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$*_p$ -MODULES AND A SPECIAL CLASS OF MODULES DETERMINED BY THE ESSENTIAL CLOSURE OF THE CLASS OF ALL $*$-RINGS

Puguh Wahyu Prasetyo[1], **Indah Emilia Wijayanti**[2] **and Halina France-Jackson**[3]

[1,2]Department of Mathematics
Universitas Gadjah Mada
Indonesia

[1]Department of Mathematics Education
Ahmad Dahlan University
Indonesia

[3]Department of Mathematics
Nelson Mandela Metropolitan University
South Africa

Abstract

A ring A is called a $*$-ring if A is a prime ring and A has no nonzero proper prime homomorphic image. The $*$-ring was introduced by Korolczuk in 1981. Since $*$-rings have an important role in radical theory of rings, the properties of $*$-ring have been being investigated

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intensively. Since every ring can be viewed as a module over itself, the generalization of $*$-ring into module theory is an interesting investigation. We would like to present the generalization of $*$-rings in module theory named $*_p$-modules. An A-module M is called a $*_p$-module if M is a prime A-module and M has no nonzero proper prime submodule. According to the result of our investigation, we show that every $*$-ring is a $*_p$-module over itself. Furthermore, let A be a ring, let M be an A-module, and let I be an ideal of A with $I \subseteq (0 : M)_A$, where $(0 : M)_A = \{a \in A | aM = \{0\}\}$. We show that M is a $*_p$-module over A if and only if M is a $*_p$-module over A/I. On the other hand, the essential closure $*_k$ of the class of all $*$-rings is a special class of rings. As the last result of our investigation, we present the special class of modules determined by $*_k$.

1. Introduction

Let A be a ring. A ring A is called a *prime ring* if $\{0\}$ is a prime ideal of A (Gardner and Wiegandt [5]). Any homomorphic image of a ring A can be represented as A/I, where I is an ideal of A. The homomorphic image A/I of A is called a *prime homomorphic image* if A/I is a prime ring. The class of rings σ is hereditary if σ contains all ideals of a ring $A \in \sigma$. The class of rings σ is essentially closed if σ is closed under essential extensions. Let π denote the class of all prime rings. A subclass μ of π is called a *special class* if μ is hereditary and μ is essentially closed. For hereditary class of rings ϱ, the upper radical $\mathcal{U}(\varrho)$ is defined as the class of all ring A such that A has no nonzero homomorphic image in ϱ. The prime radical β is the upper radical determined by the class of all prime rings π.

A prime ring A is called a *$*$-ring* if A has no nonzero proper ideal I of A such that A/I is a prime ring (Korolczuk [6]). Some properties of $*$-rings were presented in (France-Jackson [2]). $*$-rings have been being studied intensively in radical theory of rings because of Gardner's question

mentioned in (Gardner [4]). Let $*$ denote the class of all $*$-rings and let $*_k$ denote the essential closure of $*$. The essential closure $*_k$ of $*$ is a special class of rings. Gardner asked whether the prime radical β coincide with the upper radical $\mathcal{U}(*_k)$ determined by $*_k$. (France-Jackson et al. [3]) have given an alternative solution of this question to have a positive answer.

On the other hand, let M be an A-module. An A-module M is called a *prime A-module* if $AM \neq \{0\}$ and for $m \in M$ and $J \lhd A$ such that $Jm = \{0\}$ implies $m = 0$ or $JM = \{0\}$. The set $(0 : M)_A = \{a \in A | aM = \{0\}\}$ is called an *annihilator* of an A-module M. An A-module is faithful if $(0 : M)_A = \{0\}$ (Gardner and Wiegandt [5]).

Theorem 1.1 (Gardner and Wiegandt [5])**.** *Let A be a ring and let $I \trianglelefteq A$.*

(1) *If M is an A/I-module, then with scalar multiplication $am = (a + I)m$, M forms an A-module with $I \subseteq (0 : M)_A$.*

(2) *If M is an A-module and $I \subseteq (0 : M)_A$, then M is an A/I-module with the scalar multiplication $(a + I)m = am$.*

(3) *If M is an A-module and $I \subseteq (0 : M)_A$, then N is a submodule of the A/I-module if and only if N is a submodule of the A-module M.*

(4) $(0 : M)_A/I = (0 : M)_{A/I}$.

(Gardner and Wiegandt [5]) For every ring A, let $\sum_A$ denote the class of all A-modules M with $AM \neq \{0\}$, and $\sum = \bigcup \sum_A$. Let $\ker(\sum_A) = \bigcap ((0 : M)_A | M \in \sum_A)$ and we consider the class $\sum$ might satisfy the following conditions:

1. (M1) If $M \in \sum_{A/I}$, then $M \in \sum_A$.

2. (M2) If $M \in \sum_A$ and $I \trianglelefteq A$, $I \subseteq (0 : M)_A$, then $M \in \sum_{A/I}$.

3. (M3) If $\ker(\sum_A) = \{0\}$, then $\sum_B \neq \{\varnothing\}$ for all nonzero ideals B of A.

4. (M4) If $\sum_B \neq \{\varnothing\}$ whenever $\{0\} \neq B \trianglelefteq A$, then $\ker(\sum_A) = \{0\}$.

Proposition 1.2 (Gardner and Wiegandt [5])**.** *Let A be a ring and let $I \trianglelefteq A$. Then there is a prime A-module M such that $(0 : M)_A = I$ if and only if I is a prime ideal of A.*

Definition 1.3 (Gardner and Wiegandt [5])**.** For every ring A, let $\sum_A$ be a class of prime A-modules and let $\sum = \bigcup \sum_A$. The class $\sum$ is called a *special class of modules* if $\sum$ satisfies (M1), (M2), and the following conditions:

1. (SM3) If $M \in \sum_A$, $B \trianglelefteq A$ and $BM \neq \{0\}$, then $M \in \sum_B$.

2. (SM4) If $B \trianglelefteq A$ and $M \in \sum_B$, then $BM \in \sum_A$.

If $\sum$ is a special class of modules, then $\mu = \{A \mid A$ has a faithful module in $\sum_A\}$ is a special class of rings. Conversely, if μ is a special class of rings and we define $\sum_A = \{M \mid M$ is a prime A-module and $A/(0 : M)_A \in \mu\}$, then $\sum = \bigcup \sum(A)$ is a special class of modules (Nicholson and Watters [7]).

Example 1.4. Let π denote the class of all prime rings and for every ring A let $\sum_A = \{M \mid M$ is a prime A-module and $A/(0 : M)_A \in \pi\}$. Since π is a special class of rings, the class $\sum = \bigcup \sum_A$ is a special class of modules.

These basic theories motivate us to investigate the special class of modules generated by $*_k$.

2. Main Results

Let M be an A-module. A homomorphic image M/N of A-module M is called a *prime homomorphic image* of M if M/N is a prime A-module. Since every ring can be viewed over itself, we will give a new type of module

named $*_p$-module. This kind of module is motivated by the existence of $*$-ring.

Definition 2.1. Let M be an A-module. A-module M is called a $*_p$-*module* if M is a prime A-module and M has no nonzero proper prime homomorphic image.

The necessary and sufficient condition for A-module M to be a $*_p$-module is given below.

Lemma 2.2. *Let M be an A-module. The following conditions are equivalent*:

1. *M is a $*_p$-module over A.*

2. *M is a prime A-module and every proper prime submodule N of M implies $N = \{0\}$.*

Proof. (1) $\Rightarrow$ (2) Let M be a $*_p$-module over A. By the definition, we have M is a prime A-module. Furthermore, M has no nonzero proper prime image. Let N be a proper prime submodule of M. Suppose $N \neq \{0\}$. Then M/N is a nonzero proper prime homomorphic image of M, a contradiction.

(2) $\Rightarrow$ (1) Let M be a prime A-module and every proper prime submodule N of M implies $N = \{0\}$. Suppose M/N is a nonzero prime homomorphic image of M. This gives N is a proper prime submodule of M. This implies that $N = \{0\}$. So, we may conclude that M has no nonzero proper prime homomorphic image. □

Some modules are naturally $*_p$-module. In the next lemma, we show that every simple module M over a ring A is a $*_p$-module.

Lemma 2.3. *Let A be a commutative ring and M be an A-module. If M is a simple A-module, then M is a $*_p$-module over A.*

Proof. Let $a \in A$ and $m \in M$ such that $am = 0$. Suppose $a \neq 0 \Rightarrow a \in (0 : m)$. Thus, $m \in M_r$, where M_r is a torsion submodule of M. Since M is a simple A-module, we have $M_r = \{0\} \Rightarrow m = 0$ or $M_r = M \Rightarrow a \in (0 : M)$. Hence, M is a prime A-module. Since M is a simple A-module, A-module M has no nonzero proper prime homomorphic image. So, M is a $*_p$-module. □

Example 2.4. 1. (Adkins and Weintraub [1]). An abelian group A is a simple $\mathbb{Z}$-module if and only if A is a cyclic group of prime order. Hence, A is a $*_p$-module over the ring $\mathbb{Z}$ of integers if A is a cyclic group of prime order.

2. The integers modulo prime number $\mathbb{Z}_p$ is a simple $\mathbb{Z}$-module. Hence, A is a $*_p$-module over $\mathbb{Z}_p$.

3. (Adkins and Weintraub [1]). Let $V = \mathbb{R}^2 = \{(a, b) | a, b \in \mathbb{R}\}$ and consider the linier transformation $T : V \to V$ defined by $T(u, v) = (-v, u)$. Then the $\mathbb{R}[X]$-module V_T is a simple $\mathbb{R}[X]$-module. So, we may deduce that V_T is $*_p$-module over $\mathbb{R}[X]$.

The following theorem shows that every $*$-ring is a $*_p$-module.

Theorem 2.5. *Let A be a ring. If A is a $*$-ring, then A is a $*_p$-module over itself.*

Proof. We will show that A is a prime A-module. For this step, we can follow Corollary 3.14.17 in (Gardner and Wiegandt [5]) or we give the other way to proof. Since A is a prime ring, $AA = A^2 \neq \{0\}$. Suppose A is not a prime A-module. Then there exists $J \lhd A$ with $JA \neq \{0\}$ and $0 \neq a \in A$ such that $Ja = \{0\}$. Since $0 \neq a \in A$, we can construct the nonzero ideal $\langle a \rangle$ of A generated by a such that $J\langle a \rangle = \{0\}$, contrary to A is a prime ring.

Suppose A is not a $*_p$-module. Then there exists a nonzero proper prime submodule I of A. In the other words, A/I is a prime A-module. Now define $(0 : A/I)_A = \{a \in A | a(A/I) = \{0\}\}$. Clearly $(0 : A/I)_A \neq \{0\}$, because $0 \neq I \subseteq (0 : A/I)_A$. We will show that $(0 : A/I)_A$ is a prime ideal of A. Let $J, K \lhd A$ such that $JK \subseteq (0 : A/I)_A$. If $K \nsubseteq (0 : A/I)_A$, let $k \in K$, $\bar{a} = a + I \in A/I$ be such that $k\bar{a} = \{\bar{0}\}$. Then $J(k\bar{a}) \subseteq JK\bar{a} \subseteq (0 : A/I)_A \bar{a} = \{\bar{0}\}$. So, $J(A/I) = \{\bar{0}\}$. This gives $J \subseteq (0 : A/I)_A$. Hence, $(0 : A/I)_A$ is a prime ideal of A, contrary to A is a $*$-ring. □

The converse of Theorem 2.5 is not true in general.

Example 2.6. The ring $J = \{2x/2y + 1 | \gcd(2x, 2y + 1) = 1, x, y \in \mathbb{Z}\}$ is a $*$-ring. By Theorem 2.5, we have J is a $*_p$-module over J. However, the module J over itself is not a simple module.

Lemma 2.7. *Let A be a ring. If M is a $*_p$-module over A, then every nonzero proper homomorphic image of a $*_p$-module over A is not a $*_p$-module over A.*

Proof. Let A be a ring and consider M is a $*_p$-module over A. Suppose M/N is a nonzero proper homomorphic image of M. Clearly, M/N is not a prime A-module. Hence, M/N is not a $*_p$-module over A. □

In the following theorem, we give a sufficient condition for an A-module M to be a $*_p$-module over A.

Theorem 2.8. *Let I be an ideal of a ring A with $I \subseteq (0 : M)_A$ and let M be an A-module such that $AM \neq \{0\}$. If M is a $*_p$-module over the factor ring A/I, then M is a $*_p$-module over A.*

Proof. Let M be a $*_p$-module over the factor ring A/I. Then M is a prime A/I-module. By Proposition 3.14.15 in (Gardner and Wiegandt [5]), we have M is a prime A-module. Suppose there exists a nonzero proper prime homomorphic image M/N of M over A. It follows from Proposition 3.14.15 in Gardner and Wiegandt [5], we have M/N is a prime A/I-module. In the other words, M has a nonzero proper prime homomorphic image over A/I, contrary to M is a $*_p$-module. Hence, M has no nonzero proper prime homomorphic image over A. Thus, M is a $*_p$-module over A. □

The following theorem shows the consequence of the existence of a $*_p$-module M over a ring A.

Theorem 2.9. *Let I be an ideal of a ring A such that $I \subseteq (0 : M)_A$ and let M be an A-module such that $AM \neq \{0\}$. If M is a $*_p$-module over the ring A, then M is a $*_p$-module over A/I.*

Proof. Let M be a $*_p$-module over the ring A and let I be an ideal of a ring A such that $I \subseteq (0 : M)_A$. Clearly, M is a prime A/I-module. Suppose there exists a nonzero proper prime homomorphic image M/N of M over A/I. Then M/N is a prime A/I-module. By Proposition 3.14.15 in (Gardner and Wiegandt [5]), we have M/N is a prime A-module, contrary to M is a $*_p$-module. So, we may conclude that M is a $*_p$-module over A/I. □

Theorem 2.10. *Let $*_k$ be the essential closure of the class of all $*$-rings and for every ring A let $\sum_A = \{M \mid M$ is a prime A-module with $A/(0 : M)_A \in *_k\}$. Then the class $\sum = \bigcup \sum_A$ is a special class of modules.*

Proof. We can follow the construction of a special class of modules generated by a special class of rings presented in (Nicholson and Watters [7]) or we will explain the detail of proof by showing that the class $\sum = \bigcup \sum_A$ satisfies (M1), (M2), (SM3), and (SM4). Let M be an A-module such that

$M \in \sum_{A/I}$. Then M is a prime A/I-module with $(A/I)/(0:M)_{A/I} \in *_k$. By Proposition 3.14.15 in (Gardner and Wiegandt [5]), M is a prime A-module. Let $\bar{a} \in (0:M)_{A/I} \Rightarrow \bar{a}M = \{0\}$, where $\bar{a} = a + I$ for some $a \in A$. Since $\{0\} = (a+I)M = aM$, $a \in (0:M)_A$ and by the assumption $I \subseteq (0:M)_A$ implies $\bar{a} = a + I \in (0:M)_A/I$. Hence, $(0:M)_{A/I} \subseteq (0:M)_A/I$. On the other hand, let $a \in (0:M)_A \Rightarrow aM = \{0\}$. Since $\{0\} = aM = (a+I)M \Rightarrow a+I \in (0:M)_{A/I}$. Hence, $(0:M)_A/I \subseteq (0:M)_{A/I}$. So, we may conclude that $(0:M)_A/I = (0:M)_{A/I}$. This gives us the following isomorphism $A/(0:M)_A \cong (A/I)/(0:M)_A/I = (A/I)/(0:M)_{A/I} \in *_k$. We can infer that $M \in \sum_A$.

Let $M \in \sum_A$. Then M is a prime A-module with $A/(0:M)_A \in *_k$. By following Proposition 3.14.15 in (Gardner and Wiegandt [5]), we have M is a prime A/I-module, where $I \subseteq (0:M)_A$. Since $A/(0:M)_A \in *_k$ and $(A/I)/(0:M)_{A/I} \cong A/(0:M)_A$, we have $M \in \sum_{A/I}$.

Let $M \in \sum_A$ and let $B \lhd A$ such that $BM \neq \{0\}$.

By Proposition 3.14.13 in (Gardner and Wiegandt [5]), we have M is a prime B-module. Since $B/(0:M)_B = B/(B \cap (0:M)_A) \cong (B + (0:M)_A)/(0:M)_A \lhd A/(0:M)_A \in *_k$ and $*_k$ is a special class of rings, we have $B/(0:M)_B \in *_k$.

Let $B \lhd A$ and let $M \in \sum_M$. Then M is a prime B-module with $B/(0:M)_B \in *_k$. By Proposition 3.14.14 in (Gardner and Wiegandt [5]), we have BM is a prime A-module with respect to a $\sum b_i m_i = \sum (ab_i) m_i$, $a \in A, b_i \in B, m_i \in M$. We will show that $A/(0:BM)_A \in *_k$. Furthermore, $B/(0:M)_B = B/(B \cap (0:BM)_A) \cong (B + (0:BM)_A)/(0:BM)_A \in *_k$. On the other hand, $(B + (0:BM)_A)/(0:BM)_A \lhd A/(0:BM)_A$. Since $*_k$ is a special class of rings, $*_k$ satisfies the following condition:

If $\{0\} \neq I \lhd A$, $I \in *_k$ and A is a prime ring, then $A \in *_k$.

We have the following facts:

$$(B + (0 : BM)_A)/(0 : BM)_A \lhd A/(0 : BM)_A \text{ and}$$

$$(B + (0 : BM)_A)/(0 : BM)_A \text{ is a prime ring.}$$

So, we may conclude that $A/(0 : BM)_A \in *_k$. This implies $BM \in \sum_A$. Hence, the class $\sum = \bigcup \sum_A$, where $\sum_A = \{M \mid M$ is a prime A-module such that $A/(0 : M)_A \in *_k\}$, is the special class of modules determined by $*_k$. □

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