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NUMERICAL SOLUTION OF SECOND-ORDER FUZZY DIFFERENTIAL EQUATION OF INTEGER AND FRACTIONAL ORDER USING REPRODUCING KERNEL HILBERT SPACE METHOD TOOLS

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Abstract

This paper presents a numerical solution procedure for solving second-order differential equation of integer and fractional order subject to fuzzy conditions. The procedure is based on the usage of tools of

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reproducing kernel Hilbert space in which every function satisfies the initial fuzzy conditions in the second-order differential equation. The procedure produces solutions of high accuracy and examples are provided to illustrate the effectiveness of the solution procedure. The proposed procedure is flexible and has the potential to be further employed to solve problems involving other levels of order in fractional calculus subject to fuzzy conditions.

1. Introduction

The second-order differential equations of integer order have a variety of applications in areas such as mechanical vibration, electric circuits and signal processing. With the establishment of fractional differential equations around the year 1695 [1, 2], in the realm of the concept of second-order differential equations, the equations include the level of the second-order derivative where the order involves all cases of order α where $1 < \alpha \leq 2$.

With the emergence of fuzzy theory by Zadeh in 1965 [3], there exist many applications involving second-order differential equations and fuzzy values, hence requiring new solution procedure to solve the equation that has been transformed into uncertainty state. Several researchers have focused on the study of fractional calculus with fuzzy concepts [4, 5]. Fractional calculus involving fuzzy theory has found potential applications in several scientific fields such as nanotechnology, engineering, bioengineering and viscoelasticity [1, 6-10]. However, since most fuzzy fractional equations do not have exact solutions, new methods as well as improved classical methods have been developed such as Laplace transforms [11], Euler method [12] and homotopy analysis transform method [13] to obtain approximate solutions. Recently, the reproducing kernel Hilbert space has been used in a variety of applications involving differential equations including first-order differential equations with boundary values [14], linear Volterra integral equations [15, 16], second-order nonlinear oscillators with initial conditions [17], second-order differential equations with boundary values [18], Fredholm and Volterra functional integral equations with initial conditions [19] fourth-order boundary value problems of mixed type integrodifferential equations [20] second-order, two-point fuzzy boundary value problems [21] fuzzy Fredholm-Volterra integrodifferential equations [22] and fuzzy differential equations [23]. In fractional calculus, the reproducing kernel Hilbert space method has also been used to solve fractional integrodifferential equations [24], In general, the reproducing kernel Hilbert space method is a promising tool since the employment of the method in those studies has been proven to be not only efficient but also convenient. However, those studies did not consider fuzzy conditions with fractional derivative. Therefore, this study employs the reproducing kernel Hilbert space method towards finding the numerical solution of the "second-order fuzzy differential equation of integer and fractional order", an equation that includes all derivatives in the level of second-order with at least one set of fuzzy values as the initial conditions.

In this paper, we consider the following second-order differential equation of integer and fractional order subject to fuzzy conditions:

$$D^{c,\alpha}_{z\in[0,1]}p(z) = g(z,\ p(z_0),\ p^{(1)}(z_0)),$$

$$p(0) = z_{F1} = (a_1,\ b_1,\ c_1), \quad p^{(1)}(0) = z_{F2} = (a_2,\ b_2,\ c_2), \tag{1}$$

where $D^{c,\alpha}_{z\in[0,1]}$ is the derivative of order α in the sense of Caputo for $1 < \alpha \leq 2,\ \ z \in [0, 1],\ \ z_{F1}$ and z_{F2} are fuzzy initial conditions and $g(z,\ p(z_0),\ p^{(1)}(z_0))$ is a linear or nonlinear function depending on the nature of the problem.

We propose a new algorithm based on the reproducing kernel Hilbert space method tools to provide a numerical solution for equation (1).

This paper is organized as follows: First, a brief introduction of fractional calculus, fuzzy theory and existing solution methods of fractional calculus is given in Section 1. Basic definitions in fractional calculus, fuzzy theory and the reproducing kernel Hilbert space are presented in Section 2. Section 3 focuses on the theoretical aspects of the solution procedure of the proposed method based on the reproducing kernel Hilbert space. The algorithm for the solution procedure is given in Section 4. Two numerical examples are provided in Section 5 to demonstrate the effectiveness of the algorithm. Finally, the conclusion of the study is given in Section 6.

2. Basic Definitions

This section contains some main definitions related to fractional calculus and fuzzy theory.

2.1. Caputo definitions of fractional derivatives

Caputo definition is one of the most important definitions used in fractional calculus. There are two basic definitions in the sense of Caputo, which can be summarized as follows:

Definition 1 [25, 26]**.** The *left Caputo fractional derivative* is defined as:

$$D^{c,\alpha}_{z\in[a,b]}y(z) = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)}\int_a^z (z-\tau)^{\lceil\alpha\rceil-\alpha-1}y^{(\lceil\alpha\rceil)}(\tau)d\tau. \tag{2}$$

Definition 2 [25, 26]**.** The *right Caputo fractional derivative* is defined as:

$$D^{c,\alpha}_{z\in[a,b]}y(z) = \frac{(-1)^{\lceil\alpha\rceil}}{\Gamma(\lceil\alpha\rceil - \alpha)}\int_z^b (\tau-z)^{\lceil\alpha\rceil-\alpha-1}y^{(\lceil\alpha\rceil)}(\tau)d\tau, \tag{3}$$

where $\lceil\ \rceil$ denotes the largest integer near α, $z > a$ in equation (2) and $z < b$ in equation (3).

2.2. Basic definitions in fuzzy theory

The following are the basic definitions in fuzzy theory which are related to the aim of this paper where the notation R denotes real numbers and R^n_{Fi} denotes n-dimensional fuzzy number. For more details, refer to [3, 28-34].

Definition 3. The *r-cut* of $u_{Fi}(z)$ is the crisp set $[u_{Fi}(z)]$ that contains all elements with degree in $u_{Fi}(z) \geq r$ such that

$$[u_{Fi}(z)]^r = \{z \in R : u_{Fi}(z) \geq r\}, \quad \forall i = 1, 2. \tag{4}$$

For fuzzy number $u_{Fi}(z)$, its r-cut is closed and bounded interval in R and we denote it as

$$[u_{Fi}(z)]^r = [u_{1,1r}(z), u_{1,2r}(z)] \text{ for } i = 1,$$

where

$$u_{1,1r} = \min\{z : z \in [u_F(z)]^r\}$$

and

$$u_{1,2r} = \max\{z : z \in [u_F(z)]^r\} \text{ for each } r \in [0, 1]. \tag{5}$$

To fit the definition of triangular fuzzy number [35] with respect to the conditions in equation (1), we redefine equation (1) as given in the following definition.

Definition 4. For $u_{Fi}(z) \in R_{Fi}^n$, u_{Fi} is a triangular fuzzy number represented with three points (a_i, b_i, c_i), where this representation is interpreted as membership functions of the following form:

$$u_{Fi}(z) = \begin{cases} 0, & z < a_i, \\ \dfrac{z - a_i}{b_i - a_i}, & a_i \le z \le b_i, \\ \dfrac{c_i - z}{c_i - b_i}, & b_i \le z \le c_i, \\ 0, & z > c_i \end{cases} \tag{6}$$

and its r-cut is as follows:

$$[u_{Fi}]^r = [a_i + r(b_i - a_i), c_i - r(c_i - b_i)] \text{ for } r \in [0, 1], \quad i = 1, 2.$$

2.3. Basic definitions of reproducing kernel

Definition 5 [36]**.** The function space $FS_2^m[a, b]$ is defined as follows:

$$FS_2^m[a, b] = \begin{Bmatrix} u : u^{(i)} \text{ is absolutely continuous,} \\ i = 1, 2, \ldots, m-1, u^{(m)} \in L^2[a, b] \end{Bmatrix}. \tag{7}$$

The inner product in the function space $FS_2^m[a, b]$ for any functions $u(z), v(z) \in FS_2^m[a, b]$ is generally defined as

$$\langle u, v\rangle_{FS_2^m[a, b]} = \sum_{i=0}^{m-1} u^{(i)}(a)v^{(i)}(a) + \int_a^b u^{(m)}(z)v^{(m)}(z)dz. \tag{8}$$

The norm in the function space $FS_2^m[a, b]$ for any functions $u(z), v(z) \in FS_2^m[a, b]$ is defined as

$$\| u \|_{FS_2^m[a, b]} = \sqrt{\langle u, u\rangle_{FS_2^m[a, b]}}. \tag{9}$$

3. The Solution Procedure

To solve equation (1) by reproducing kernel tools [36], it is necessary to homogenize the fuzzy initial conditions $P(z_0) = z_{F1},\ P^{(1)}(z_0) = z_{F2}$, and to do so, we consider the following condition:

$$p^H(z) = p(z) - (zz_{F2} + (z_{F1} - z_0 z_{F2})). \tag{10}$$

Then equation (1) can be formulated as

$$D_{z\in[0,1]}^{c,\alpha}(p^H(z)) = G(z,\ p^H(z),\ p^{H(1)}(z)),$$

$$p^H(z_{F1}) = 0, \quad p^{H(1)}(z_{F1}) = 0, \tag{11}$$

where

$$G(z,\ p^H(z),\ p^{H(1)}(z))$$

$$= g(z,\ p^H(z) + (zz_{F2} + (z_{F1} - z_0 z_{F2})),\ p^{H(1)}(z_0) + z_{F2}). \tag{12}$$

Combining equation (11) and equation (12) yields

$$D_{z\in[a,b]}^{c,\alpha}(p^H(z))$$

$$= g(z,\ p^H(z) + (zz_{F2} + (z_{F1} - z_0 z_{F2})),\ p^{H(1)}(z_0) + z_{F2})$$

subject to the initial conditions

$$p^H(z_{F1}) = 0, \quad p^{H(1)}(z_{F1}) = 0. \tag{13}$$

Since z_{F1} and z_{F2} are fuzzy numbers, equation (13) can be formulated in a new form as follows:

$$D^{c,\alpha}_{z\in[a,b]}(p^H(z)) = g(z,\ p^H(z) + (z(a_2, b_2, c_2) + ((a_1, b_1, c_1) - z_0(a_2, b_2, c_2))),\ p^{H(1)}(z_0) + z_{F2})$$

subject to the initial conditions

$$p^H(z_{F1}) = 0, \quad p^{H(1)}(z_{F1}) = 0. \tag{14}$$

By substituting $z_{F1} = (a_1, b_1, c_1)$ and $z_{F2} = (a_2, b_2, c_2)$, we get the following new formula for equation (14):

$$\left[D^{c,\alpha}_{z\in[a,b]}(p^H(z)) = g\begin{pmatrix} z,\ p^H(z) + (z(a_2, b_2, c_2) + ((a_1, b_1, c_1) \\ - z_0(a_2, b_2, c_2))),\ p^{H(1)}(z) + (a_2, b_2, c_2)\end{pmatrix}\right]^r$$

subject to the initial conditions

$$p^H(z_{F1}) = 0, \quad p^{H(1)}(z_{F1}) = 0. \tag{15}$$

By using r-cut definition, we get the following new formula for equation (15):

$$D^{c,\alpha}_{z\in[a,b]}(p^H_{1,1r}(z))$$

$$= g(z,\ p^H_{1,j1}(z) + (zz_{F2,jr} + (z_{F1,jr} - z_0 z_{F2,jr})),\ p^{H(1)}_{1,jr}(z) + z_{F2,jr}), \quad \text{for } j = 1,$$

$$D^{c,\alpha}_{z\in[a,b]}(p^H_{1,2r}(z))$$

$$= g(z,\ p^H_{1,jr}(z) + (zz_{F2,jr} + (z_{F1} - z_0 z_{F2,jr})),\ p^{H(1)}_{1,jr}(z) + z_{F2,jr}), \quad \text{for } j = 2, \tag{16}$$

where

$$z_{F1,1r} = a_1 + r(b_1 - a_1), \quad z_{F2,1r} = a_2 + r(b_2 - a_2),$$

$$z_{F1,2r} = c_1 - r(c_1 - b_1), \quad z_{F2,1r} = c_2 - r(c_2 - b_2).$$

To solve by reproducing tools equation (16), we define the next operator $L_{1,jr} : FS_2^3[a, b] \to FS_2^1[a, b], \ \forall j = 1, 2$ such that $L_{1,jr} p_{1,jr}^H(z) = D_{z\in[a,b]}^{c,\alpha} p_{1,jr}^H(z), 1 < \alpha \le 2.$ Hence, we can write equation (16) as follows:

$$L_{1,jr} p_{1,1r}^H(z) = g_{1,1r}(z, p_{1,j1}^H(z) + (zz_{F2,jr} + (z_{F1,jr} - z_0 z_{F2,jr})), p_{1,jr}^{H(1)}(z) + z_{F2,jr}), \quad \forall j = 1, 2, \quad (17)$$

where

$$z_{F1,1r} = a_1 + r(b_1 - a_1), \quad z_{F2,1r} = a_2 + r(b_2 - a_2),$$

$$z_{F1,2r} = c_1 - r(c_1 - b_1), \quad z_{F2,1r} = c_2 - r(c_2 - b_2),$$

$$z \in [a, b], \quad p_{1,jr}^H(z) \in FS_2^3[a, b]$$

and

$$g_{1,jr}(z, p_{1,jr}^H(z) + (zz_{F2,jr} + (z_{F1,jr} - z_0 z_{F2,jr})), p_{1,jr}^{H(1)}(z) + z_{F2,jr}) \in FS_2^1[a, b], \quad \forall j = 1, 2.$$

We need to construct an orthogonal function system $FS_2^3[a, b]$. To achieve that we take a countable dense set $\{z_k\}_{k=1}^{\infty}$ of $[a, b]$, let $e_{k,1,jr}(z) = R_1 K_{z_k}(z)$ and $\psi_{k,1,jr}(z) = L_{k,jr}^{ad} e_{k,1,jr}(z)$, where $R_1 K_z(y)$ is the reproducing kernel of $FS_2^1[a, b]$ and $L_{k,jr}^{ad}$ is the adjoint operator of $L_{k,jr}$, $\psi_{k,1,jr}(z) \in FS_2^3$.

By the properties of $R_3K_z(y)$, we have

$$\langle p^H_{1,\,jr}(z),\ \psi_{k,1,\,jr}(z)\rangle_{FS^3_2[a,b]}$$

$$= \langle p^H_{1,\,jr}(z),\ L^{ad}_{1,\,jr}\psi_{k,1,\,jr}(z)\rangle_{FS^3_2[a,b]}$$

$$= \langle L_{1,\,jr}\, p^H_{1,\,jr}(z),\ \psi_{k,1,\,jr}(z)\rangle_{FS^1_2[a,b]}$$

$$= L_{1,\,jr}\, p^H_{1,\,jr}(z),$$

where $R_3K_z(y)$ is the reproducing kernel of $FS^3_2[a,\,b]$.

Theorem 1 [36]**.** *The reproducing kernel* $R_3K_z(y)$ *can be expressed as*:

$$R_3K_z(y) = \frac{-1}{120}\begin{cases} (a-y)^2 * \begin{pmatrix} 6a^3 + 5z * y^2 - y^3 - 10z^2 \\ * (3+y) - 3a^2(10 + 5z + y) \\ + 2a(5z^2 - y^2 + 5z(6+y)) \end{pmatrix}, & z \le y, \\ (a-y)^2 * \begin{pmatrix} 6a^3 + 5y * z^2 - z^3 - 10y^2 \\ * (3+z) - 3a^2(10 + 5y + z) \\ + 2a(5y^2 - z^2 + 5y(6+z)) \end{pmatrix}, & z > y. \end{cases} \tag{18}$$

Theorem 2 [37]**.** *The reproducing kernel* $R_1K_z(z)$ *can be expressed as*:

$$R_1K_z(y) = \begin{cases} \frac{1}{2}\operatorname{csc} h(b-a)(\cosh(z+y-b-a) \\ \quad + \cosh(z-y-b+a)), & z \le y, \\ \frac{1}{2}\operatorname{csc} h(b-a)(\cosh(y+z-b-a) \\ \quad + \cosh(y-z-b+a)), & z > y. \end{cases} \tag{19}$$

Lemma 1. $\psi_{k,1,\,jr}(z)$ *can be expressed in the form of* $\psi_{k,1,\,jr}(z) = L_{y,1,\,jr}R_3K_z(y)\,|_{y=z_k}$, *where the subscript y of L refers to the application of the operator L to the function y*.

Proof.

$$\psi_{k,1,\,jr}(z) = L^{ad}_{1,\,jr}e_{k,1,\,jr}(z) = \langle L^{ad}_{1,\,jr}e_{k,1,\,jr}(y),\, R_3K_z(y)\rangle_{FS^3_2[a,b]}$$

$$= \langle e_{k,1,\,jr}(y),\, L_{y,1,\,jr}R_3K_z(y)\rangle_{FS^1_2[a,b]} = L_{y,1,\,jr}R_3K_z(y)\,|_{y=z_k}.$$

□

Lemma 2. *If equation* (1) *has fractional derivatives, then* $\psi_{k,1,\,jr}(z)$ *can be expressed as follows*:

$$\psi_{k,1,\,jr}(z) = \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\int_a^{z_k}(z_k-\tau)^{\lceil\alpha\rceil-\alpha-1}R_3K_z(\tau)^{(\lceil\alpha\rceil)}d\tau,\ k = 1, 2, \ldots.$$

Theorem 3. *Suppose that the inverse operator* $L^{-1}_{1,\,jr}$ *for equation* (17) *exists. If* $\{z_k\}_{k=1}^{\infty}$ *is dense in* $[a, b]$, *then* $\{\psi_{k,1,\,jr}(z)\}_{(k,1,\,jr)}^{(\infty,1,2r)}$ *is the complete fuzzy function system of* $FS^3_2[a, b]$.

Proof. $\forall p^H_{1,\,jr}(z) \in FS^3_2[a, b]$, let $\langle p^H_{1,\,jr}(z), \psi_{k,1,\,jr}(z)\rangle = 0$, for $k = 1, 2, \ldots$. Then

$$\begin{aligned}\langle p^H_{1,\,jr}(z), \psi_{k,1,\,jr}(z)\rangle_{FS^3_2[a,b]} &= \langle p^H_{1,\,jr}(z), L^{ad}_{1,\,jr}e_{k,1,\,jr}(z)\rangle_{FS^3_2[a,b]}\\ &= \langle L_{1,\,jr}p^H_{1,\,jr}(z), e_{k,1,\,jr}(z)\rangle_{FS^1_2[a,b]}\\ &= L_{1,\,jr}p^H_{1,\,jr}(z)\\ &= 0,\end{aligned}$$

where $\{z_k\}_{k=1}^{\infty}$ is dense in $[a, b]$, then $L_{1,\,jr}p^H_{1,\,jr}(z) = 0$ from the existence of inverse and the continuity of $p^H_{1,\,jr}(z)$. □

By Gram-Schmidt orthogonalization process of $\{\psi_{k,1,\,jr}(z)\}_{(k,1,\,jr)}^{(\infty,1,2r)}$, we derive an orthonormal system $\{\overline{\psi_{k,1,\,jr}(z)}\}_{(k,1,1r)}^{(\infty,1,2r)}$:

$$\overline{\psi_{k,1,jr}(z)} = \sum_{s=1}^{k} \beta_{ks} \psi_{s,1,jr}(z), \quad \forall k = 1, 2, \ldots \quad \text{and} \quad j = 1, 2, \qquad (20)$$

where $\beta_{11} = \dfrac{1}{\| \psi_{1,1,jr} \|}$, $\beta_{kk} = \dfrac{1}{M_{k,1,jr}}$ and

$$\beta_{kp,1,jr} = \frac{-1}{M_k} \left(\sum_{s=p}^{k-1} d_{ks,1,jr} \beta_{sp,1,jr} \right), \text{ for } p < k.$$

Then $M_{k,1,jr} = \sqrt{\| \psi_{1,1,jr} \|^2 - \sum_{s=p}^{k-1} (d_{ks,1,jr})^2}$, where

$$d_{ks,1,jr} = \langle \psi_{k,1,jr}, \overline{\psi_{k,1,jr}} \rangle_{FS_2^3[a,b]}.$$

Theorem 4. *If* $\{z_k\}_{k=1}^{\infty}$ *is in* $[a, b]$ *and the solution* $p_{1,jr}^H(z)$ *is unique on* $FS_2^3[a, b]$, *then the exact solution of equation* (16) *is given by*:

$$p_{1,jr}^H(z)$$

$$= \sum_{k=1}^{\infty} \sum_{s=1}^{k} \beta_{ks,1,jr} g_{1,1r}(z, p_{1,j1}^H(z) + (zz_{F2,jr}$$

$$+ (z_{F1,jr} - z_0 z_{F2,jr})), p_{1,jr}^{H(1)}(z) + z_{F2,jr}) \overline{\psi}_{k,1,jr}(z),$$

$$\forall j = 1, 2. \qquad (21)$$

Proof. By Theorem 3, $\{\overline{\psi}_{k,m,jr}\}_{(k,m,jr)=(1,1,1r)}^{(\infty,1,2r)}$ is the complete orthonormal basis of the $FS_2^3[a, b]$. Therefore, $p_{1,jr}^H(z)$ can be expanded into Fourier series about the orthonormal system $\{\overline{\psi}_{k,m,jr}\}_{(k,m,jr)=(1,1,1r)}^{(\infty,1,2r)}$ as

$$p_{1,jr}^H(z) = \sum_{k=1}^{\infty} \langle p_{1,jr}^H(z), \overline{\psi}_{k,1,jr} \rangle \psi_{k,1,jr}(z), \quad \forall j = 1, 2,$$

where we define $FS_2^3[a, b]$ as reproducing kernel Hilbert space. Therefore,

$$p_{1,jr}^{H}(z) = \sum_{k=1}^{\infty} \langle p_{1,jr}^{H}(z), \overline{\psi}_{k,m,jr} \rangle \psi_{k,m,jr}(z), \quad \forall m = 1, \quad j = 1, 2$$

is convergent in the sense of the norm in Definition 5. Hence

$$\begin{aligned}
& p_{1,jr}^{H}(z) \\
&= \sum_{k=1}^{\infty} \langle p_{1,jr}^{H}(z), \overline{\psi}_{k,1,jr}(z) \rangle \overline{\psi}_{k,1,jr}(z) \\
&= \sum_{k=1}^{\infty} \left\langle p_{1,jr}^{H}(z), \sum_{s=1}^{k} \beta_{ks,m,jr}(z) \Psi_{s,m,jr}(z) \right\rangle_{FS_2^3[0,1]} \overline{\psi}_{k,1,jr}(z) \\
&= \sum_{k=1}^{\infty} \sum_{s=1}^{k} \beta_{ks,m,jr}(z) \langle p_{1,jr}^{H}(z), \Psi_{s,m,jr}(z) \rangle_{FS_2^3[0,1]} \overline{\psi}_{k,1jr}(z) \\
&= \sum_{k=1}^{\infty} \sum_{s=1}^{k} \beta_{ks,m,jr}(z) \langle p_{1,jr}^{H}(z), L_{1,jr}^{ad} e_{s,1,jr}(z) \rangle_{FS_2^3[0,1]} \overline{\psi}_{k,1,jr}(z) \\
&= \sum_{k=1}^{\infty} \sum_{s=1}^{k} \beta_{ks,m,jr}(z) \langle L_{1,jr} p_{1,jr}^{H}(z), e_{s,1,jr}(z) \rangle_{FS_2^1[0,1]} \overline{\psi}_{k,m,jr}(z), \\
&= \sum_{k=1}^{\infty} \sum_{s=1}^{k} \beta_{ks,1,jr}(z) \left\langle \begin{matrix} g_{1,jr}(z, p_{1,j1}^{H}(z) + (zz_{F2,jr} + (z_{F1,jr} - z_0 z_{F2,jr})), \\ p_{1,jr}^{H(1)}(z) + z_{F2,jr}), e_{s,1,jr}(z) \end{matrix} \right\rangle \overline{\psi}_{k,1,jr}(z) \\
&= \sum_{k=1}^{\infty} \sum_{s=1}^{k} \beta_{ks,1,jr}(z) g_{1,jr}(z_s, p_{1,j1}^{H}(z_s) + (zz_{F2,jr} \\
&\qquad + (z_{F1,jr} - z_0 z_{F2,jr})), p_{1,jr}^{H(1)}(z_s) + z_{F2,jr}) \overline{\psi}_{k,1,jr}(z), \\
&\qquad \forall j = 1, 2. \quad \square
\end{aligned}$$

By taking finite terms, the following approximate solution for equation (16) is obtained:

$$p_{m,\,jr}^{H}(z)$$

$$= \sum_{k=1}^{N}\sum_{s=1}^{k}\beta_{ks,1,\,jr}(z)\,g_{1,\,jr}(z_s,\ p_{1,\,jr}^{H}(z_s) + (zz_{F2,\,jr}$$

$$+ (z_{F1,\,jr} - z_0 z_{F2,\,jr})),\ p_{1,\,jr}^{H(1)}(z_s) + z_{F2,\,jr})\overline{\psi}_{k,1,\,jr}$$

$$\text{for } j = 1, 2. \quad (22)$$

Using equation (10), the approximate solution of $p(z)$ in equation (1) is as follows:

$$p_{1,\,jr}(z)$$

$$= \sum_{k=1}^{N}\sum_{s=1}^{k}\beta_{ks,1,\,jr}(z)\,g_{1,\,jr}(z_s,\ p_{1,\,jr}^{H}(z_s)$$

$$+ (zz_{F2,\,jr} + (z_{F1,\,jr} - z_0 z_{F2,\,jr})),\ p_{1,\,jr}^{H(1)}(z_s) + z_{F2,\,jr})\overline{\psi}_{k,1,\,jr}$$

$$+ (zz_{F2,\,jr} + (z_{F1,\,jr} - z_0 z_{F2,\,jr})). \quad (23)$$

4. The Algorithm

To implement the algorithm to solve equation (1) according to the solution procedure, the input and output are as follows:

Input. Kernel function $R_3K_z(y)$, interval $[0, 1]$, the integer N and m, the differential operator $L_{y,1,\,jr}$. Defined the inner product for $\langle\psi_i(z), \overline{\psi}_i(z)\rangle$, fuzzy initial conditions $z_{F1} = (a_1, b_1, c_1)$, $z_{F2} = (a_2, b_2, c_2)$, order of derivatives $\alpha \in (1, 2]$ and the function.

Output. Approximate solution of $p_{1,\,jr}(z)$.

The following are the steps to implement the algorithm:

Steps

1. Fix z in $[0, 1]$ and set $y \in [0, 1]$.

If $z \leq y$, set

$$R_3K_z(y) = \frac{-1}{120}(a-y)^2 * \begin{pmatrix} 6a^3 + 5z * y^2 - y^3 - 10z^2 * (3+y) \\ -3a^2(10 + 5z + y) \\ + 2a(5z^2 - y^2 + 5z(6+y)) \end{pmatrix}.$$

Otherwise,

$$R_3K_z(y) = \frac{-1}{120}(a-z)^2 * \begin{pmatrix} 6a^3 + 5y * z^2 - z^3 - 10y^2 * (3+z) \\ -3a^2(10 + 5y + z) \\ + 2a(5y^2 - z^2 + 5y(6+z)) \end{pmatrix}.$$

2. $\forall j = 1, 2$ do steps 3-6.

3. For $k = 1, 2, \ldots, N$, $j = 1, 2$, $d = 1, 2, \ldots, m$ do the following:

Set $z_0 = 0$

Set $z_k = z_{k-1} + \frac{1}{N}$

Set $r_0 = 0$

Set $r_d = r_{d-1} + \left(\frac{d}{m}\right)$.

If $j = 1$; Set $z_{F1, jr_d} = a_1 + r_d(b_1 - a_1)$

Set $z_{F2, jr_d} = a_2 + r_d(b_2 - a_2)$.

If $j = 2$; Set $z_{F1, jr_d} = c_1 - r_d(c_1 - b_1)$

Set $z_{F2, jr_d} = c_2 - r_d(c_2 - b_2)$

Set $z_{F2, jr_d} = a_2 + r_d(b_2 - a_2)$

Set $\psi_{k,1, jr_d}(z) = L_{y,1, jr} R_3K_z(y)\big|_{y=z_k}$.

The output is $\{\psi_{k,1, jr_d}(z)\}_{(k,1, jr_d)}^{(N,1, 2r_d)}$.

4. For $k = 2, ..., N$, $s = 1, 2, ..., k$ do the following:

$$\text{Set } \beta_{11} = \frac{1}{\| \psi_{1,1,\, jr_d} \|}$$

$$\text{Set } \beta_{kk} = \frac{1}{M_{k,1,\, jr_d}}$$

$$\text{Set } \beta_{kp,1,\, jr_d} = \frac{-1}{M_k}\left(\sum_{s=p}^{k-1} d_{ks,1,\, jr}\beta_{sp,1,\, jr_d}\right),\ p < k$$

$$\text{Set } M_{k,1,\, jr_d} = \sqrt{\| \psi_{1,1,\, jr} \|^2 - \sum_{s=p}^{k-1} (d_{ks,1,\, jr_d})^2},$$

$$d_{ks,1,\, jr} = \langle \psi_{k,1,\, jr_d}(z), \overline{\psi}_{k,1,\, jr_d}(z) \rangle_{FS_2^3[a,b]}.$$

The output is $\beta_{ks,1,\, jr_d}$.

5. For $k = 1, 2, ..., N$,

$$\text{Set } \overline{\psi_{k,1,\, jr_d}(z)} = \sum_{s=1}^{k} \beta_{ks}\psi_{s,1,\, jr_d}(z).$$

The output is $\overline{\psi_{k,1,\, jr_d}(z)}$.

6. For $k = 1, 2, ..., N$,

$$\text{Set } p_{1,\, jr_d}^{H}(z_0) = 0,\ p_{1,\, jr_d}^{H(1)}(z_0) = 0,$$

$$p_{1,\, jr}^{H}(z)$$

$$= \sum_{k=1}^{N}\sum_{s=1}^{k} \beta_{kl,1,\, jr}(z) g_{1,\, jr}(z_s,\ p_{1,\, j1}^{H}(z_s)$$

$$+ (zz_{F2,\, jr} + (z_{F1,\, jr} - z_0 z_{F2,\, jr})),\ p_{1,\, jr}^{H(1)}(z_s) + z_{F2,\, jr})\overline{\psi}_{k,1,\, jr}(z)$$

$$\text{Set } p_{1,\, jr_d}(z) = p_{1,\, jr_d}^{H}(z) + (zz_{F2,\, jr_d} + (z_{F1,\, jr_d} - z_0 z_{F2,\, jr_d})).$$

The output is $p_{1,\, jr_d}(z) = [p_{1,1r_d}(z),\ p_{1,2r_d}(z)]$.

5. Numerical Examples

Example 1. Consider the following fuzzy fractional differential equations of order α:

$$D^{c,\alpha}_{z\in[0,1]}p(z) = -15p^{(1)}(z) + p(z) + 0.3\cos(z),$$

$$p(z_0) = (1, 2, 3), \quad p^{(1)}(z_0) = (1, 2, 3), \quad \alpha = 2.$$

Solution. By applying fuzzy theory after homogenizing the initial conditions, we obtain the following system:

$$D^{c,\alpha}_{z\in[0,1]}p^{*}_{1,1r}(z) = -15p^{*(1)}_{1,1r} + p^{*}_{1,1r}(z) + 0.3\cos(z) - 15(a_2 + r(b_2 - a_2))$$

$$+ [(a_2 + r(b_2 - a_2))z$$

$$+ [(a_1 + r(b_1 - a_1) - z_0(a_2 + r(b_2 - a_2)))]],$$

$$D^{c,\alpha}_{z\in[0,1]}p^{*}_{1,2r}(z) = -15p^{*(1)}_{1,2r} + p^{*}_{1,2r}(z) + 0.3\cos(z) - 15(c_2 - r(c_2 - b_2))$$

$$+ [(c_2 - r(c_2 - b_2))z$$

$$+ [(c_1 - r(c_1 - b_1) - z_0(c_2 - r(c_2 - b_2)))]],$$

$$p^{*}_{1,1r}(z) = 0, \quad p^{*}_{1,2r}(z) = 0, \quad p^{*(1)}_{1,1r}(z) = 0, \quad p^{*(1)}_{1,2r}(z) = 0,$$

$$(r_0, [p_{1,1r}(0) = 1.00,\ p_{1,2r}(0) = 3.00,\ p^{(1)}_{1,1r}(0) = 1.00,\ p^{(1)}_{1,2r}(0) = 3.00])^T,$$

$$(r_2, [p_{1,1r}(0) = 2.00,\ p_{1,2r}(0) = 2.00,\ p^{(1)}_{1,1r}(0) = 2.00,\ p^{(1)}_{1,2r}(0) = 2.00])^T.$$

The exact solutions of the system at $r = 0$ are:

$$p_{1,1r}(z) = 1.063e^{-15.0664z}(-0.057 + 1.e^{15.1327z} - 0.002e^{15.0664z}\cos[z]$$

$$+ 0.018e^{15.0664z}\sin[z]),$$

$$p_{1,2r}(z) = 3.186e^{-15.0664z}(-0.058 + 1.e^{15.1324z} - 0.0008e^{15.0664z}\cos[z]$$

$$+ 0.006e^{15.0664z}\sin[z]).$$

The exact solutions of the system at $r = 1.0$ are:

$$p_{1,1r}(z) = p_{1,2r}(z)$$

$$= 2.125e^{-15.0664z}(-0.058 + 1.e^{15.134z}$$

$$-0.0012e^{15.0664z}\cos[z] + 0.0092e^{15.0664z}\sin[z]).$$

Using the proposed algorithm and taking $N = 100$ in the interval $[0, 1]$, the results of the approximate solutions are obtained and presented in Tables 1.1-1.3 as well as in Figures 1.1-1.3. The approximate solutions are compared with the exact solutions within the same interval and within the same fuzzy initial values.

Table 1.1. Comparison of solutions of $p_{1,1r}(z)$ for Example 1 when $r = 0$

z	Exact solution $p_{1,1r}(z)$	Approximate solution $p_{1,1r}(z),\ r = 0$	Error
0.0	1.0	1.0	0
0.1	1.0560548492202497	1.056046406	$8.443576012 \times 10^{-6}$
0.2	1.075580238403811	1.075564918	$1.531998245 \times 10^{-5}$
0.3	1.0870309508357805	1.087012921	$1.802977793 \times 10^{-5}$
0.4	1.0966990764967495	1.096680106	$1.897072812 \times 10^{-5}$
0.5	1.1059645558430546	1.105945223	$1.933284215 \times 10^{-5}$
0.6	1.1151189829045023	1.115099457	$1.952578409 \times 10^{-5}$
0.7	1.1242133119945217	1.124193639	$1.967270826 \times 10^{-5}$
0.8	1.1332458945594859	1.133226086	$1.980809789 \times 10^{-5}$
0.9	1.1422042901738525	1.142184349	$1.994129163 \times 10^{-5}$
1.0	1.1510746573241644	1.151054582	$2.007518326 \times 10^{-5}$

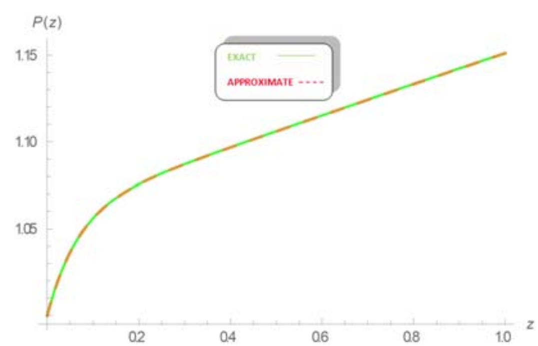

Figure 1.1. The exact solutions and approximate solutions of $p_{1,1r}(z)$ for Example 1 when $r = 0$.

Table 1.2. Comparison of solutions of $p_{1,2r}(z)$ for Example 1 when $r = 0$

z	Exact solution $p_{1,2r}(z)$	Approximate solution $p_{1,2r}(z)$, $r = 0$	Error
0.0	3	3	0
0.1	3.1662366706950786	3.166157278	$7.939228089 \times 10^{-5}$
0.2	3.2212846442096557	3.221140568	$1.440766711 \times 10^{-4}$
0.3	3.251783337499188	3.251613775	$1.695621865 \times 10^{-4}$
0.4	3.2769221944280775	3.276743786	$1.784087082 \times 10^{-4}$
0.5	3.3009404021469866	3.30075859	$1.818122682 \times 10^{-4}$
0.6	3.3247638172370437	3.324580191	$1.836258756 \times 10^{-4}$
0.7	3.348584483065339	3.348399476	$1.850071319 \times 10^{-4}$
0.8	3.3724325295567237	3.37224625	$1.862799039 \times 10^{-4}$
0.9	3.3963030689952367	3.396115538	$1.875311865 \times 10^{-4}$
1.0	3.420184446225242	3.419995661	$1.887849507 \times 10^{-4}$

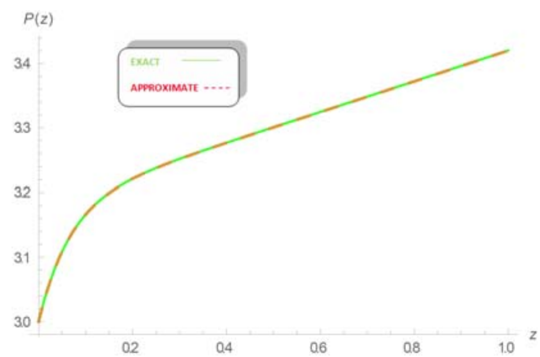

Figure 1.2. The exact solutions and approximate solutions of $p_{1,2r}(z)$ for Example 1 when $r = 0$.

Table 1.3. Comparisons of solutions of $p_{1,1r}(z) = p_{1,2r}(z)$ for Example 1 when $r = 1.0$

z	Exact solution $p_{1,1r}(z)$	Approximate solution $p_{1,1r}(z),\ p_{1,2r}(z),\ r = 1.0$	Error
0.0	2.0	2.0	0
0.1	2.111145759957664	2.11109302	$5.273955193 \times 10^{-5}$
0.2	2.1484324413067335	2.148336733	$9.570880142 \times 10^{-4}$
0.3	2.1694071441674843	2.169294506	$1.126386119 \times 10^{-4}$
0.4	2.1868106354624137	2.18669212	$1.185152966 \times 10^{-4}$
0.5	2.2034524789950205	2.203331703	$1.207762732 \times 10^{-4}$
0.6	2.2199414000707733	2.219819419	$1.219810603 \times 10^{-4}$
0.7	2.2363988975299303	2.236275999	$1.228986467 \times 10^{-4}$
0.8	2.2528392120581042	2.252715468	$1.237441999 \times 10^{-4}$
0.9	2.269253679584544	2.269129104	$1.245756262 \times 10^{-4}$
1.0	2.285629551774703	2.285504142	$1.254093765 \times 10^{-4}$

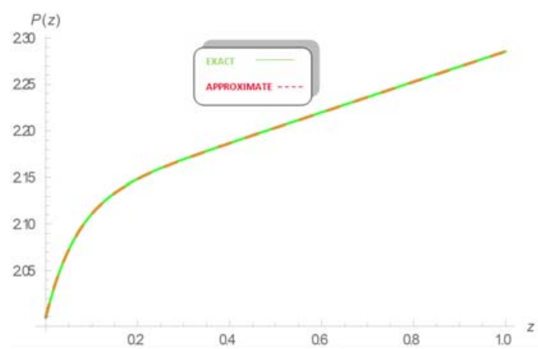

Figure 1.3. The exact solutions and the approximate solutions of $p_{1,1r}(z) = p_{1,2r}(z)$ for Example 1 when $r = 1.0$.

Notably, Tables 1.1-1.3 and Figures 1.1-1.3 depict the accuracy of the solution of second-order fuzzy differential equations of integer and fractional order obtained as well as the efficiency of the proposed solution procedure based on reproducing kernel Hilbert space.

Example 2. Consider the following fuzzy fractional differential equation of order $\alpha = 1.7$:

$$D^{c,\alpha}_{z\in[0,1]} p(z) = -15 p^{(1)}(z) + p(z) + 0.3\cos(z),$$

$$p(z_0) = (1,\ 2,\ 3), \quad p^{(1)}(z_0) = (0,\ 0,\ 0).$$

Solution. By applying fuzzy theory after homogenizing the initial conditions, we obtain

$$D^{c,\alpha}_{z\in[0,1]} p^{*}_{1,1r}(z)$$

$$= -15 p^{*(1)}_{1,1r} + p^{*}_{1,1r}(z) + 0.3\cos(z) - 15(a_2 + r(b_2 - a_2))$$

$$+ [(a_2 + r(b_2 - a_2))z + [(a_1 + r(b_1 - a_1) - z_0(a_2 + r(b_2 - a_2)))]],$$

$$D^{c,\alpha}_{z\in[0,1]} p^{*}_{1,2r}(z)$$

$$= -15 p^{*(1)}_{1,2r} + p^{*}_{1,2r}(z) + 0.3\cos(z) - 15(c_2 - r(c_2 - b_2))$$

$$+ [(c_2 - r(c_2 - b_2))z + [(c_1 - r(c_1 - b_1) - z_0(c_2 - r(c_2 - b_2)))]],$$

$$p^{*}_{1,1r}(z) = 0, \quad p^{*}_{1,2r}(z) = 0, \quad p^{*(1)}_{1,1r}(z) = 0, \quad p^{*(1)}_{1,2r}(z) = 0.$$

Results for Example 2 are obtained using the proposed algorithm for $N = 50$ in the interval $[0, 1]$. Table 2.1 shows the numerical solutions at $\alpha = 1.7$ for $r = 0,$ 0.5 and 1.0. It is clear that for each $r = 0$ and $r = 0.5,$ the solution is in the form of an interval, which can also be observed in Figures 2.1 and 2.2, respectively. On the other hand, for $r = 1,$ the solution is in the form of a point as depicted in Figure 2.3.

Table 2.1. Numerical solutions of $p_{1,1r}(z)$ and $p_{1,2r}(z)$ for Example 2 with different values of r

z	Approximate solution $p_{1,1r}(z)$, $r = 0$, $\alpha = 1.7$	Approximate solution $p_{1,2r}(z)$, $r = 0$, $\alpha = 1.7$	Approximate solution $p_{1,1r}(z)$, $r = 0.5$, $\alpha = 1.7$	Approximate solution $p_{1,2r}(z)$, $r = 0.5$, $\alpha = 1.7$	Approximate solution $p_{1,1r}(z) = p_{1,2r}(z)$, $r = 1.0$, $\alpha = 1.7$
0.0	0.0	3.0	1.5	2.5	2.0
0.1	1.040386965	3.118544818	1.559926428	2.599005355	2.079465891
0.2	1.059052554	3.170978863	1.587034131	2.642997286	2.115015709
0.3	1.074368264	3.213254893	1.609089921	2.678533236	2.143811579
0.4	1.088230848	3.251182866	1.628968852	2.710444861	2.169706857
0.5	1.101253663	3.286666582	1.647606893	2.740313352	2.193960123
0.6	1.113709269	3.320571877	1.665424921	2.768856225	2.217140573
0.7	1.125739539	3.353365614	1.682646058	2.796459096	2.239552577
0.8	1.137425359	3.385327379	1.699400864	2.823351874	2.261376369
0.9	1.148815508	3.416637003	1.715770882	2.849681629	2.282726255
1.0	1.159940525	3.447416409	1.731809496	2.875547438	2.303678467

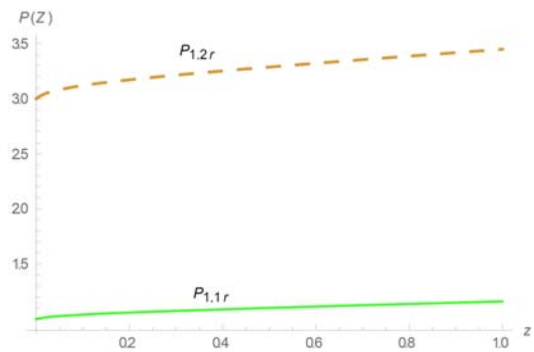

Figure 2.1. Graphs of $p_{1,1r}(z)$ and $p_{1,2r}(z)$ when $r = 0$ for Example 2.

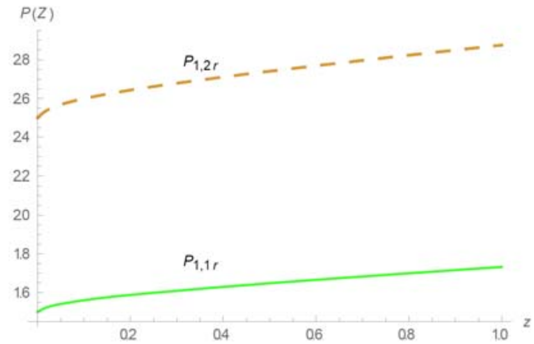

Figure 2.2. Graphs of $p_{1,1r}(z)$ and $p_{1,2r}(z)$ when $r = 0.5$ for Example 2.

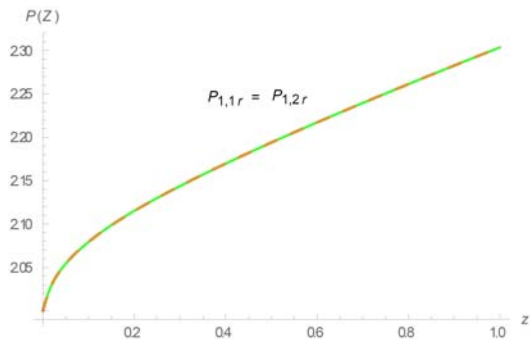

Figure 2.3. Graphs of $p_{1,1r}(z) = p_{1,2r}(z)$ when $r = 1.0$ for Example 2.

6. Conclusion

In this study, we have proposed a new solution procedure based on reproducing kernel theory to solve initial value problems of second-order fuzzy differential equations of integer and fractional order with the focus on order α in $(1, 2]$ in the sense of Caputo fractional derivatives. This solution approach is considerably convenient since it requires less effort without having to resort to more advanced mathematical tools. The accuracy of the results obtained from the illustrated examples indicates the effectiveness of the proposed procedure. Further research may include the utilization of this new method in solving other types of problems in fuzzy environment involving fractional calculus.

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