n,j+1}$ are identical for all $1 \le j \le n+1$ by (11). Hence

$$\hat{\lambda}_{n,j-1}(\lambda) = \lambda_{n,j}(\lambda) = \lambda\left(\frac{\hat{\lambda}_n(\lambda)}{\lambda}\right)^{j-1} \ge \frac{\lambda}{(1+\lambda)^{j-1}}.$$

In Proposition 2.1, we proved that the values $\hat{\lambda}_{n,j-1}(\lambda) = \lambda_{n,j}(\lambda)$ decay as n increases, hence the limit of $\lambda_{n,j}(\lambda)$ as $n \to \infty$ exists and equals at least the given quantity. We have to show equality. Again, as all the quotients $\lambda_{n,j-1}/\lambda_{n,j}$ are identical, it obviously suffices to show this for $j = 2$. For λ, $\hat{\lambda}_n(\lambda)$ as above, define $\alpha(n)$ implicitly by

$$\hat{\lambda}_n(\lambda) = \alpha(n)\frac{\lambda}{1+\lambda}. \tag{38}$$

Then the sequence $\alpha(n) \ge 1$ decreases to some limit at least 1 and we have to show $\lim_{n\to\infty} \alpha(n) = 1$. Observe a rearrangement of (29) and (38) yields

$$\hat{\lambda}_{n,n+1}(\lambda) = \lambda\left(\frac{\hat{\lambda}_n(\lambda)}{\lambda}\right)^{n+1} = \lambda\left(\frac{\alpha(n)}{1+\lambda}\right)^{n+1}.$$

Inserting the right hand side in the identity (26), elementary rearrangements lead to

$$\alpha(n) = 1 + \lambda\left(\frac{\alpha(n)}{1+\lambda}\right)^{n+1}. \tag{39}$$

If we had $\lim_{n\to\infty} \alpha(n) \ge \lambda + 1$, then $\hat{\lambda}_n(\lambda) \ge \lambda = \lambda_n(\lambda)$, a contradiction.

Thus, $\lim_{n\to\infty} \alpha(n) < \lambda + 1$. Hence the right hand side of (39) converges to 1 as $n \to \infty$, and thus the left hand side does as well. This completes the proof. $\square$

For the proof of Theorem 2.4, we consider λ in small intervals of the form $(1/n, 1/n + \varepsilon)$.

Proof of Theorem 2.4. Clearly, $\lambda_{n,j} = 1/n$ for all $1 \le j \le n+2$ if $\lambda = 1/n$. Further, observe that by the most left inequalities of (9) and (10), and (8), we have $\lambda_{n,n+1}(\lambda) = \hat{w}_n(\lambda)^{-1} \le 1/n$ and $\lambda_{n,n+2} = w_n(\lambda)^{-1} \le 1/n$. Equality holds only if the quantities equal $1/n$ anyway, where we put $w_n(\lambda)$ for the value w_n induced for the regular graph with parameter $\lambda_{n,1} = \lambda$. Thus, we can restrict to $2 \le j \le n$.

So let $n \ge 1$ and $2 \le j \le n$ be arbitrary but fixed. Write $\lambda_n = \lambda = \alpha/n$ for $\alpha > 1$, where we consider only α slightly larger than 1. Then (28) becomes

$$\frac{\left(1 + \frac{\alpha}{n}\right)^{n+1}}{\frac{\alpha}{n}} = \frac{\left(1 + \left(\frac{\alpha}{n}\right)^{1-\frac{n+1}{j-1}} \lambda_{n,j}\left(\frac{\alpha}{n}\right)^{\frac{n+1}{j-1}}\right)^{n+1}}{\left(\frac{\alpha}{n}\right)^{1-\frac{n+1}{j-1}} \lambda_{n,j}\left(\frac{\alpha}{n}\right)^{\frac{n+1}{j-1}}}. \tag{40}$$

We ask for which values of j it is possible to have $\lambda_{n,j}\left(\frac{\alpha}{n}\right) = 1/n$ for some $\alpha > 1$. So we insert $\lambda_{n,j}\left(\frac{\alpha}{n}\right) = 1/n$ in (40), and rearrange (40) in the following way. We multiply with α/n, then divide by the denominator of the right hand side and take the $(n+1)$st root. After further elementary rearrangements and simplification, we end up with the equivalent identity

$$n = \frac{\alpha - \alpha^{\frac{j-n-1}{j-1}}}{\alpha^{\frac{1}{j-1}} - 1}. \tag{41}$$

Let $\theta := \alpha^{\frac{1}{j-1}}$. Clearly, $\theta > 1$ is equivalent to $\alpha > 1$. Furthermore, (41) is equivalent to

$$n = \theta^{j-1-n}\frac{\theta^n - 1}{\theta - 1} = \theta^{j-2} + \theta^{j-3} + \cdots + \theta^{j-1-n} =: \chi_{n,j}(\theta). \tag{42}$$

By construction, $\chi_{n,j}(1) = n$. First consider $n \le 2j - 3$ or equivalently $j \ge \frac{n+3}{2}$. We calculate

$$\chi'_{n,j}(t) = (j-2)t^{j-3} + (j-3)t^{j-4} + \cdots + (j-n-1)t^{j-n-2}$$

and

$$\chi''_{n,j}(t) = (j-2)(j-3)t^{j-4} + (j-3)(j-4)t^{j-5}$$
$$+ \cdots + (j-1-n)(j-2-n)t^{j-n-3}.$$

It is easy to verify $\chi''_{n,j}(t) > 0$ for all $t > 0$. Indeed, any expression in the sum is non-negative, and for $j \ge 4$ the first and for $j = 3$ the last is strictly positive. Hence it suffices to show $\chi'_{n,j}(1) > 0$ to see that $\chi_{n,j}(t) > n$ for all $t > 1$. Indeed, for $j \ge \frac{n+3}{2}$ we verify

$$\chi'_{n,j}(1) = (j-2) + (j-3) + \cdots + (j-n-1)$$
$$= nj - \sum_{i=2}^{n+1} i = nj - \frac{n^2 + 3n}{2} \ge 0. \tag{43}$$

We conclude $\lambda_{n,j}(\lambda) \ne 1/n$ for all $\lambda > 1/n$. By the continuity of $\lambda_{n,j}$, we must have either $\lambda_{n,j}(\lambda) < 1/n$ for all $\lambda > 1/n$ or $\lambda_{n,j}(\lambda) > 1/n$ for all $\lambda > 1/n$. However, since $j \geq 3$, we can exclude the latter since in Proposition 2.3 we showed

$$\lim_{\lambda \to \infty} \lambda_{n,j}(\lambda) = 0, \quad j \geq 3. \tag{44}$$

We have proved all claims for $j \geq \frac{n+3}{2}$. Now let $j < \frac{n+3}{2}$, which is equivalent to $n \geq 2j - 2$. Then

$$\chi'_{n,j}(1) = nj - \frac{n^2 + 3n}{2} < 0.$$

Hence, since $\chi''_{n,j}(t) > 0$ for all $t > 0$, there exists precisely one value $\mu_0 > 1$ for which $\chi_{n,j}(\mu_0) = n$, or equivalently precisely one $\widetilde{\lambda} > 1/n$ with $\lambda_{n,j}(\widetilde{\lambda}) = 1/n$. Again, by (44) and continuity, we must have $\lambda_{n,j}(\lambda) < 1/n$ for $\lambda > \widetilde{\lambda}$. Moreover, again by intermediate value theorem, either $\lambda_{n,j}(\lambda) > 1/n$ for all $\lambda \in (1/n, \widetilde{\lambda})$ or $\lambda_{n,j}(\lambda) < 1/n$ for all $\lambda \in (1/n, \widetilde{\lambda})$. Suppose conversely to the claim of the theorem, the latter is true. Recall the implicit equation (28) involving $\lambda_n = \lambda$ and $\lambda_{n,j}(\lambda)$. Denote

$$F(x) = \frac{(1+x)^{n+1}}{x}, \quad G(x, y) = \frac{(1+xy)^{n+1}}{xy},$$

such that (28) becomes $F(\lambda) = G(\lambda^{1-\frac{n+1}{j-1}}, \lambda_{n,j}(\lambda)^{1-\frac{n+1}{j-1}})$. Proceeding as above, we will show next that for λ close to $1/n$ we have

$$F(\lambda) = G(\lambda^{1-\frac{n+1}{j-1}}, \lambda_{n,j}(\lambda)^{\frac{n+1}{j-1}}) < G(\lambda^{1-\frac{n+1}{j-1}}, (1/n)^{\frac{n+1}{j-1}}). \tag{45}$$

Observe that with $\lambda = \alpha/n$, inequality (45) is equivalent to

$$n > \frac{\alpha - \alpha^{\frac{j-n-1}{j-1}}}{\alpha^{\frac{1}{j-1}} - 1}. \tag{46}$$

Proceeding as above subsequent to (41), we see that for (46) the condition $\chi'_{n,j}(1) > 0$ is sufficient. We readily verify that for $j < \frac{n+3}{2}$ and α sufficiently close to 1, with a very similar calculation as in (43). Thus, we have shown (45). Hence, if $\lambda_{n,j}(\lambda) < 1/n$ for such λ, then by intermediate value theorem of differentiation, we must have

$$\frac{dG}{dy}(\lambda^{1-\frac{n+1}{j-1}}, \eta) > 0 \tag{47}$$

for some pair (λ, η) with $\lambda \geq 1/n$ and $\eta \in (\lambda_{n,j}(\lambda)^{\frac{n+1}{j-1}}, (1/n)^{\frac{n+1}{j-1}})$. We disprove this. We calculate

$$\frac{dG(x, y)}{dy} = (nxy - 1)(1 + xy)^n \frac{1}{xy^2}.$$

Hence the sign of the partial derivative of G in (47) equals that of $nxy - 1$. Our hypothesis yields

$$n\lambda\eta \leq n\left(\frac{\alpha}{n}\right)^{1-\frac{n+1}{j-1}}\left(\frac{1}{n}\right)^{\frac{n+1}{j-1}} = \alpha^{1-\frac{n+1}{j-1}} < 1$$

since $\alpha > 1$ and the exponent is negative. Hence $dG(\lambda^{1-\frac{n+1}{j-1}}, \eta)/dy < 0$ for all $\eta \in (\lambda_{n,j}(\lambda)^{\frac{n+1}{j-1}}, (1/n)^{\frac{n+1}{j-1}})$. This contradicts (47). Hence the hypothesis was wrong and we must have $\lambda_{n,j}(\lambda) > 1/n$ for all $\lambda \in (1/n, \tilde{\lambda})$.

Finally, the fact that $\tilde{\lambda} < n$ follows from combination of $\lambda_{n,j}(\tilde{\lambda}) = 1/n$ and $\lambda_{n,j}(\tilde{\lambda}) < \tilde{\lambda}^{2-j} \leq \tilde{\lambda}^{-1} < n$ for $\tilde{\lambda} > 1/n$ and $j \geq 3$, see the proof of Proposition 2.3. $\square$

5.3. Proofs of Section 3

We turn towards the proof of Theorem 3.1. We briefly outline a sketch of the proof. The essential tools for the proof of Theorem 3.1 are special cases of [6, Theorems 2.2, 2.3 and 2.4] comprised in Theorem 5.2.

Theorem 5.2 (Bugeaud and Schleischitz)**.** *Let* $n \geq 2$ *and* ζ *be real transcendental. We have*

$$\hat{w}_n^*(\zeta) \leq \frac{nw_n(\zeta)}{w_n(\zeta) - n + 1}. \tag{48}$$

If $w_n(\zeta) > w_{n-1}(\zeta)$, *then we have the stronger estimate*

$$\hat{w}_n(\zeta) \leq \frac{nw_n(\zeta)}{w_n(\zeta) - n + 1}. \tag{49}$$

If otherwise for $m < n$ *we have* $w_m(\zeta) = w_n(\zeta)$, *then*

$$\hat{w}_n(\zeta) \leq m + n - 1 \leq 2n - 2. \tag{50}$$

Throughout assume Conjecture 1.1 holds. Before we prove Theorem 3.1, we want to provide some better numeric results for not too large n. We point out that the functions ϕ_n are increasing. This fact is rather obvious from the definition of the regular graph, we omit a rigorous proof. Let $\widetilde{w}_n(\zeta)$ be the solution of the implicit equation

$$\phi_n(\widetilde{w}_n(\zeta)) = \frac{n\widetilde{w}_n(\zeta)}{\widetilde{w}_n(\zeta) - n + 1}. \tag{51}$$

Since ϕ_n increases whereas the right hand side of (51) decreases, it follows from (15) and Theorem 5.2 that the corresponding value $\phi_n(\widetilde{w}_n(\zeta))$ is an upper bound for $\hat{w}_n^*(\zeta)$, and in case of $\phi_n(\widetilde{w}_n(\zeta)) \geq 2n - 2$ for $\hat{w}_n(\zeta)$ as well. For $n \in \{2, 3\}$, this procedure leads precisely to the bounds $(3 + \sqrt{5})/2$ and $3 + \sqrt{2}$ in (16) and (17), respectively. For $n \geq 4$ not too large, Mathematica can determine a numerical solution of (51). We provide the implied bounds

$$\hat{w}_4(\zeta) < 6.2875, \quad \hat{w}^*_{20}(\zeta) < 37.8787, \quad \hat{w}^*_{50}(\zeta) < 97.7996. \tag{52}$$

Unless ζ satisfies $w_{n-2}(\zeta) < w_{n-1}(\zeta) = w_n(\zeta)$, the above bounds for $n \in \{20, 50\}$ are valid for $\hat{w}_n(\zeta)$ as well, and we believe the additional condition is, in fact, not necessary. The numeric data suggests that $2n - \phi_n(\widetilde{w}_n(\zeta))$ converges to some constant not much larger than the value approximately 0.2004 we compute with the given bound for $n = 50$ above. In view of this indication, Theorem 3.1 is rather satisfactory. Its proof essentially relies on the above idea, along with asymptotic estimates for the values $\phi_n(\widetilde{w}_n(\zeta))$ for large *n*. For these estimates, we will frequently use the well-known fact that

$$\lim_{n\to\infty} (1 + x/n)^n = e^x \tag{53}$$

for real *x*, where the left hand side sequence is monotonic increasing. We shall also use the variation of (53) that for $n \geq 1$, $\theta > 1$ we have

$$\theta^{-1/(n+1)} < \frac{1}{1 + \frac{\log(\theta)}{n+1}} = 1 - \frac{\log(\theta)}{\log(\theta) + n + 1}. \tag{54}$$

Proof of Theorem 3.1. First we show (18). From the assumption of Conjecture 1.1 together with Proposition 5.1 and (15), we obtain

$$\hat{w}^*_n(\zeta) \leq \hat{w}_n(\zeta) \leq \phi_n(w_n(\zeta)). \tag{55}$$

Together with (48), we derive

$$\hat{w}^*_n(\zeta) \leq \min\left\{\frac{nw_n(\zeta)}{w_n(\zeta) - n + 1}, \phi_n(w_n(\zeta))\right\}. \tag{56}$$

Let $D \in (1, \Delta)$ be fixed and consider large *n*, in particular, $n > 3D$. Let

$$\kappa_n := \frac{(2n - D)(n - 1)}{n - D}.$$

First assume $w_n(\zeta) \geq \kappa_n$. Then $nw_n(\zeta)/(w_n(\zeta) - n + 1) \leq 2n - D$ such that (18) follows from (56). Since all ϕ_n are increasing, it only remains to be shown that $\phi_n(\kappa_n) \leq 2n - D$ for large n, to derive (56) in case of $n \leq w_n(\zeta) < \kappa_n$ as well. Hence we may assume $w_n(\zeta) = \kappa_n$. It is easy to check

$$\kappa_n = 2n - 2 + (2 - 2/n)D + O(1/n) = 2n + 2D - 2 + O(1/n). \tag{57}$$

In particular, $\kappa_n = 2n + o(n)$. Let

$$\varphi_n(x) = \frac{(x+1)^{n+1}}{x^n} = (1 + 1/x)^n (1 + x).$$

With (53), we infer

$$\varphi_n(\kappa_n) = \left(1 + \frac{1}{\kappa_n}\right)^n (\kappa_n + 1) = \left(1 + \frac{1}{2n + o(n)}\right)^n (2n + o(n))$$

$$= (2\sqrt{e} + o(1))n.$$

From (31), we further deduce

$$\varphi_n(\hat{w}_{n,n+1}(\zeta)) = \varphi_n(w_n(\zeta)) = \varphi_n(\kappa_n) = (2\sqrt{e} + o(1))n.$$

We noticed preceding the theorem that $\hat{w}_{n,n+1}(\zeta) \leq n$. Thus, if we write $\hat{w}_{n,n+1}(\zeta) = bn$, then $b = b(n) \in [0, 1]$, and again (53) yields that b satisfies $be^{1/b} = 2\sqrt{e} + o(1)$ as $n \to \infty$. This yields $b(n) = \tau + o(1)$ as $n \to \infty$, where $\tau \approx 0.5693$ is the solution $y \in (0, 1)$ to $ye^{1/y} = 2\sqrt{e}$. Together with (57), we infer

$$\phi(\kappa_n) = w_n(\zeta)\left(\frac{\hat{w}_{n,n+1}(\zeta)}{w_n(\zeta)}\right)^{1/(n+1)}$$

$$= (2n + 2D - 2 + o(1))\left(\frac{\tau}{2} + o(1)\right)^{1/(n+1)}. \tag{58}$$

Inserting (54) with $\theta := 2/\tau \approx 3.5128$ in (58) yields

$$\phi(\kappa_n) \leq (2n + 2D - 2 + o(1))\left(1 - \frac{\log(2/\tau + o(1))}{\log(2/\tau + o(1)) + n + 1}\right). \tag{59}$$

One checks that if $D < \Delta = \log(2/\tau) + 1$ and n is large, then the right hand side of (59) is smaller than $2n - D$. To finish the proof of (18), let D tend to Δ.

Now we show the estimates for $\hat{w}_n(\zeta)$. In case of $w_{n-2}(\zeta) = w_n(\zeta)$, from (50) with $m = n - 2$ we derive $\hat{w}_n(\zeta) \leq 2n - 3 < 2n - \Delta$, which proves the claim. In case of $w_{n-1}(\zeta) < w_n(\zeta)$, we may apply (49) and obtain the same bounds for $\hat{w}_n$ as in (56), and can proceed as in the proof of (18). Hence only possibly in case of $w_{n-2}(\zeta) < w_{n-1}(\zeta) = w_n(\zeta)$ the bounds may fail, as asserted. Finally, for (19), we need precise error terms in dependence of n. First observe that (55) and Theorem 5.2 imply

$$\hat{w}_n(\zeta) \leq \min\left\{\max\left\{2n - 2, \frac{nw_n(\zeta)}{w_n(\zeta) - n + 1}\right\}, \phi_n(w_n(\zeta))\right\}. \tag{60}$$

To derive (19), we use (33) directly. With above argument applied to $D = 2$, we see that $w_n(\zeta) \geq 2(n-1)^2/(n-2)$ implies $nw_n(\zeta)/(w_n(\zeta) - n + 1) \leq 2n - 2$. Thus, (60) implies (19). Hence again since ϕ_n are monotonic increasing, it remains to be checked that $\phi_n(w) \leq 2n - 2$ for $n \geq 10$, where $w := 2(n-1)^2/(n-2)$. Let

$$H(x, y) = x - y + 1 - \left(\frac{x}{y}\right)^n.$$

Recall $(w_n(\zeta), \phi_n(w_n(\zeta))) = (w_n(\zeta), \hat{w}_n(\zeta))$ satisfy (33). In particular, $H(w, \phi(w)) = 0$ or $\phi_n(w)$ is the solution $y_0 < w$ of

$$H(w, y_0) = \frac{2(n-1)^2}{n-2} - y_0 + 1 + \left(\frac{2(n-1)^2}{(n-2)y_0}\right)^n = 0.$$

Some elementary calculation shows

$$H(w,\, 2n-2) = \frac{2}{n-2} + 3 - \left(\frac{n-1}{n-2}\right)^{n}$$

$$= \frac{2}{n-2} + 3 - \left(1 + \frac{1}{n-2}\right)^{n-2}\left(1 + \frac{1}{n-2}\right)^{2}.$$

Together with (53) and some computation for small n, the right hand side can be easily checked to be positive for $n \geq 10$. On the other hand, we have

$$\frac{dH}{dy}(w,\, y) = -1 + nw^{n}y^{-n-1},$$

which is positive for any $y < \phi_n(w)$ by (20). Thus, indeed the root $y_0 = \phi_n(w)$ of $H(w,\, y_0) = 0$ must be smaller than $2n-2$. This finishes the proof. □

5.4. Proofs of Section 4

In the proof of Theorem 4.2, we will apply the transference inequality

$$\frac{\hat{w}_n(\underline{\zeta})}{\hat{w}_n(\underline{\zeta}) - n + 1} \geq \hat{\lambda}_n(\underline{\zeta}) \geq \frac{\hat{w}_n(\underline{\zeta}) - 1}{(n-1)\hat{w}_n(\underline{\zeta})} \tag{61}$$

due to German [8], valid for all $n \geq 1$ and $\underline{\zeta} \in \mathbb{R}^n$ that are $\mathbb{Q}$-linearly independent together with $\{1\}$.

Proof of Theorem 4.2. Too keep the notation simple, we restrict to vectors $(\zeta,\, \zeta^2,\, \ldots,\, \zeta^n)$, the proof can be readily generalized to linear forms in arbitrary $\underline{\zeta}$. Let $\varepsilon > 0$. By definition of $\hat{w}_n(\zeta)$, for any sufficiently large k we have

$$|P_{k+1}(\zeta)| < |P_k(\zeta)| < H(P_{k+1})^{-\hat{w}_n(\zeta)+\varepsilon}. \tag{62}$$

On the other hand, it follows from the definitions of $w_n(\zeta)$ and $\hat{w}_n(\zeta)$ that for large l two successive best approximations $P_l,\, P_{l+1}$ satisfy

$\log H(P_{l+1})/\log H(P_l) \leq w_n(\zeta)/\hat{w}_n(\zeta) + \varepsilon$, or equivalently

$$\log H(P_l)/\log H(P_{l+1}) \geq \hat{w}_n(\zeta)/w_n(\zeta) - \tilde{\varepsilon},$$

where $\tilde{\varepsilon}$ tends to 0 as ε does. This same argument applied repeatedly for l from $k+1$ to $k+i-2$ shows that

$$\frac{\log H(P_{k+1})}{\log H(P_{k+i-1})} \geq \left(\frac{\hat{w}_n(\zeta)}{w_n(\zeta)}\right)^{i-2} - \tilde{\varepsilon}_1, \tag{63}$$

for some $\tilde{\varepsilon}_1$ which depends on ε and tends to 0 as ε tends to 0. Combination of (62) and (63) yields

$$-\frac{\log|P_k(\zeta)|}{\log H(P_{k+i-1})} = -\frac{\log|P_k(\zeta)|}{\log H(P_{k+1})} \cdot \frac{\log H(P_{k+1})}{\log H(P_{k+i-1})}$$

$$\geq (\hat{w}_n(\zeta) - \varepsilon)\left(\left(\frac{\hat{w}_n(\zeta)}{w_n(\zeta)}\right)^{i-2} - \tilde{\varepsilon}_1\right).$$

Since $|P_k(\zeta)| > |P_{k+1}(\zeta)| > \cdots > |P_{k+n}(\zeta)|$, we infer that

$$-\frac{\log|P_{k+j}(\zeta)|}{\log H(P_{k+i-1})} \geq \hat{w}_n(\zeta)\left(\frac{\hat{w}_n(\zeta)}{w_n(\zeta)}\right)^{n-1} + \tilde{\varepsilon}_2, \quad 0 \leq j \leq i-1,$$

for some $\tilde{\varepsilon}_2$ which again depends on ε and tends to 0 as ε does. Moreover, by our assumption, we can find arbitrarily large k such that the polynomials $P_k, P_{k+1}, \ldots, P_{k+n}$ are linearly independent. Hence and since $H(P_{k+n}) \geq H(P_{k+j})$ for $0 \leq j \leq n$, we obtain (23) as we may take ε arbitrarily small. The estimate (22) is unconditioned since for $i = 3$ Conjecture 4.1 is unconditioned, see Remark 4.

Finally, (24) follows from (23) with $i = n+1$ combined with

$$w_{n,n+1}(\zeta) = \frac{1}{\hat{\lambda}_n(\zeta)} \leq \frac{(n-1)\hat{w}_n(\zeta)}{\hat{w}_n(\zeta) - 1},$$

by elementary rearrangements. The right above inequality is obtained from (61) by taking reciprocals. The dual estimates for the constants $\lambda_{n,j}, \hat{\lambda}_{n,j}$

are obtained very similarly, where for (25) we applied

$$\lambda_{n,n+1}(\zeta) = \frac{1}{\hat{w}_n(\zeta)} \leq \frac{1-\hat{\lambda}_n(\zeta)}{n-1},$$

where again we used (61). □

Proof of Theorem 4.3. By assumption and Theorem 4.2, inequality (24) holds, which is stronger than (20). As mentioned at the end of Section 3, this estimation in turn implies the claim (21). □

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References

[1] V. I. Bernik and K. Tishchenko, Integral polynomials with an overfall of the coefficient values and Wirsing's problem, Dokl. Akad. Nauk Belarusi 37(5) (1993), 9-11 (in Russian).

[2] Y. Buegaud, Approximation by algebraic numbers, Cambridge Tracts in Mathematics, Cambridge, 2004.

[3] Y. Bugeaud, On simultaneous rational approximation to a real number and its integral powers, Ann. Inst. Fourier (Grenoble) 60(6) (2010), 2165-2182.

[4] Y. Bugeaud and M. Laurent, Exponents in Diophantine approximation, Diophantine Geometry Proceedings, Scuola Normale Superiore Pisa, Ser. CRM, No. 4, 2007, pp. 101-121.

[5] Y. Bugeaud and M. Laurent, Exponents of Diophantine approximation and Sturmian continued fractions, Ann. Inst. Fourier (Grenoble) 55(3) (2005), 773-804.

[6] Y. Bugeaud and J. Schleischitz, On uniform approximation to real numbers, Acta Arith., arXiv:1512.00780 (to appear).

[7] H. Davenport and W. M. Schmidt, Approximation to real numbers by algebraic integers, Acta. Arith. 15 (1969), 393-416.

[8] O. German, On Diophantine exponents and Khintchine's transference principle, Mosc. J. Comb. Number Theory 2 (2012), 22-51.

[9] O. German and N. G. Moshchevitin, A simple proof of Schmidt-Summerer's inequality, Monatsh. Math. 170(3-4) (2013), 361-370.

[10] V. Jarnik, Zum Khintchinschen "Ubertragungssatz", Trav. Inst. Math. Tbilissi 3 (1938), 193-212.

[11] M. Laurent, Exponents of Diophantine approximation in dimension two, Canadian J. Math. 61(1) (2009), 165-189.

[12] K. Mahler, The successive minima in the geometry of numbers and the distinction between algebraic and transcendental numbers, J. Number Theory 22 (1986), 147-160.

[13] N. G. Moshchevitin, Best Diophantine approximations: the phenomenon of degenerate dimension, London Math. Soc. Lecture Note Ser. 338, Cambridge Univ. Press, Cambridge, 2007, pp. 158-182.

[14] N. G. Moshchevitin, Proof of W. M. Schmidt's conjecture concerning successive minima of a lattice, J. London Math. Soc. (2) 86(1) (2012), 129-151.

[15] N. G. Moshchevitin, On some open problems in Diophantine approximation, arXiv:1202.4539.

[16] D. Roy, Construction of points realizing the regular systems of Wolfgang Schmidt and Leonhard Summerer, J. Théor. Nombres Bordeaux 27(2) (2015), 591-603.

[17] D. Roy, On the continued fraction expansion of a class of numbers, Diophantine Approximation, Festschrift for Wolfgang Schmidt, Developments in Math., Vol. 16, H. P. Schlickewei, K. Schmidt and R. Tichy, eds., Springer-Verlag, 2008, pp. 347-361.

[18] J. Schleischitz, Two estimates concerning classical Diophantine approximation constants, Publ. Math. Debrecen 84(3-4) (2014), 415-437.

[19] J. Schleischitz, Diophantine approximation and special Liouville numbers, Comm. Math. 21 (2013), 39-76.

[20] W. M. Schmidt, Open problems in Diophantine approximation, Diophantine approximation and transcendental numbers, Luminy 1982, Progr. Math. 31, Birkhäuser, 1983, pp. 271-289.

[21] W. M. Schmidt and L. Summerer, Parametric geometry of numbers and applications, Acta Arith. 140(1) (2009), 67-91.

[22] W. M. Schmidt and L. Summerer, Diophantine approximation and parametric geometry of numbers, Monatsh. Math. 169(1) (2013), 51-104.

[23] W. M. Schmidt and L. Summerer, Simultaneous approximation to three numbers, Mosc. J. Comb. Number Theory 3(1) (2013), 84-107.

[24] K. Tishchenko, On approximation of real numbers by algebraic numbers of bounded degree, J. Number Theory 123(2) (2007), 290-314.

[25] E. Wirsing, Approximation mit algebraischen Zahlen beschränkten Grades, J. Reine Angew. Math. 206 (1961), 67-77.