Far East Journal of Mathematical Sciences (FJMS)

http://www.pphmj.com
http://dx.doi.org/10.17654/MS101050983
Volume 101, Number 5, 2017, Pages 983-992 **ISSN: 0972-0871**

CERTAIN INTEGRAL AND SERIES REPRESENTATIONS OF LEGENDRE FUNCTIONS DERIVABLE FROM THE REPRESENTATION OF THE GROUP $SO(2, 1)$

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Received: November 16, 2016; Accepted: December 26, 2016
2010 Mathematics Subject Classification: Primary 33C10, 33C80; Secondary 33B15, 33C05.
Keywords and phrases: Legendre function, integral representation, series representation, group $SO(2, 1)$, Cartan decomposition, matrix elements of group representation, bases transforms.
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Abstract

Earlier, we have noted application of matrix elements in the image of a circle rotation with respect to the hyperbolic basis in the representation space of the group $SO(2, 1)$. In this sequel, we aim to use the matrix elements of the image of a hyperbolic rotation with respect to the circle basis to derive certain integral and series representations for the Legendre functions P_{μ}^{ν}.

1. Introduction

This investigation is a sequel of our work [8], in which we have obtained several integral and series representations involving Whittaker or Bessel functions, or their products, which have been derived from certain connections between so-called circle and parabolic bases in relation to a representation of the group $SO(2, 1)$ and values of matrix elements of the representation in some particular cases with respect to the above bases. We have used the above-mentioned approach for $SO(2, 1)$ in [5], for $SO(2, 2)$ in [7], and for $SO(3, 1)$ in [6]. In particular, in [6], the classical (basic and modified) Bessel functions have disappeared but the general hypergeometric functions ${}_0F_3$, which are Delerue's multi-index generalizations of the modified Bessel functions of the first and second kinds (see [1]), have emerged.

Here, by using the circle and hyperbolic bases in connection with the group $SO(2, 1)$, we aim to derive certain integral and series representations for Legendre functions P_{μ}^{ν} and P_{μ}. It is noted that a different technique, with a connection between analogical bases for the group $SO(3, 1)$ and the index general Mehler-Fock integral transform [11], has been used in [4].

2. Notations and Definitions

We choose to recall some necessary notations and definitions which have been introduced in the authors' earlier works (see [4-8]). In this sequel, we deal with the group $SO(2, 1)$ consisting of 3×3-matrices g satisfying $g\,\mathrm{diag}(1, -1, -1)\,g^T = \mathrm{diag}(1, -1, -1)$. We use the representations

$$SO(2, 1) \to GL(\mathfrak{D}) \quad \text{and} \quad SO(2, 1) \to GL(\mathfrak{D}^\bullet),$$

where $\mathfrak{D}$ and $\mathfrak{D}^\bullet$ are the spaces of infinitely differentiable functions defined on the cone

$$\Lambda := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1^2 - x_2^2 - x_3^2 = 0\}$$

and satisfy the condition of σ- and $(-\sigma - 1)$-homogeneity $(\sigma \in \mathbb{C} \backslash \mathbb{Z})$, respectively, and $GL(\mathfrak{D})$ defines the corresponding group of linear operators whose determinants are not equal to zero. Here and in the following, let $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Z}$ be the sets of complex, real numbers and integers, respectively. The image of T consists of linear operators $T(g) : f(x) \mapsto f(g^{-1}x)$. Likewise the representation $T^\bullet$ is defined. More precisely, we deal with the subgroups H_1 and H_2 acting transitively on the circle section $\gamma_1 : x_1 = 1$ and the hyperbolic section $\gamma_2 : x_2 = \pm 1$ of the cone Λ, respectively. The following parameterizations are used:

$$\gamma_1 = \{x(\varphi) = (1, \cos\varphi, \sin\varphi) \,|\, \varphi \in [-\pi, \pi)\}$$

and

$$\gamma_2 = \{x(s) = (\cosh s, \pm 1, \sinh s) \,|\, s \in \mathbb{R}\}.$$

The functionals

$$\mathsf{F}_i : \mathfrak{D} \times \mathfrak{D}^\bullet \to \mathbb{C}, \quad (f, g) \mapsto \int_{\gamma_i} f(x) g(x) (dx)_i,$$

where $(dx)_i$ are H_i-invariant measures and which coincide and are invariant with respect to the pair $(T, T^\bullet)$.

In Sections 3 and 4, we obtain matrix elements of some representation operators with respect to the *circle* and *hyperbolic* bases, for example, whose definitions for the space $\mathfrak{D}^{\bullet}$ look as follows (see [10]):

$$B_1^{\bullet} := \{ f_k^{\bullet}(x) = x_1^{-k-\sigma-1}(x_2 + \mathbf{i}x_3)^k \,|\, k \in \mathbb{Z}\}$$

and

$$B_2^{\bullet} := \{ f_{\rho,\pm}^{*\bullet}(x) = (x_2)_{\pm}^{-1-\sigma-\mathbf{i}\rho}(x_1 + x_3)^{\mathbf{i}\rho} \,|\, \rho \in \mathbb{R}\}.$$

3. Matrix Elements of the Linear Operators $B_2^{\bullet} \to B_1^{\bullet}$ and $T^{\bullet}|_{H_2}$ and the Corresponding Integral Representation of the Legendre Function $P_{\sigma}^{k}(\cosh\theta)$

Express an arbitrary function belonging to the basis $B_1^{\bullet}$ as a linear combination of the functions belonging to $B_2^{\bullet}$:

$$f_k^{\bullet}(x) = \int_{\mathbb{R}} [m_{k,\rho,+} f_{\rho,+}^{*\bullet}(x) + m_{k,\rho,-} f_{\rho,-}^{*\bullet}(x)]d\rho. \tag{3.1}$$

Since

$$(f_{\rho,\pm}^{*\bullet}(x)\, f_{\hat{\rho},\pm}^{*}(x))|_{\gamma_{\pm}} = \exp(\mathbf{i}[\rho + \hat{\rho}]s)s = 2\pi\delta(\rho + \hat{\rho})$$

and

$$(f_{\rho,\pm}^{*\bullet}(x)\, f_{\hat{\rho},\pm}^{*}(x))|_{\gamma_{\mp}} = (f_{\rho,\pm}^{*\bullet}(x)\, f_{\hat{\rho},\mp}^{*}(x))|_{\gamma_{\mp}} = 0,$$

we have

$$m_{k,\rho,\pm} = \frac{1}{2\pi}\mathsf{F}_i(f_k^{\bullet},\ f_{-\rho,\pm}^{*}).$$

Lemma 1. *The following identity holds true*: *For* $\Re(\sigma) > -1$,

$$m_{k,\rho,\pm} = 2^{-\sigma-1}\pi^{-1}\mathbf{i}^k F(\pm k),$$

where

$$F(k) = (-1)^k \exp\left(-\frac{\pi}{2}[\rho + \mathbf{i}(\sigma+1)]\right) B(-k-\sigma, 1+\sigma-\mathbf{i}\rho)$$

$$\times {}_2F_1(-\sigma - \mathbf{i}\rho, -k-\sigma; -k-\mathbf{i}\rho+1; -1) + \exp\left(\frac{\pi}{2}[-\rho + \mathbf{i}(\sigma+1)]\right)$$

$$\times B(-k-\sigma, 1+\sigma+\mathbf{i}\rho)\, {}_2F_1(-\sigma + \mathbf{i}\rho, -k-\sigma; \mathbf{i}\rho + 1 - k; -1),$$

where B *and* ${}_2F_1$ *denote the Beta function and hypergeometric function, respectively* (*see*, *e.g*., [9, *Sections* 1.1 *and* 1.5]).

Proof. By using the substitution $\varphi = \frac{\pi}{2} - 2t,$ we express the integral

$$m_{k,\rho,\pm} = \frac{1}{2\pi}\int_{\pm\frac{\pi}{2}}^{\pi\pm\frac{\pi}{2}} \exp(\mathbf{i}k\varphi)(\pm\cos\varphi)^{\sigma+\mathbf{i}\rho}(1+\sin\varphi)^{-\mathbf{i}\rho} d\varphi$$

in the form

$$m_{k,\rho,\pm} = 2^{\sigma}\pi^{-1}\mathbf{i}^k \int_0^{\frac{\pi}{2}} (\sin t)^{\sigma+\mathbf{i}\rho}(\cos t)^{\sigma-\mathbf{i}\rho} \exp(\mp 2kt)\, dt,$$

for which a known integral formula (see [2, Entry 2.5.32.6]) is used. □

In particular, using a known formula (see, e.g., [3, Entry 7.3.6.2]))

$${}_2F_1(a, b; 1+a-b; -1) = 2^{-a}\sqrt{\pi}\,\Gamma\begin{bmatrix} 1+a-b \\ \frac{a+1}{2}, \frac{a}{2}+1-b \end{bmatrix},$$

we have

$$m_\rho \equiv m_{0,\rho,\pm}$$

$$= \pi^{-\frac{1}{2}}\left\{2^{-1+\mathbf{i}\rho} \exp\left(-\frac{\pi}{2}[\rho + \mathbf{i}(\sigma+1)]\right)\Gamma\begin{bmatrix} -\sigma, 1+\sigma-\mathbf{i}\rho \\ \frac{1-\sigma-\mathbf{i}\rho}{2}, 1+\frac{\sigma-\mathbf{i}\rho}{2} \end{bmatrix}\right.$$

$$\left. + 2^{-1-\mathbf{i}\rho} \exp\left(\frac{\pi}{2}[-\rho + \mathbf{i}(\sigma+1)]\right)\Gamma\begin{bmatrix} -\sigma, 1+\sigma+\mathbf{i}\rho \\ \frac{1-\sigma+\mathbf{i}\rho}{2}, 1+\frac{\sigma+\mathbf{i}\rho}{2} \end{bmatrix}\right\}.$$

Theorem 1. *The following representation holds true*: *For* $\Re(\sigma) > -1$,

$$\int_{-\infty}^{+\infty} \exp(-\mathbf{i}\theta\rho) m_\rho (m_{-\hat{k}, -\rho, +} + m_{-\hat{k}, -\rho, -}) d\rho = \{\mathbf{i}^{\hat{k}} (\sigma + 1)_{\hat{k}}\}^{-1} P_\sigma^{\hat{k}} (\cosh \theta), \tag{3.2}$$

where $P_\sigma^{\hat{k}}$ *are the Legendre functions of the first kind* (*see, e.g.*, [9, *p*. 68]).

Proof. Consider the following expression:

$$T^\bullet(g)[f_k^\bullet] = \sum_{\hat{k} \in \mathbb{Z}} t_{k\hat{k}}(g) f_{\hat{k}}^\bullet.$$

It is clear that

$$t_{k\hat{k}}(g) = \frac{1}{2\pi} \mathsf{F}_1(T^\bullet(g)[f_k^\bullet], f_{-\hat{k}}). \tag{3.3}$$

In case where g is a hyperbolic rotation in $(x_1 x_3)$-plane through angle θ, we have

$$\begin{aligned} & t_{k\hat{k}}(g) \\ & = \frac{1}{2\pi} \sum_{s_1, s_2 \in \{+, -\}} \iint_{(\rho, \hat{\rho}) \in \mathbb{R}^2} m_{k, \rho, s_1} m_{-\hat{k}, \hat{\rho}, s_2} \mathsf{F}_2(T^\bullet(g)[f_{\rho, s_1}^{*\bullet}], f_{\hat{\rho}, s_2}^*) d\rho d\hat{\rho} \\ & = \int_{-\infty}^{+\infty} \exp(-\mathbf{i}\theta\rho)(m_{k, \rho, +} m_{-\hat{k}, -\rho, +} + m_{k, \rho, -} m_{-\hat{k}, -\rho, -}) d\rho. \end{aligned} \tag{3.4}$$

On the other hand, let us represent g as a product $g = g_1 g_2 g_3$, where g_1 and g_3 are the trigonometric rotations in the $(x_2 x_3)$-plane through angles θ_1 and θ_3, respectively, and g_2 is a hyperbolic rotation in the $(x_1 x_2)$-plane through angle θ_2 (Cartan decomposition). Then

$$\begin{aligned} t_{0\hat{k}}(g) & = \frac{\exp(-\mathbf{i}\hat{k}\theta_3)}{\pi} \int_0^\pi (\cosh \theta_2 - \sinh \theta_2 \cos \varphi)^\sigma \cos(\hat{k}\varphi) d\varphi \\ & = \{(\sigma + 1)_{\hat{k}}\}^{-1} \exp(-\mathbf{i}\hat{k}\theta_1) P_\sigma^{\hat{k}} (\cosh \theta_2), \end{aligned} \tag{3.5}$$

where we have used a known formula (see [2, Entry 2.5.16.28]). By solving the matrix equation $g_1^{-1}g = g_2 g_3$, we obtain $\theta_1 = \frac{\pi}{2}$, $\theta_2 = \theta$, $\theta_3 = \frac{3\pi}{2}$. Now the desired representation (3.2) is seen to follow from (3.4) and (3.5). □

4. Matrix Elements of the Linear Operators $B_1^{\bullet} \to B_2^{\bullet}$ and $T^{\bullet}|_{H_1}$ and the Associated Series Representation of the Legendre Function $P_\sigma(|\cos\vartheta|)$

Throughout this section, let h be a trigonometric rotation in the $(x_2 x_3)$-plane through angle ϑ which is not an integral multiple of π. We denote by $t_{\rho,\alpha,\hat{\rho},\beta}(g)$ $(\alpha, \beta \in \{+, -\})$ the matrix elements of the linear operator $T^{\bullet}(h)$ with respect to the basis $B_2^{\bullet}$, that is,

$$T^{\bullet}(h)[f_{\rho,\pm}^{*,\bullet}] = \int_{\mathbb{R}} (t_{\rho,\pm,\hat{\rho},+}(h)\, f_{\hat{\rho},+}^{*\bullet} + t_{\rho,\pm,\hat{\rho},-}(h)\, f_{\hat{\rho},-}^{*\bullet})\, d\rho.$$

Lemma 2. *The following identity holds true*: *For* $\Re(\sigma) > -1$,

$$t_{0,\pm,0,+}(h) = t_{0,\pm,0,-}(h) = -\csc\sigma \cdot {}_2F_1\left(\frac{\sigma+1}{2}, -\frac{\sigma}{2}; 1; \sin^2\vartheta\right).$$

Proof. We have

$$t_{0,\pm,0,+}(h) = \frac{(\sin\vartheta)^\sigma}{2\pi} \sum_{l=0}^{1} \int_{(-1)^l \cot\vartheta}^{+\infty} (\sinh s - (-1)^l \cot\vartheta)^\sigma ds$$

and

$$t_{0,\pm,0,-}(h) = \frac{(\sin\vartheta)^\sigma}{2\pi} \sum_{l=0}^{1} \int_{-\infty}^{(-1)^l \cot\vartheta} (-\sinh s + (-1)^l \cot\vartheta)^\sigma ds.$$

In order to evaluate the first $(l = 0)$ and the second $(l = 1)$ integrals in $t_{0,\pm,0,\pm}(h)$, we introduce new variable $u = \cot\vartheta \pm \sinh s$ and $u = -\cot\vartheta \pm \sinh s$, respectively. Thus, we have

$$t_{0,\pm,0,\pm}(h)=\frac{(\sin\vartheta)^{\sigma}}{2\pi}\sum_{l=0}^{1}\int_{0}^{+\infty}u^{\sigma}(u^{2}+(-1)^{l}\cdot 2u\cot\vartheta+1+\cot^{2}\vartheta)^{-\frac{1}{2}}du.$$

Finally, using a known formula (see [2, Entry 2.2.9.7])

$$\int_{0}^{+\infty}\frac{x^{\alpha-1}dx}{(ax^{2}+2bx+c)^{\mu}}=\frac{B(\alpha,\,2\rho-\alpha)}{a^{\frac{a}{2}}c^{\mu-\frac{\alpha}{2}}}\,{}_{2}F_{1}\left(\frac{\alpha}{2},\,\mu-\frac{\alpha}{2};\,\mu+\frac{1}{2};\,1-\frac{b^{2}}{ac}\right)$$

$$(a>0,\,b^{2}<ac,\,0<\Re(\alpha)<2\Re(\mu)),$$

we obtain the desired formula. □

A modification of Lemma 2 with the help of a known formula [3, Entry 7.3.1.41] gives another identity asserted by the following lemma:

Lemma 3. *The following identity holds true*: *For* $\Re(\sigma)>-1,\ \ \vartheta\neq\pi k$ $(k\in\mathbb{Z})$,

$$t_{0,\pm,0,+}(h)=t_{0,\pm,0,-}(h)=-\csc\sigma\cdot P_{\sigma}(|\cos\vartheta|),$$

where P_{σ} *are Legendre functions* (*see*, *e.g.*, [3, *Entry* 7.3.1.41]).

Theorem 2. *Let*

$$M(k)=\begin{cases}\sin^{2}\dfrac{\sigma\pi}{2} & (k\equiv 0\ (\mathrm{mod}\,2)),\\[2mm] \cos^{2}\dfrac{\sigma\pi}{2} & (k\equiv 1\ (\mathrm{mod}\,2)).\end{cases}$$

Then

$$\sum_{k\in\mathbb{Z}}\csc(\sigma\pi)\left\{\Gamma\left(\frac{1-\sigma-k}{2}\right)\Gamma\left(\frac{k-\sigma-1}{2}+1\right)\Gamma\left(\frac{\sigma+k}{2}+1\right)\right\}^{-1}$$

$$\times\left\{\Gamma\left(\frac{\sigma-k}{2}+1\right)\right\}^{-1}M(k)\exp(-\mathbf{i}\vartheta k)$$

$$=\frac{\csc^{2}\sigma}{4\pi}P_{\sigma}(|\cos\vartheta|),$$

where $\Re(\sigma)>-1$ *and* $\vartheta\neq\pi t$ $(t\in\mathbb{Z})$.

Proof. Consider the matrix element $m_{k,\rho,\pm}$, which is defined in Section 3, as a function of σ, that is, $m_{k,\rho,\pm} \equiv m_{k,\rho,\pm}(\sigma)$. It is easy to establish that the coefficients of the expression

$$f^{*\bullet}_{\rho,\pm}(x) = \sum_{k \in \mathbb{Z}} \tilde{m}_{\rho,\pm,k}(\sigma) f^{\bullet}_k(x)$$

can be expressed in the form

$$\tilde{m}_{\rho,\pm,k}(\sigma) = \frac{1}{2\pi} \mathsf{F}_i(f^{*\bullet}_{\rho,\pm}, f_{-k}) = m_{-k,-\rho,\pm}(-\sigma - 1). \tag{4.1}$$

By using the well-known formula connecting the matrices of the linear operator $T^{\bullet}(h)$ with respect to bases $B^{\bullet}_1$ and $B^{\bullet}_2$, and taking that $B^{\bullet}_1$ consists of the eigenfunctions of the operator $T^{\bullet}(h)$, that is, the first of the above matrices being diagonal, we have

$$t_{\rho,\pm,\hat{\rho},s}(h) = \sum_{k \in \mathbb{Z}} m_{k,\rho,\pm} \tilde{m}_{\hat{\rho},s,k} \exp(-\mathbf{i}\vartheta k), \quad s \in \{+, -\}.$$

Putting here $\rho = \hat{\rho} = 0$ and using Lemmas 1 and 2, and (4.1), the proof is complete. □

References

[1] P. Delerue, Sur le calcul symbolique an variables et les fonctions hyperbesseliennes, Ann. Soc. Sci. Bruxelles 67(3) (1953), 229-274.

[2] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, Integrals and Series, Vol. 1: Elementary Functions, OPA (Overseas Publishers Association), Amsterdam B.V. Published under the license of Gordon and Breach Science Publishers, 1986.

[3] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, Integrals and Series, Vol. 3. More Special Functions, OPA (Overseas Publishers Association), Amsterdam B.V. Published under the license of Gordon and Breach Science Publishers, 1986.

[4] I. A. Shilin and J. Choi, Certain connections between the spherical and hyperbolic bases on the cone and formulas for related special functions, Integral Transforms Spec. Funct. 25(5) (2014), 374-383.

[5] I. A. Shilin and J. Choi, Certain relations between Bessel and Whittaker functions related to some diagonal and block-diagonal 3×3-matrices, submitted.

[6] I. A. Shilin and J. Choi, Improper integrals associated with the Bessel functions and their multi-index analogues, Appl. Math. Sci. 10(34) (2016), 1683-1692.

[7] I. A. Shilin and J. Choi, Integral and series representations of special functions related to the group $SO(2, 2)$, Ramanujan J. (in press).

[8] I. A. Shilin and J. Choi, On a family of series and integrals involving Whittaker or Bessel functions or their products, submitted.

[9] H. M. Srivastava and J. Choi, Zeta and q-Zeta Functions and Associated Series and Integrals, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[10] N. J. Vilenkin and M. A. Sleinikova, Integral relations for the Whittakers functions and the representations of the three-dimensional Lorentz group, Math. USSR Sb. 10 (1970), 173-180.

[11] S. Yakubovich, Index Transforms, World Scientific, 1996.