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A NOTE ON A PARAMETERIZED FAMILY OF QUARTIC THUE EQUATIONS

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Abstract

In this paper, by the method of Tzanakis, we give all integral solutions of parameterized quartic Thue equation

$$x^4 - 4sx^3y + (12s - 4)x^2y^2 - 8sxy^3 + 4y^4 = 1, \quad s > 18,$$

which are $(x, y) = (1, 1), (-1, -1), (1, 0), (-1, 0)$.

1. Introduction

Let $F(x, y) \in \mathbb{Z}[X, Y]$ be a homogeneous, irreducible polynomial of

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degree $n \geq 3$, and let μ be a nonzero integer. The equation $F(x, y) = \mu$ called a *Thue equation* [6] attracts a lot of attention of mathematicians.

Since 1990, Thomas [5] firstly solved a family of Thue equations with positive discriminant, many experts have studied parameterized Thue equations with different degrees such as [2, 3, 7], and so on.

Tzanakis [7] reduced quartic Thue equations of certain type to a system of Pellian equations. Using the method of Tzanakis, Dujella and Jadrijević [2, 3] solved the parameterized Thue equation

$$x^4 - 4sx^3y + (6s + 2)x^2y^2 - 4sxy^3 + y^4 = \mu. \tag{1.1}$$

Ziegler [8] studied more general equation

$$x^4 - 4sx^3y - (2ab + 4(a + b)s)x^2y^2 - 4absxy^3 + a^2b^2y^4 = \mu,$$

$$\mu \in \{1, -1\}, \tag{1.2}$$

where $a, b \in \frac{1}{4}\mathbb{Z}$, $s \in \mathbb{Z}$. He indicated that when $a \neq b$, $|a| > |b|$, $s > 7.23 \cdot 10^{10}|a|^{\frac{29+\sqrt{241}}{4}}$, if equation (1.2) is solvable, then $\mu = 1$; and listed integral solutions to equation (1.2) in the cases of a, b fixed and s sufficiently large.

In this paper, we solve the case $a = -2$, $b = -1$ in equation (1.2) without restriction of s large enough, and prove the following theorem:

Theorem 1.1. *Let* (x, y) *be an integral solution to equation*

$$x^4 - 4sx^3y + (12s - 4)x^2y^2 - 8sxy^3 + 4y^4 = 1, \quad s > 18. \tag{1.3}$$

Then $(x, y) = (1, 1), (-1, -1), (1, 0), (-1, 0)$.

2. Tzanakis Method and Some Refined Lemmas

Let us follow notations in [8], and put

$$a_0 = 1, \quad a_1 = -s, \quad a_2 = \frac{6s-2}{3}, \quad a_3 = -2s, \quad a_4 = 4.$$

Applying Tzanakis method to (1.3), we have

$$g_2 = \frac{16}{3} - 8s + 4s^2, \quad g_3 = -\frac{64}{27} + \frac{16}{3}s - \frac{8}{3}s^2.$$

$$\rho_1 = -s + \frac{2}{3}, \quad \rho_2 = s - \frac{4}{3}, \quad \rho_3 = \frac{2}{3}.$$

Then by $\frac{a_1^2}{a_0} - a_2 \geq \max\{\rho_1, \rho_2, \rho_3\}$ it follows that equation (1.3) is reduced to

$$(s-2)U^2 - sV^2 = -2, \quad (s-1)U^2 - sZ^2 = -1, \tag{2.1}$$

where

$$U = x^2 - 2y^2, \quad V = x^2 - 4xy + 2y^2, \quad Z = x^2 - 2xy + 2y^2.$$

We next discuss Pellian system of $(s-2)U^2 - sV^2 = -2$ and $(s-1)U^2 - sZ^2 = -1$. Observing that if (U, V, Z) is a solution to (2.1), then also $(\pm U, \pm V, \pm Z)$ is a solution to (2.1), we may assume $U, V, Z > 0$ without loss of generality.

Let (U_n, Z_n) be a positive solution of $(s-1)U^2 - sZ^2 = -1$. Then

$$\begin{aligned} Z_n\sqrt{s} + U_n\sqrt{s-1} &= (\sqrt{s} + \sqrt{s-1})(2s - 1 + 2\sqrt{s(s-1)})^n \\ &= (\sqrt{s} + \sqrt{s-1})^{2n+1}, \end{aligned}$$

and

$$U_n = \frac{1}{2\sqrt{s-1}}[(\sqrt{s} + \sqrt{s-1})^{2n+1} - (\sqrt{s} + \sqrt{s-1})^{-(2n+1)}]. \tag{2.2}$$

Let (U'_m, V_m) be a positive solution of $(s-2)U^2 - sV^2 = -2$. Then

$$V_m\sqrt{s}+U'_m\sqrt{s-2}=(\sqrt{s}+\sqrt{s-2})(s-1+\sqrt{s(s-2)})^m$$

$$=\sqrt{2}\left(\frac{\sqrt{s}+\sqrt{s-2}}{\sqrt{2}}\right)^{2m+1},$$

and

$$U'_m=\frac{1}{\sqrt{2(s-2)}}\left[\left(\frac{\sqrt{s}+\sqrt{s-2}}{\sqrt{2}}\right)^{2m+1}-\left(\frac{\sqrt{s}+\sqrt{s-2}}{\sqrt{2}}\right)^{-(2m+1)}\right]. \quad (2.3)$$

Let (U, V, Z) be a positive solution to equation (2.1). Thus there exist nonnegative integers *m* and *n*, such that $U=U_n=U'_m$, that is,

$$\frac{\alpha^{2n+1}-\alpha^{-(2n+1)}}{2\sqrt{s-1}}=\frac{\beta^{2m+1}-\beta^{-(2m+1)}}{\sqrt{2(s-2)}}, \quad (2.4)$$

where $\alpha=\sqrt{s}+\sqrt{s-1}$ and $\beta=\dfrac{\sqrt{s}+\sqrt{s-2}}{\sqrt{2}}$.

Define

$$\Lambda=(2m+1)\log\beta-(2n+1)\log\alpha+\log\left(\frac{\sqrt{2(s-1)}}{\sqrt{s-2}}\right).$$

Lemma 2.1. *If* $U_n=U'_m$ *with* m, n *positive, then* $1<\dfrac{2m+1}{2n+1}<\dfrac{\log\alpha}{\log\beta}$ *and*

$$0<\Lambda<\frac{\beta^2}{\beta^2-1}\cdot\beta^{-2(2m+1)}.$$

Proof. It is trivial that $m>n$. Now we can assert $\beta^{2m+1}<\alpha^{2n+1}$, which directly yields $1<\dfrac{2m+1}{2n+1}<\dfrac{\log\alpha}{\log\beta}$. Otherwise $\beta^{2m+1}\geq\alpha^{2n+1}$ and $2\sqrt{s-1}>\sqrt{2(s-2)}$ lead to $U_n<U'_m$. Thus we have

$$0 < \Lambda = \log\left(\frac{1-\alpha^{-2(2n+1)}}{1-\beta^{-2(2m+1)}}\right) < -\log(1-\beta^{-2(2m+1)}) < \frac{\beta^2}{\beta^2-1}\cdot\beta^{-2(2m+1)}.$$

□

Lemma 2.2. $U_n \equiv (-1)^n (1-2(n+1)s) \pmod{16s^2}$, $U'_m \equiv (-1)^m (1-m(m+1)s) \pmod{4s^2}$.

Proof. From recurrence relations it follows that

$$U_n = (4s-2)U_{n-1} - U_{n-2}, \quad U'_m = (2s-2)U'_{m-1} - U'_{m-2}.$$

We induce on n and m, respectively. The lemma is trivial. □

Lemma 2.3. *If* $U_n = U'_m$ *with* $m, n \neq 0$, *then* $m > 1.1547\sqrt{s} - 1$ *and* $n > 0.95825\sqrt{s} - 0.91493$.

Proof. From Lemma 2.2, $U_n = U'_m \pmod{4s^2}$ and thus

$$(-1)^n(1-2n(n+1)s) \equiv (-1)^m(1-m(m+1)s) \pmod{4s^2}.$$

We assume that $m \leq 1.1547\sqrt{s} - 1$ and define $\mathcal{A} = (-1)^n(1-2n(n+1)s) - (-1)^m(1-m(m+1)s)$. Then, we have

$$0 < |\mathcal{A}| < 3m(m+1)s + 2 < 3(m+1)^2 s.$$

From assumption of m it follows that $0 < |\mathcal{A}| < 4s^2$, which contradicts $\mathcal{A} \equiv 0 \pmod{4s^2}$. Thus $m > 1.1547\sqrt{s} - 1$ holds.

Since $1 < \frac{2m+1}{2n+1} < \frac{\log\alpha}{\log\beta}$, by Lemma 2.1 and then $1 < \frac{2m+1}{2n+1} < 1.205$, we obtain $n > 0.95825\sqrt{s} - 0.91493$. □

Lemma 2.4. *Let* (U, V, Z) *be a positive solution to equation* (2.1) *with* $s > 18$. *Then*

$$\left|\sqrt{\frac{s-1}{s}}-\frac{Z}{U}\right|<\frac{1}{U^2\sqrt{s(s-1)}},\quad \left|\sqrt{\frac{s-2}{s}}-\frac{V}{U}\right|<\frac{2}{U^2\sqrt{s(s-2)}}.$$

Proof.

$$\left|Z-\sqrt{\frac{s-1}{s}}U\right|=\frac{\left|Z-\sqrt{\frac{s-1}{s}}U\right|\left|Z+\sqrt{\frac{s-1}{s}}U\right|}{\left|Z+\sqrt{\frac{s-1}{s}}U\right|}<\frac{1}{U\sqrt{s(s-1)}}.$$

Dividing U on both sides of the equation, we obtain the first inequality. We can prove the second inequality by the same technique. □

Lemma 2.5 [1, Theorem 3.2]**.** *If* a_i, p_i, q *and* N *are integers for* $0 \le i \le 2$, *with* $a_0 < a_1 < a_2$, $a_j = 0$ *for some* $0 \le j \le 2$, q *nonzero and* $N > M^9$, *where* $M = \max_{0\le i\le 2}\{|a_i|\}$, *then we have*

$$\max_{0\le i\le 2}\left\{\left|\sqrt{1+\frac{a_i}{N}}-\frac{p_i}{q}\right|\right\}>(130N\gamma)^{-1}q^{-\lambda},$$

where

$$\lambda=1+\frac{\log(33N\gamma)}{\log\left(1.7N^2\prod_{0\le i<j\le 2}(a_i-a_j)^{-2}\right)},$$

and

$$\gamma=\begin{cases}\dfrac{(a_2-a_0)^2(a_2-a_1)^2}{2a_2-a_0-a_1} & a_2-a_1\ge a_1-a_0,\\[2ex] \dfrac{(a_2-a_1)^2(a_1-a_0)^2}{a_1+a_2-2a_0} & a_2-a_1<a_1-a_0.\end{cases}$$

Applying Lemma 2.3, we have

Lemma 2.6. *If* $U_n = U'_m$ *with* m, n *nonnegative and* $s > 512$, *then* $m = n = 0$.

Proof. We assume m, $n > 0$ and take $a_0 = -2$, $a_1 = -1$, $a_2 = 0$, $p_0 = V$, $p_1 = Z$, $p_2 = U$, $q = U$, and $N = s$, $M = \max_{0\le i\le 2}\{|a_i|\} = 2$. $N > M^9$ implies $s > 512$.

We have $\gamma = \dfrac{4}{3}$ since $a_2 - a_1 \ge a_1 - a_0$. From $\prod_{0\le i<j\le 2}(a_i - a_j)^2 = 4$, we have $\lambda = 1 + \dfrac{\log(44s)}{\log(0.425s^2)}$. If $2 - \lambda > 0$, $s \ge 104$. Combining Lemmas 2.4 and 2.5, we obtain

$$(130N\gamma)^{-1}U^{-\lambda} < \max_{0\le i\le 2}\left\{\left|\sqrt{1+\frac{a_i}{N}} - \frac{p_i}{q}\right|\right\} < \frac{2}{U^2\sqrt{s(s-2)}}.$$

Taking logarithms and solving for $\log U$ yields

$$\log U < \frac{\log(1040s/3) - \frac{1}{2}\log((s-2)s)}{2-\lambda}. \tag{2.5}$$

Note that $U_n > (4s-3)^n$, $U'_m > (2s-3)^m$. By Lemma 2.3 it follows that

$$\log U > n\log(4s-3) > \left(\sqrt{\frac{4}{3}s} - 1\right)\log(4s-3). \tag{2.6}$$

Thus, we combine (2.5) and (2.6) to deduce $s \le 141$ which contradicts $s > 512$. Therefore, the lemma is proved. □

Next our aim is to discuss $18 < s \le 512$. We will employ Baker's method (cf. an improvement of Matveev [4]) to determine the upper bounds of m and n, and use Baker-Davenport reduction method to decide existence of solutions for many cases.

Lemma 2.7 [4, Corollary 2.3]**.** *Let* $\alpha_1, \alpha_2, ..., \alpha_l$ *be algebraic numbers, not* 0, 1, *and let* $\log\alpha_1, \log\alpha_2, ..., \log\alpha_l$ *represent determinations of their logarithms. Let* D *be the degree over* $\mathbb{Q}$ *of the number field* $\mathbb{K} = \mathbb{Q}(\alpha_1, \alpha_2, ..., \alpha_l)$, *let* $b_1, b_2, ..., b_l$ *be rational integers. Define* $B =$

$\max\{|b_1|, |b_2|, \ldots, |b_l|\}$, *and* $A_i = \max\{Dh(\alpha_i), |\log(\alpha_i)|, 0.16\}$, $(1 \le i \le l)$, *where* $h(\alpha)$ *denotes the absolute logarithmic Weil height of* α. *Assume that the number* $\Lambda = b_1 \log\alpha_1 + b_2 \log\alpha_2 + \cdots + b_l \log\alpha_l$ *does not vanish, then*

$$|\Lambda| \ge \exp\{-C(l, \chi) D^2 A_1 A_2 \cdots A_l \log(eD)\log(eB)\},$$

where $\chi = 1$ *if* $\mathbb{K} \subset \mathbb{R}$ *and* $\chi = 2$ *otherwise, and*

$$C(l, \chi) = \min\left\{\frac{1}{\chi}\left(\frac{1}{2}el\right)^{\chi} 30^{l+3} l^{3.5},\ 2^{6l+20}\right\}.$$

Recall

$$\Lambda = (2m+1)\log\beta - (2n+1)\log\alpha + \log\left(\frac{\sqrt{2(s-1)}}{\sqrt{s-2}}\right).$$

We apply Lemma 2.7 to Λ. It is obvious that $l = 3$, $\chi = 1$, $D = 4$, $b_1 = 2m + 1$, $b_2 = -(2n+1)$, $b_3 = 1$, $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $\alpha_3 = \dfrac{\sqrt{2(s-1)}}{\sqrt{s-2}}$. Accordingly we obtain that $B = 2m+1$, $h(\alpha_1) = \dfrac{1}{2}\log\alpha$, $h(\alpha_2) = \dfrac{1}{2}\log\beta$, $h(\alpha_3) = \dfrac{1}{4}\log(2(s-1)(s-2))$, and therefore $A_1 = 2\log\alpha$, $A_2 = 2\log\beta$, $A_3 = \log(2(s-1)(s-2))$.

A direct computation by Lemma 2.7 shows that

$$C(3, 1) = \min\left\{\frac{1}{2} 3e \cdot 30^6 3^{3.5},\ 2^{38}\right\} = 1.39007 \cdot 10^{11},$$

and

$$\log\Lambda \ge -2.12295 \cdot 10^{13} \cdot \log\alpha \log\beta \log(2(s-1)(s-2))\log((2m+1)e). \tag{2.7}$$

By Lemma 2.1, we have

$$\log\Lambda < \log\frac{\beta^2}{\beta^2 - 1} - 2(2m+1)\log\beta < 0.03 - 2(2m+1)\log\beta. \tag{2.8}$$

Combining (2.7) and (2.8), we have

$$\frac{2(2m+1)\log\beta - 0.03}{\log((2m+1)e)} < 2.12295 \cdot 10^{13} \log\alpha \log\beta \log(2(s-1)(s-2)).$$

For $18 < s \leq 512,$ we solve the above inequality to get $m < 1.84544 \cdot 10^{15}$.

Lemma 2.8 [3]. *Assume that* M' *is a positive integer. Let* $\frac{p}{q}$ *be the convergent of the continued fraction expansion of* κ *such that* $q > 10M'$ *and let* $\varepsilon = \| \mu' q \| - M' \cdot \| \kappa q \|,$ *where* $\| \vartheta \|$ *denotes the distance from* ϑ *and its nearest integer. If* $\varepsilon > 0,$ *then there is no solution of the inequality*

$$0 < m'\kappa - n' + \mu' < AB^{-m'}$$

in integers m' *and* n' *with* $\frac{\log(Aq/\varepsilon)}{\log B} \leq m' \leq M.$

We put

$$\kappa = \frac{\log\beta}{\log\alpha}, \quad \mu = \frac{\log\sqrt{\frac{2(s-1)}{s-2}}}{\log\alpha}, \quad A = \frac{\beta^2}{(\beta^2-1)\log\alpha}, \quad B = \beta^2,$$

with $m' = 2m+1$, $n' = 2n+1$, and $M' = 1.84544 \cdot 10^{15}$. By running PARI/GP program, we can reduce the size of the bound of m to 15. Thus the left work is to check some special cases for m.

3. Proof of Theorem 1.1

In Section 2, we actually prove that $U_n = U'_m$ holds if and only if $m = n = 0$ whether $s > 512$ or $s \leq 512$. So equation (2.1) admits the only trivial positive solution $(U, V, Z) = (1, 1, 1)$. By assumptions on $U, V, Z > 0$ we know $(U, V, Z) = (\pm 1, \pm 1, \pm 1)$ are all integral solutions to (2.1).

We look back $x^2 - 2y^2 = \pm 1,$ $x^2 + 4xy + 2y^2 = \pm 1,$ $x^2 - 2xy + 2y^2$

$= \pm 1$. Note that $x^2 - 2xy + 2y^2 = (x - y)^2 + y^2 = \pm 1$. Hence $y = \pm 1$ and $x - y = 0$, or $x - y = \pm 1$ and $y = 0$, which can verify $(x, y) = (1, 1)$, $(-1, -1)$, $(1, 0)$, $(-1, 0)$.

So, the theorem is proved.

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