Advances and Applications in Discrete Mathematics

http://www.pphmj.com
http://dx.doi.org/10.17654/DM018010045
Volume 18, Number 1, 2017, Pages 45-56 **ISSN: 0974-1658**

THE *k*-INDEPENDENT GRAPH OF A GRAPH

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Abstract

Let $G = (V, E)$ be a simple graph. A set $I \subseteq V$ is an independent set, if no two of its members are adjacent in G. The k-independent graph of G, $I_k(G)$, is defined to be the graph whose vertices correspond to the independent sets of G that have cardinality at most k. Two vertices in $I_k(G)$ are adjacent if and only if the corresponding independent sets of G differ by either adding or deleting a single vertex. In this paper, we obtain some properties of $I_k(G)$ and compute it for some graphs.

Received: August 16, 2016; Accepted: November 5, 2016
2010 Mathematics Subject Classification: 05C60, 05C69.
Keywords and phrases: independence number, k-independent graph, reconfiguration.
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1. Introduction

Given a simple graph $G = (V, E)$, a set $I \subseteq V$ is an independent set of G, if there is no edge of G between any two vertices of I. A maximal independent set is an independent set that is not a proper subset of any other independent set. A maximum independent set is an independent set of greatest cardinality for G. This cardinality is called *independence number of* G, and is denoted by $\alpha(G)$. Reconfiguration problems have been studied often in recent years. These arise in settings where the goal is to transform feasible solutions to a problem in a step-by-step manner, while maintaining a feasible solution throughout.

For the study of dominating set reconfiguration problem: given two dominating sets S and T of a graph G, both of size at most k, is it possible to transform S into T by adding and removing vertices one-by-one, while maintaining a dominating set of size at most k throughout? Recently the k-dominating graph of a graph G has been defined in [9]. The k-dominating graph of G, $D_k(G)$, is defined to be the graph whose vertices correspond to the dominating sets of G that have cardinality at most k. Two vertices in $D_k(G)$ are adjacent if and only if the corresponding dominating sets of G differ by either adding or deleting a single vertex. Authors in [9] gave conditions that ensure $D_k(G)$ is connected. In [1], authors proved that if G is a graph without isolated vertices of order $n \geq 2$ and with $G \cong D_k(G)$, then $k = 2$ and $G = K_{1, n-1}$ for some $n \geq 4$. It is also proved that for a given r, there exist only a finite number of r-regular, connected dominating graphs of connected graphs [1].

One of the most well-studied problems in reconfiguration problems is the reconfiguration of independent sets. For a graph G and integer k, the independent sets of size at least/exactly k of G form the feasible solutions. Independent sets are also called *token configurations*, where the independent set vertices are viewed as tokens [4]. Deciding for existence of a reconfiguration between two k-independent sets with at most ℓ operations is strongly NP-complete [10]. Bonamy and Bousquet [3] considered the k-TAR reconfiguration graph, $TAR_k(G)$, as follows:

A k-independent set of G is a set $S \subseteq V$ with $|S| \geq k$, such that no two elements of S are adjacent. Two k-independent sets I and J are adjacent if they differ on exactly one vertex. This model is called the *Token Addition and Removal* (*TAR*). Authors in [3] provided a cubic-time algorithm to decide whether $TAR_k(G)$ is connected when G is a graph which does not contain induced paths of length 4. Their work solves an open question in [4]. Also, they described a linear-time algorithm which decides whether two elements of $TAR_k(G)$ are in the same connected component. As usual, we denote the complete graph, path and cycle of order n by K_n, P_n and C_n, respectively. Also, $K_{1,n}$ is the star graph with $n+1$ vertices.

In the next section, we study the k-independent graph of a graph G. In Section 3, we study the α-independent graph of a graph. Finally, in Section 4, we exclude the empty set from the family set of independent sets of G, denote the new k-independent graph of G by $I_k^*(G)$ and study its connectedness.

2. The k-independent Graph of a Graph

In this section, we shall study the k-independent graph of a graph G. First we rewrite the definition of the reconfiguration graph $TAR_k(G)$ as follows. For a graph G and a non-negative integer k, the k-independent graph of G, $I_k(G)$, is defined to be the graph whose vertices correspond to the independent sets of G that have cardinality at most k. Two vertices in $I_k(G)$ are adjacent if and only if the corresponding independent sets of G differ by either adding or deleting a single vertex. As an example, Figure 1 shows $I_3(K_{1,3})$.

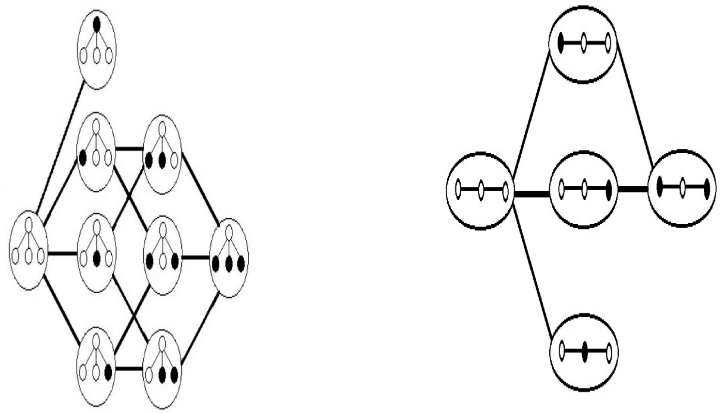

Figure 1. Graphs $I_3(K_{1,3})$ and $I_2(P_3)$, respectively.

Note that *k*-dominating and *k*-independent graph are similar to recent work in graph colouring, too. Given a graph *H* and a positive integer *k*, the *k*-colouring graph of *H*, denoted $G_k(H)$, has vertices corresponding to the (proper) *k*-vertex-colourings of *H*. Two vertices in $G_k(H)$ are adjacent if and only if the corresponding vertex colourings of *G* differ on precisely one vertex. Authors in [5-8] studied the connectedness of *k*-colouring graphs. Also they studied their hamiltonicity. Let to introduce a notation. Let *A* and *B* be independent sets of *G* of cardinality at most *k*. We use the notation $A \leftrightarrow B$, if there is a path in $I_k(G)$ joining *A* and *B*. It is easy to see that for every $A, B \in I_k(G)$, $A \leftrightarrow B$ if and only if $B \leftrightarrow A$ and if $A \supseteq B$, then $A \leftrightarrow B$ and $B \leftrightarrow A$.

The following theorem gives some properties of the *k*-independent graph of a graph:

Theorem 2.1. (i) *If G is a graph of order n*, *then* $I_1(G) \cong K_{1,n}$.

(ii) *For every graph G and every* $0 \leq k \leq \alpha(G)$, *the independent graph* $I_k(G)$ *is connected and* $\Delta(I_k(G)) = |V(G)|$.

(iii) *For every graph G*, *the independent graph* $I_k(G)$ *is a bipartite graph*.

(iv) *If* $G \ncong \overline{K_n}$, *then* $I_k(G)$ *is not a regular graph*.

(v) *If* $G \ncong \overline{K_n}$, *then* $I_k(G)$ *is not a vertex-transitive graph*, *and so is not a Cayley graph*.

Proof. (i) It follows from the definition.

(ii) It is straightforward.

(iii) Let X be the set of independent sets of size less than $k+1$ of G with odd cardinality and Y be the set of independent sets of size less than $k+1$ with even cardinality. It is clear that $X \cup Y = V(I_k(G))$ and $X \cap Y = \phi$. Suppose that $A, B \in X$, then $(A \backslash B) \cup (B \backslash A)$ cannot be a vertex of $I_k(G)$. Because $|A| = |B|$ or $||A| - |B|| \geq 2$. So AB is not an edge of $I_k(G)$ and with similar argument we have this for two vertices in Y. Therefore, $I_k(G)$ is a bipartite graph with parts X and Y.

(iv) Let G be a graph of order n. The empty set is an independent set of G which has degree n in $I_k(G)$. Let I_1 be an independent set of G with $|I_1| = \alpha(G)$. We know that I_1 is adjacent to α independent sets. Since $G \ncong \overline{K_n}$, we have $\alpha(G) \neq n$. Therefore, $I_k(G)$ is not a regular graph.

(v) It follows from Part (iv). □

Theorem 2.2. (i) *Let* G *be a graph of order* n. *There is no integer* k, *such that* $I_k(G) \cong G$.

(ii) *If* $G \ncong K_n$, *then the girth of* $I_k(G)$ *is* 4.

(iii) *Let* $G \neq K_n$ *be a graph*. *Then for all integers* $k \geq 2$, $I_k(G)$ *is not a tree*.

Proof. (i) Since for every integer number $k \geq 1$, $|V(I_k(G))| \geq n+1$, so we have the result.

(ii) Let v_1 and v_2 be two non-adjacent vertices of graph G. So $\{v_1\}$ and $\{v_2\}$ are two independent sets of G and therefore two vertices of $I_k(G)$.

Now, $\varnothing, \{v_1\}, \{v_1, v_2\}, \{v_2\}, \varnothing$ is a cycle in $I_k(G)$ and this is the shortest cycle in $I_k(G)$. Therefore, the girth of $I_k(G)$ is 4.

(iii) It follows from part (ii). □

3. The α-independent Graph of Some Graphs

Let G be a simple graph with independence number α. Looks that in the among of k-independent graph of G, the α-independent graph of G is more important. In this section, we study the α-independent graph of some graphs. To study the α-independent graph of G, we are interested to know the order of $I_\alpha(G)$. Let i_k be the number of independent sets of cardinality k in G. The polynomial

$$I(G, x) = \sum_{k=0}^{\alpha(G)} i_k x^k,$$

is called the *independence polynomial of G* [2]. Obviously, $I(G, 1)$ gives the number of all independent sets of a graph G. In other words, $|V(I_\alpha(G))| = I(G, 1)$. Since $I(K_n, x) = 1 + nx$, we have $I(K_n, 1) = n + 1$. Therefore, we have the following easy result:

Theorem 3.1. *For any integer $k > 1$, there is some connected graph G such that $|V(I_\alpha(G))| = k$.*

The following theorem is about the α-independent graph of stars:

Theorem 3.2. (i) *The n-independent graph of $K_{1,n}$, i.e., $I_n(K_{1,n})$ is a bipartite graph with parts X and Y, with $|X| = 2^{n-1}$ and $|Y| = 2^{n-1} + 1$.*

(ii) *The n-independent graph $I_n(K_{1,n})$ is not Hamiltonian.*

Proof. (i) Let X be the set of independent sets of $K_{1,n}$ with even cardinality and Y be the set of independent sets of odd cardinality. By Theorem 2.1(iii), $I_n(K_{1,n})$ is a bipartite graph with parts X and Y. Obviously

$|X| = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k}$ and since the number of independent sets of $K_{1,n}$ is $I(K_{1,n}, 1) = 2^n + 1$, we have $|Y| = 1 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k-1}$. Therefore, we have the result.

(ii) Since a bipartite graph with different number of vertices in its parts is not a Hamiltonian graph, so the n-independent graph $I_n(K_{1,n})$ is not a Hamiltonian graph. □

Here we consider the α-independent of some another graphs. Figure 1 shows the $I_2(P_3)$.

Theorem 3.3. *For every* $n \in \mathbb{N}$, $\delta(I_\alpha(P_n)) = \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. The minimum degree of vertices of $I_{\lceil \frac{n}{2} \rceil}(P_n)$ is due to maximal independent sets of P_n with minimum cardinality. These vertices are adjacent to $n - \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor$ of independent sets with less cardinality. □

Here we shall obtain information on the Hamiltonicity of α-independent of some specific graphs. Using the value of the independence polynomial at −1, we have $I(G; -1) = i_0 - i_1 + i_2 - \cdots + (-1)^\alpha i_\alpha = f_0(G) - f_1(G)$, where $f_0(G) = i_0 + i_2 + i_4 + \cdots$, $f_1(G) = i_1 + i_3 + i_5 + \cdots$ are equal to the numbers of independent sets of even size and odd size of G, respectively. $I(G, -1)$ is known as the alternating number of independent sets. We need the following theorem:

Theorem 3.4 [11]**.** *For* $n \geq 1$, *the following hold*:

(i) $I(P_{3n-2}; -1) = 0$ *and* $I(P_{3n-2}; -1) = I(P_{3n}; -1) = (-1)^n$;

(ii) $I(C_{3n}; -1) = 2(-1)^n$, $I(C_{3n+1}; -1) = (-1)^n$ *and* $I(C_{3n+2}; -1) = (-1)^{n+1}$;

(iii) $I(W_{3n+1}; -1) = 2(-1)^n - 1$ *and* $I(W_{3n}; -1) = I(W_{3n+2}; -1) = (-1)^n - 1$.

Corollary 3.5. *For all positive integer n, the graphs* $I_\alpha(P_{3n-1})$, $I_\alpha(P_{3n})$, $I_\alpha(C_n)$ *and* $I_\alpha(W_n)$ *are not Hamiltonian.*

Proof. We know that $I_\alpha(P_n)$, $I_\alpha(C_n)$ and $I_\alpha(W_n)$ are bipartite graphs with parts containing the independent sets of even and odd cardinality. By Theorem 3.4, theses bipartite graphs have parts with different cardinality. Therefore, we have the result. □

4. Connectedness of $I_k^*(G)$

As we have seen in Section 2, since the empty set is an independent set of any graph, so the *k*-independent graph $I_k(G)$ is a connected graph. We do not consider empty set in the study of *k*-independent graph.

Suppose that $\mathcal{I}$ is a family of all independent sets of graph *G*. If we put $V(I_k(G)) = \mathcal{I} \backslash \varnothing$, then we denote the *k*-independent graph of *G*, by $I_k^*(G)$. Note that in this case, for some *k* and *G*, $I_k^*(G)$ is disconnected and for some *k* and *G* is connected. For example, Figure 2 shows $I_3^*(K_{1,3})$ and $I_2^*(C_4)$, which are disconnected graphs with two components. Also, Figure 3 shows $I_2^*(W_5)$ and $I_3^*(P_5)$, respectively. Observe that $I_3^*(P_5)$ is connected and $I_2^*(W_5)$ is disconnected with three components. Theorem 2.2 implies that for any graph $G \neq K_n$, and for all integers $k \geq 2$. $I_k(G)$ is not a tree, but as we see in Figure 3, the graph $I_k^*(G)$ can be a forest. This naturally raises the question: For which graph *G*, the component of $I_k^*(G)$ is a forest? What is the number of components?

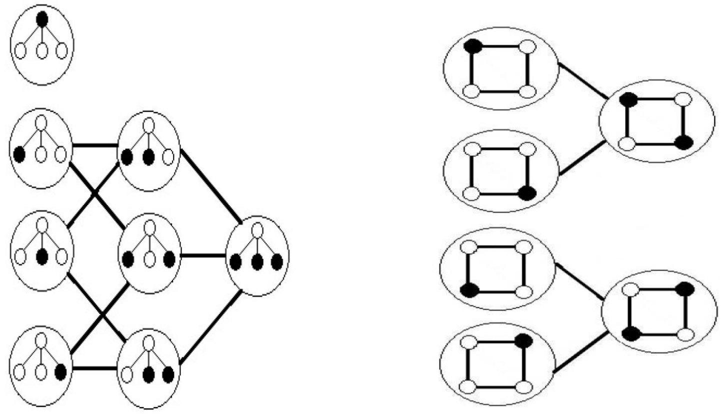

Figure 2. Graphs $I_3^*(K_{1,3})$ and $I_2^*(C_4)$, respectively.

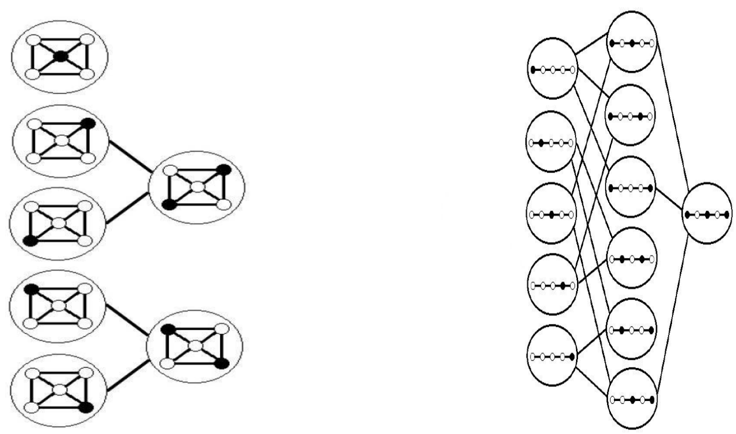

Figure 3. Graphs $I_2^*(W_5)$ and $I_3^*(P_5)$, respectively.

The following theorem is a sufficient condition for disconnectedness of $I_\alpha^*(G)$.

Theorem 4.1. *If a graph G of order n has a vertex of degree* $n-1$, *then* $I_\alpha^*(G)$ *is disconnected*.

Proof. Let v be a vertex of degree $n-1$. Obviously $\{v\}$ is a nonempty independent set of G, and so is an isolated vertex of $I_\alpha^*(G)$. □

Note that the converse of Theorem 4.1 is not true. For example, $I_\alpha^*(C_4)$ has two components, but C_4 is 2-regular (Figure 3). Now, we state the following theorem:

Theorem 4.2. *Let* $K_{n_1, n_2, \ldots, n_m}$ *be a complete m-partite graph. Then* $I_\alpha^*(K_{n_1, n_2, \ldots, n_m})$ *has m connected components.*

Proof. Let X_1 and X_2 be two arbitrary parts of $K_{n_1, n_2, \ldots, n_m}$. Suppose that I_1 contains all nonempty subsets of part X_1 and I_2 contains all nonempty sets of part X_2. Obviously, each member of I_1 and each member of I_2 are independent sets of $K_{n_1, n_2, \ldots, n_m}$ and so they are vertices of $I_\alpha^*(K_{n_1, n_2, \ldots, n_m})$. No member of I_1 is adjacent to a member of I_2 in $I_\alpha^*(K_{n_1, n_2, \ldots, n_m})$. So $I_\alpha^*(K_{n_1, n_2, \ldots, n_m})$ is a disconnected graph. Since the members of I_1 (and the members of I_2) form a connected graph, therefore we have *m* components. □

It is obvious that, for all graph *G* with $\alpha(G) = 2$, $I_2^*(G)$ is a forest.

Theorem 4.3. *For a graph G with* $\alpha(G) > 2$, *the components of* $I_k^*(G)$, $2 \leq k \leq \alpha$, *are not forest.*

Proof. We consider following two cases:

Case 1. If $k = 2$. Let $\{v_1, v_2, v_3\}$ be an independent set of *G*. So $\{v_1\}$, $\{v_2\}$, $\{v_3\}$, $\{v_1, v_2\}$, $\{v_1, v_3\}$ and $\{v_2, v_3\}$ are independent sets of *G* and vertices of $I_k^*(G)$. Therefore, $\{v_1\}$, $\{v_1, v_2\}$, $\{v_2\}$, $\{v_2, v_3\}$, $\{v_3\}$, $\{v_1, v_3\}$, $\{v_1\}$ make a cycle in $I_k^*(G)$.

Case 2. If $k > 2$. Let $\{v_1, v_2, v_3\}$ be an independent set of *G*. So $\{v_1\}$, $\{v_1, v_2\}$ and $\{v_1, v_3\}$ are independent sets of *G* and vertices of $I_k^*(G)$.

Therefore, $\{v_1\}$, $\{v_1, v_2\}$, $\{v_1, v_2, v_3\}$, $\{v_1, v_3\}$, $\{v_1\}$ make a cycle in $I_k^*(G)$ and so $I_k^*(G)$ is not a forest. □

Note that if G is a graph of order n with $\alpha(G) > 2$, then similar to Theorem 4.3, $I_k^*(G)$ cannot be a path, cycle and a chordal graph.

Theorem 4.4. *Let G be a* (*non-complete*) *bipartite graph of order* $n > 4$. *Then* $I_k^*(G)$ *is connected*.

Proof. Let I_1 and I_2 be two independent sets of G and $|I_1|, |I_2| \leq k$, so I_1 and I_2 are two vertices of $I_k(G)$. If $I_1 \cap I_2 \neq \phi$, then $I_1 \leftrightarrow I_1 \cap I_2 \leftrightarrow I_2$. If $I_1 \cap I_2 = \phi$, then we consider following two cases:

Case 1. There are $v_1 \in I_1$ and $v_2 \in I_2$ such that v_1 and v_2 are not adjacent, then $I_1 \leftrightarrow \{v_1\} \leftrightarrow \{v_1, v_2\} \leftrightarrow \{v_2\} \leftrightarrow I_2$.

Case 2. For all $v_1 \in I_1$ and $v_2 \in I_2$, v_1 is adjacent to v_2. So $I_1 \subset A$ and $I_2 \subset B$, where A and B are two parts of G. Since G is not complete bipartite graph so $I_1 \neq A$ and $I_2 \neq B$ and there are $v_3 \in A$ and $v_4 \in B$ such that $v_3 \notin I_1$ and v_3 is not adjacent to v_4. We put $I_3 = (I_1 \backslash \{v_1\}) \cup \{v_3\}$. So $|I_3| = |I_1|$ and $I_1 \leftrightarrow I_1 \backslash \{v_1\} \leftrightarrow I_3$ and $I_3 \leftrightarrow \{v_3\} \leftrightarrow \{v_3, v_4\} \leftrightarrow \{v_4\} \leftrightarrow I_2$. Therefore, $I_1 \leftrightarrow I_2$. □

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