JP Journal of Geometry and Topology

http://www.pphmj.com
http://dx.doi.org/10.17654/GT019040281
Volume 19, Number 4, 2016, Pages 281-289 **ISSN: 0972-415X**

REAL HYPERSURFACE WITH DERIVATIVES OF STRUCTURE LIE OPERATOR ON HOLOMORPHIC DISTRIBUTION $\mathcal{D}$ IN A COMPLEX SPACE FORM

Dong Ho Lim[1], Hoon Joo Kim[1] and Hong Ja Cho[1,2]

[1]Department of Mathematics Education
Sehan University
YoungAm-Gun
Jeollanam-do, 526-702
Republic of Korea
e-mail: dhlim@sehan.ac.kr
hoonjoo@sehan.ac.kr

[2]Department of Early Childhood Education
Sehan University
YoungAm-Gun
Jeollanam-do, 526-702
Republic of Korea
e-mail: h.j.cho@sehan.ac.kr

Abstract

We classify the real hypersurfaces in terms of Lie derivative of structure Lie operator in the direction of the Reeb vector field coincides with the covariant derivative of it in the same direction on holomorphic distribution $\mathcal{D}$ in a nonflat complex space form $M_n(c)$.

Received: July 29, 2016; Revised: September 17, 2016; Accepted: September 24, 2016
2010 Mathematics Subject Classification: Primary 53C40; Secondary 53C15.
Keywords and phrases: holomorphic distribution, structure Lie operator, Lie derivative, model space of type A.
This paper was supported by the Sehan University Research Fund in 2016.

1. Introduction

A complex n-dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. As is well-known, a complete and simply connected complex space form is complex analytically isometric to a complex projective space $P_n\mathbf{C}$, a complex Euclidean space $\mathbf{C}^n$ or a complex hyperbolic space $H_n\mathbf{C}$, according to $c > 0$, $c = 0$ or $c < 0$.

We consider a real hypersurface M in a complex space form $M_n(c)$, $c \neq 0$. Then M has an almost contact metric structure (ϕ, g, ξ, η) induced from the Kaehler metric and complex structure J on $M_n(c)$. The structure vector field ξ is said to be *principal* if $A\xi = \alpha\xi$ is satisfied, where A is the shape operator of M and $\alpha = \eta(A\xi)$. In this case, it is known that α is locally constant [4] and that M is called a *Hopf hypersurface*.

Takagi [13] completely classified homogeneous real hypersurfaces in such hypersurfaces as six model spaces A_1, A_2, B, C, D and E. Berndt [1] classified all homogeneous Hopf hyersurfaces in $H_n\mathbf{C}$ as four model spaces which are said to be A_0, A_1, A_2 and B.

Theorem 1 [13]**.** *Let M be a homogeneous real hypersurface of $P_n\mathbf{C}$. Then M is tube of radius r over one of the following Kaehlerian submanifolds*:

(A_1) *a hyperplane* $P_{n-1}\mathbf{C}$, *where* $0 < r < \frac{\pi}{\sqrt{c}}$;

(A_2) *a totally geodesic* $P_k\mathbf{C}$ $(1 \leq k \leq n-2)$, *where* $0 < r < \frac{\pi}{\sqrt{c}}$;

(B) *a complex quadric* $\mathbf{Q}_{n-1}$, *where* $0 < r < \frac{\pi}{2\sqrt{c}}$;

(C) $P_1(C) \times P_{\frac{n-1}{2}}(C)$, *where* $0 < r < \frac{\pi}{2\sqrt{c}}$ *and* $n \geq 5$ *is odd*;

(D) *a complex Grassmann* $G_{2.5}C$, *where* $0 < r < \frac{\pi}{2\sqrt{c}}$ *and* $n = 9$;

(E) *a Hermitian symmetric space* $SO(10)/U(5)$, *where* $0 < r < \frac{\pi}{2\sqrt{c}}$ *and* $n = 15$.

Theorem 2 [1]**.** *Let M be a real hypersurface in* $H_n\boldsymbol{c}$. *Then M has constant principal curvatures and* ξ *is principal if and only if M is locally congruent to one of the followings*:

(A_0) *a self-tube, that is, a horosphere*;

(A_1) *a geodesic hypersphere*;

(A_2) *a tube over a totally geodesic* $H_k(C)(1 \leq k \leq n-1)$;

(B) *a tube over a totally real hyperbolic space* $H_n(R)$.

A real hypersurface of A_1 or A_2 in $P_n\mathbf{C}$ or A_0, A_1, A_2 in $H_n\mathbf{C}$, then *M* is said to be a type *A* for simplicity.

The induced operator L_ξ on real hypersurface *M* from the 2-form $\mathcal{L}_\xi g$ is defined by $(\mathcal{L}_\xi g)(X, Y) = g(L_\xi X, Y)$ for any vector field *X* and *Y* on *M*, where $\mathcal{L}_\xi$ denotes the operator of the Lie derivative with respect to the structure vector field ξ. This operator L_ξ is given

$$L_\xi = \phi A - A\phi$$

on *M*, and call it structure Lie operator of *M*.

As a typical characterization of real hypersurfaces of type *A*, the following is due to Okumura [12] for $c > 0$ and Montiel and Romero [8] for $c < 0$.

Theorem 3 [8, 12]**.** *Let M be a real hypersurface of* $M_n(c)$, $c \neq 0$, $n \geq 2$. *It satisfies* $L_\xi = 0$ *on M if and only if M is locally congruent to one of the model spaces of type A.*

One of the most interesting problems in the study of real hypersurfaces M in $M_n(c)$ is to investigate a geometric characterization of these model spaces. Recently, some works have studied several conditions on the structure Lie operator L_ξ and given some results on the classification of real hypersurfaces of type A in $M_n(c)$ ([3], [5] and [6], etc.).

The Lie derivative of the shape operator, Ricci operator and Jacobi operator was investigated by Ki et al. [5], Kimura and Maeda [3], Perez et al. [2]. As for the covariant derivative, Lim [8] obtained the following:

Theorem 4 [8]**.** *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 2$. It satisfies $(\mathcal{L}_\xi L_\xi)X = (\nabla_\xi L_\xi)X$ on M if and only if M is locally congruent to one of the model spaces of type A.*

The holomorphic distribution $\mathcal{D}$ of real hypersurface M in $M_n(c)$ is defined by

$$\mathcal{D} = \{X \in T_P(M) \mid g(X, \xi)_p = 0\}. \tag{1}$$

In this paper, we shall study geometric characterizations of real hypersurfaces M on holomorphic distribution $\mathcal{D}$ in a non-flat complex space form $M_n(c)$ with Lie ξ-parallel and ξ-parallel of structure Lie operator. More specifically, we prove the following:

Theorem A. *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$. If M satisfies $(\mathcal{L}_\xi L_\xi)X = (\nabla_\xi L_\xi)X$ on $\mathcal{D}$ in $M_n(c)$. Then M is a Hopf hypersurface on $\mathcal{D}$ in $M_n(c)$.*

Theorem B. *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$. Then M satisfies $(\mathcal{L}_\xi L_\xi)X = (\nabla_\xi L_\xi)X$ on $\mathcal{D}$ in $M_n(c)$ if and only if M is locally congruent to one of the model spaces of type A.*

All manifolds in the present paper are assumed to be connected and of class C^∞ and the real hypersurfaces supposed to be oriented.

2. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M_n(c)$, and N be a unit normal vector field of M. By $\tilde{\nabla}$ we denote the Levi-Civita connection with respect to the Fubini-Study metric tensor $\tilde{g}$ of $M_n(c)$. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \tilde{\nabla}_X N = -AX$$

for any vector fields X and Y tangent to M, where g denotes the Riemannian metric tensor of M induced from $\tilde{g}$, and A is the shape operator of M in $M_n(c)$. For any vector field X on M we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi,$$

where J is the almost complex structure of $M_n(c)$. Then we see that M induces an almost contact metric structure (ϕ, g, ξ, η), that is,

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \tag{2}$$

for any vector fields X and Y on M. Since the almost complex structure J is parallel, we can verify from the Gauss and Weingarten formulas the followings:

$$\nabla_X \xi = \phi AX, \tag{3}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{4}$$

Since the ambient manifold is of constant holomorphic sectional curvature c, we have the following Gauss and Codazzi equations respectively:

$$R(X, Y)Z = \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y$$

$$- 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \tag{5}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\} \tag{6}$$

for any vector fields X, Y and Z on M, where R denotes the Riemannian curvature tensor of M.

By use of (3), we have $(\mathcal{L}_\xi g)(X, Y) = g((\phi A - A\phi)X, Y)$ for any vector fields X and Y on M, and hence the induced operator L_ξ from $\mathcal{L}_\xi g$ is given by

$$L_\xi X = (\phi A - A\phi)X.$$

$$(\mathcal{L}_\xi L_\xi)X = [\xi, L_\xi X] - L_\xi[\xi, X] \tag{7}$$

for any vector fields X in $\mathcal{D}$ and $[\ ,\ ]$ denotes Lie bracket on M.

Let W be a unit vector field on M with the same direction of the vector field $-\phi\nabla_\xi\xi$, and let μ be the length of the vector field $-\phi\nabla_\xi\xi$ if it does not vanish, and zero (constant function) if it vanishes. Then it is easily seen from (1) that

$$A\xi = \alpha\xi + \mu W, \tag{8}$$

where $\alpha = \eta(A\xi)$. We notice here that W is orthogonal to ξ.

We put

$$\Omega = \{p \in M \mid \mu(p) \neq 0\}. \tag{9}$$

Then Ω is an open subset of M.

3. Proof of Theorems

In this section, we shall prove Theorems A and B. Now we prepare without proof the following Lemmas 3.1 and 3.2 in order to prove the results.

Lemma 3.1 [4]**.** *If ξ is a principal curvature vector and the corresponding principal curvature α is locally constant*.

Lemma 3.2 [10]**.** *Assume that ξ is a principal curvature vector and the corresponding principal is α. Then we have*

$$A\phi A - \frac{\alpha}{2}(A\phi + \phi A) - \frac{c}{2}\phi = 0. \tag{10}$$

Proof of Theorem A. Let M be a real hypersurfaces in a complex space form $M_n(c)$, $c \neq 0$, satisfying $(\mathcal{L}_\xi L_\xi)X = (\nabla_\xi L_\xi)X$. We assume that the open set Ω given in (9) is not empty. Then the above condition together with (2) and (8) implies that

$$\phi A^2 \phi X = -A^2 X + \alpha\eta(AX)\xi + \mu\eta(AX)W \tag{11}$$

for any vector field X on $\mathcal{D}$. If we put $X = W$ into (11) and making use of (2) and (8), then we have

$$\phi A^2 \phi W = -A^2 W + \alpha^2\xi + \mu^2 W. \tag{12}$$

If we apply to ϕ to (12) and the first and second equation (2), then we get

$$\phi A^2 W - A^2\phi W = \eta(A^2\phi W)\xi - \mu^2\phi W. \tag{13}$$

Putting $X = \phi W$ into (11) and using the first and fifth equation (2), we obtain

$$\phi A^2 W - A^2\phi W = 0. \tag{14}$$

If we compare (13) and (14), then we obtain

$$\eta(A^2\phi W)\xi - \mu^2\phi W = 0.$$

Taking inner product of the above equation with ϕW, we have $\mu = 0$, and it is contradiction. Thus the set Ω is empty, and hence M is a Hopf hypersurface.

□

Proof of Theorem B. By Theorem A, M is a Hopf hypersurface on $\mathcal{D}$ in $M_n(c)$. Since ξ is a Reeb vector field, the assumption $(\mathcal{L}_\xi L_\xi)X = (\nabla_\xi L_\xi)X$ is equivalent to

$$\phi A^2 \phi X = -A^2 X + \eta(AX)\xi. \tag{15}$$

For any vector field X on M such that $AX = \lambda X$, it follows from (9) that

$$\left(\lambda - \frac{\alpha}{2}\right)A\phi X = \frac{1}{2}\left(\alpha\lambda + \frac{c}{2}\right)\phi X. \tag{16}$$

If $\lambda \neq \frac{\alpha}{2}$, then we see from (16) that ϕX is also a principal direction, say $A\phi X = \mu\phi X$. From (15), we have

$$(\lambda - \mu)(\lambda + \mu) = 0. \tag{17}$$

If $\lambda = \mu$, then by using (16) and the kind of principal curvature and multiplicity, M is locally congruent to A_2. Thus, we have $\phi AX = A\phi X$ for any vector field X on M. On the other hand, if $\lambda = -\mu$ and make use of (16), then we obtain

$$\lambda^2 + \frac{c}{4} = 0. \tag{18}$$

Thus, real hypersurface M does not exists in $P_n(c)$. Next, we assume that $c = -4$ in $H_n(c)$. Then we see from (18) that $\lambda = \pm 1$ and $\mu = \mp 1$. Since M has at most three distinct principal curvatures, we assume that M be locally congruent to one of type B. If $\lambda = \tanh u$, then we have $\tanh u = 1$ or $\tanh u = -1$. Since $\tanh u = \frac{e^u - e^{-u}}{e^u + e^{-u}}$, it is contradiction by the direct calculation. Therefore, Hopf hypersurface M satisfying $\lambda = -\mu$ does not exist. If $\lambda = \frac{\alpha}{2}$, then it is easily seen that $A\phi X = \frac{\alpha}{2}\phi X = \phi AX$ for any vector fields X. Therefore, we have $L_\xi = \phi A - A\phi = 0$ on M. The statement of Theorem B follows immediately from Theorem 1. □

Acknowledgement

The authors thank the anonymous referees for their valuable suggestions which led to the improvement of the manuscript.

References

[1] J. Berndt, Real hypersurfaces with constant principal curvatures in complex hyperbolic space, J. Reine Angew. Math. 395 (1989), 132-141.

[2] J. De Dios Pérez and F. G. Santos, Real hypersurfaces in complex projective space whose structure Jacobi operator satisfies $\mathcal{L}_\xi R_\xi = \nabla_\xi R_\xi$, Rocky Mountain J. Math. 39 (2009), 1293-1301.

[3] M. Kimura and S. Maeda, Lie derivative on hypersurfaces in a complex projective space, Czechoslovak Math. J. 45(120) (1995), 135-148.

[4] U-H. Ki and Y. J. Suh, On real hypersurfaces of a complex space form, Math. J. Okayama Univ. 32 (1990), 207-221.

[5] U-H. Ki, S. Maeda and Y. J. Suh, Lie derivatives on homogeneous real hypersurfaces of type *B* in a complex projective space, Yokohama Math. J. 42 (1994), 121-131.

[6] D. H. Lim, W. H. Sohn and C. G, Jeong, On real hypersurfaces of complex space form with recurrent structure Lie operator, Houston. J. Math. (2016), to appear.

[7] D. H. Lim and W. H. Sohn, On characterizations of real hypersurface in complex space form with Codazzi type structure Lie operator, Monatsh. Math. 173 (2014), 371-378.

[8] D. H. Lim, On real hypersurfaces with derivative the condition of structure Lie operator in a nonflat complex space form, Revista Matemaca Iberoamericana (submitted).

[9] S. Montiel and A. Romero, On some real hypersurfaces of a complex hyperbolic space, Geom. Dedicata 20 (1986), 245-261.

[10] R. Niebergall and P. J. Ryan, Real hypersurfaces in complex space forms, Tight and Taut Submanifolds, Cambridge Univ. Press, 1998, pp. 233-305.

[11] M. Ortega, J. de Dios Pérez and F. G. Santos, Non-existence of real hypersurfaces with parallel structure Jacobi operator in a nonflat complex space forms, Rocky Mountain J. Math. 36 (2006), 1603-1613.

[12] M. Okumura, On some real hypersurfaces of a complex projective space, Trans. Amer. Math. Soc. 212 (1975), 355-364.

[13] R. Takagi, On homogeneous real hypersurfaces in a complex projective space, Osaka J. Math. 10 (1973), 495-506.