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COLLOCATION METHOD WITH QUINTIC *B*-SPLINE METHOD FOR SOLVING COUPLED BURGERS' EQUATIONS

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Abstract

In the present paper, a numerical method is proposed for the numerical solution of a coupled system of Burgers' equation by using the quintic *B*-spline collocation scheme on the uniform mesh points. The method is shown to be unconditionally stable using von Neumann technique. To test accuracy the error norms L_2, L_∞ are computed and give some examples to illustrate the sufficiency of the method for solving such nonlinear partial differential equations. Computed results are depicted graphically and are compared with those already available in the literature.

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1. Introduction

The coupled Burgers' equations as are given in [1]:

$$u_t - u_{xx} + k_1 uu_x + k_2(uv)_x = 0, \tag{1}$$

$$v_t - v_{xx} + k_1 vv_x + k_3(uv)_x = 0, \tag{2}$$

where k_1, k_2 and k_3 are real constants and subscripts x and t denote differentiation w.r.t. the distance x and time t, respectively, with the following boundary conditions:

$$u(a, t) = f_1(a, t), \quad u(b, t) = f_2(b, t),$$

$$v(a, t) = g_1(a, t), \quad v(b, t) = g_2(b, t), \quad 0 \le t \le T \tag{3}$$

and initial conditions

$$u(x, 0) = f(x),$$

$$v(x, 0) = g(x), \quad a \le x \le b. \tag{4}$$

Numerical solutions for above have been provided by various authors, including Radwan [2], Khater et al. [3], Ali et al. [4], Rashid and Ismail [5], Liu and Hou [6], Mittal and Arora [7], Mokhtari et al. [8], Sadek and Kucuk [9], Mittal and Jiwar [10], Kutluay and Ucar [11], Srivastava et al. [12], Kumar and Pandit [13], Srivastava et al. [14], Mittal and Tripathi [15], Abdou and Soliman [16], Dehghan et al. [17]. We have studied coupled nonlinear Burgers' equations by using non-polynomial spline method [18]. Also, we take linearization of nonlinear term using finite difference approximation and applying Crank-Nicolson scheme. Quintic *B*-spline collocation method is used to find numerical solutions of some nonlinear equations in [19-22]. The short outline of this paper is as follows: In Section 2, quintic *B*-spline collocation scheme is explained. In Sections 3 and 4, the method is illustrated and applied to the coupled Burgers' equations. In Section 5, a stability of the method is present. In Section 6, numerical examples are included to verify the applicability and accuracy of the proposed method computationally. In Section 7, the conclusion gives a summary of what has been done in this paper.

2. Quintic *B*-spline Functions

To construct numerical solution, consider nodal points (x_j, t_n) defined in the region $[a, b] \times [0, T]$, where

$$a = x_0 < x_1 < \cdots < x_N = b, \quad h = x_{j+1} - x_j = \frac{b-a}{N}, \quad j = 0, 1, ..., N,$$

$$0 = t_0 < t_1 < \cdots < t_n < \cdots < T, \quad t_{j+1} - t_j = \Delta t, \quad t_n = n\Delta t, \quad n = 0, 1, \ldots.$$

The quintic *B*-spline basis functions at knots are given by:

$$B_j(x) = \frac{1}{h^5}\begin{cases} (x - x_{j-3})^3 & x_{j-3} \le x \le x_{j-2} \\ (x - x_{j-3})^5 - 6(x - x_{j-2})^5, & x_{j-2} \le x \le x_{j-1} \\ (x - x_{j-3})^5 - 6(x - x_{j-2})^5 + 15(x - x_{j-1})^5, & x_{j-1} \le x \le x_j \\ (-x + x_{j-3})^5 + 6(x - x_{j+2})^5 - 15(x - x_{j+1})^5, & x_j \le x \le x_{j+1} \\ (-x + x_{j+3})^5 + 6(x - x_{j+2})^5, & x_{j+1} \le x \le x_{j+2} \\ (-x + x_{j+3})^5, & x_{j+2} \le x \le x_{j+3} \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Using quintic *B*-spline basis function (5) the values of $B_j(x)$ and its derivatives at the knots points can be calculated, which are tabulated in Table 1.

3. Solution of Coupled Burgers' Equations

To apply the proposed method, we rewrite (1) and (2) as

$$\frac{\partial^2 u(x, t)}{\partial x^2} = \frac{\partial u(x, t)}{\partial t} + k_1 u(x, t)\frac{\partial u(x, t)}{\partial x} + k_2[u(x, t)v(x, t)]_x,$$

$$\frac{\partial^2 v(x, t)}{\partial x^2} = \frac{\partial v(x, t)}{\partial t} + k_1 v(x, t)\frac{\partial v(x, t)}{\partial x} + k_3[u(x, t)v(x, t)]_x,$$

and consider the approximations $u(x, t) = U_j^n$ and $v(x, t) = V_j^n$, then from famous Cranck-Nicolson scheme and forward finite difference approximation for the derivative t, [23], we get

$$\left[\frac{U_{xxj}^{n+1} + U_{xxj}^{n}}{2}\right] = \frac{U_j^{n+1} - U_j^n}{k} + k_1\left[\frac{(UU_x)_j^{n+1} + (UU_x)_j^n}{2}\right] + k_2\left[\frac{(UV_x)_j^{n+1} + (UV)_{xj}^n}{2}\right], \tag{6}$$

$$\left[\frac{V_{xxj}^{n+1} + V_{xxj}^{n}}{2}\right] = \frac{V_j^{n+1} - V_j^n}{k} + k_1\left[\frac{(VV_x)_j^{n+1} + (VV_x)_j^n}{2}\right] + k_3\left[\frac{(UV)_{xj}^{n+1} + (UV)_{xj}^n}{2}\right], \tag{7}$$

where $k = \Delta t$ is the time step.

Table 1. The values of quintic B-spline and its first and second derivatives at the knots points

x	x_{j-3}	x_{j-2}	x_{j-1}	x_j	x_{j+1}	x_{j+2}	x_{j+3}
B_j	0	1	26	66	26	1	0
B_j'	0	$\frac{-5}{h}$	$\frac{-50}{h}$	0	$\frac{50}{h}$	$\frac{5}{h}$	0
B_j''	0	$\frac{20}{h^2}$	$\frac{40}{h^2}$	$\frac{-120}{h^2}$	$\frac{40}{h^2}$	$\frac{20}{h^2}$	0

In the Crank-Nicolson scheme, the time stepping process is half explicit and half implicit. So the method is better than simple finite difference method.

The nonlinear terms in equations (6) and (7) are linearized using the form given by Rubin and Graves [24] as: we take linearization of the nonlinear term as follows:

$$(UU_x)_j^{n+1} = U_j^n U_{xj}^{n+1} + U_j^{n+1} U_{xj}^n - U_j^n U_{xj}^n,$$

$$(VV_x)_j^{n+1} = V_j^n V_{xj}^{n+1} + V_j^{n+1} V_{xj}^n - V_j^n V_{xj}^n. \tag{8}$$

Similarly the linearized form for $(UV)_x$ can be obtained. Expressing $U(x, t)$ and $V(x, t)$ by using quintic *B*-spline functions $B_j(x)$ and the time dependent parameters $c_j(t)$ and $\delta_j(t)$, for $U(x, t)$ and $V(x, t)$, respectively, the approximate solution can be written as:

$$U_N(x, t) = \sum_{j=-2}^{N+2} c_j(t) B_j(x), \qquad V_N(x, t) = \sum_{j=-2}^{N+2} \delta_j(t) B_j(x). \tag{9}$$

Using approximate function (9) and quintic *B*-spline functions (5), the approximate values $U(x)$, $V(x)$ and their derivatives up to second order are determined in terms of the time parameters $c_j(t)$ and $\delta_j(t)$, respectively, as

$$U_j = U(x_j) = c_{j-2} + 26c_{j-1} + 66c_j + 26c_{j+1} + c_{j+2},$$

$$U'_j = U'(x_j) = \frac{5}{h}(c_{j+2} + 10c_{j+1} - 10c_{j-1} - c_{j-2}),$$

$$U''_j = U''(x_j) = \frac{20}{h^2}(c_{j-2} + 2c_{j-1} - 6c_j + 2c_{j+1} + c_{j+2}),$$

$$V_j = V(x_j) = \delta_{j-2} + 26\delta_{j-1} + 66\delta_j + 26\delta_{j+1} + \delta_{j+2},$$

$$V'_j = V'(x_j) = \frac{5}{h}(\delta_{j+2} + 10\delta_{j+1} - 10\delta_{j-1} - \delta_{j-2}),$$

$$V''_j = V''(x_j) = \frac{20}{h^2}(\delta_{j-2} + 2\delta_{j-1} - 6\delta_j + 2\delta_{j+1} + \delta_{j+2}). \tag{10}$$

On substituting the approximate solution for *U*, *V* and its derivatives from equation (10) at the knots in equations (6) and (7) yields the following difference equation with the variables $c_j(t)$s and $\delta_j(t)$,

$$A_1 c_{j-2}^{n+1} + A_2 c_{j-1}^{n+1} + A_3 c_j^{n+1} + A_4 c_{j+1}^{n+1} + A_5 c_{j+2}^{n+1} + A_6 \delta_{j-2}^{n+1} + A_7 \delta_{j-1}^{n+1}$$

$$+ A_8 \delta_j^{n+1} + A_9 \delta_{j+1}^{n+1} + A_{10} \delta_{j+2}^{n+1}$$

$$= A_{11} c_{j-2}^{n} + A_{12} c_{j-1}^{n} + A_{13} c_j^{n} + A_{12} c_{j+1}^{n} + A_{11} c_{j+2}^{n}, \tag{11}$$

$$B_1 \delta_{j-2}^{n+1} + B_2 \delta_{j-2}^{n+1} + B_3 \delta_j^{n+1} + B_4 \delta_{j+1}^{n+1} + B_5 \delta_{j+2}^{n+1} + B_6 \delta_{j-2}^{n+1} + B_7 c_{j-1}^{n+1}$$

$$+ B_8 c_j^{n+1} + B_9 c_{j+1}^{n+1} + B_{10} c_{j+1}^{n+1}$$

$$= A_{11} \delta_{j-2}^{n} + A_{12} \delta_{j-1}^{n} + A_{13} \delta_j^{n} + A_{12} \delta_{j+1}^{n} + A_{11} \delta_{j+2}^{n}, \tag{12}$$

where

$$A_1 = 1 - \frac{10\Delta t}{h^2} - \frac{5k_1\Delta t}{2h} z_1 + \frac{5k_1\Delta t}{2h} z_2 - \frac{5k_2\Delta t}{2h} z_3 + \frac{5k_2\Delta t}{2h} z_4,$$

$$A_2 = 26 - \frac{20\Delta t}{h^2} - \frac{25k_1\Delta t}{h} z_1 + \frac{65k_1\Delta t}{h} z_2 - \frac{25k_2\Delta t}{h} z_3 + \frac{65k_2\Delta t}{h} z_4,$$

$$A_3 = 66 + \frac{60\Delta t}{h^2} + \frac{165k_1\Delta t}{h} z_2 + \frac{165k_2\Delta t}{h} z_4,$$

$$A_4 = 26 - \frac{20\Delta t}{h^2} + \frac{25k_1\Delta t}{h} z_1 + \frac{65k_1\Delta t}{h} z_2 + \frac{25k_2\Delta t}{h} z_3 + \frac{65k_2\Delta t}{h} z_4,$$

$$A_5 = 1 - \frac{10\Delta t}{h^2} + \frac{5k_1\Delta t}{2h} z_1 + \frac{5k_1\Delta t}{2h} z_2 + \frac{5k_2\Delta t}{2h} z_3 + \frac{5k_2\Delta t}{2h} z_4,$$

$$A_6 = \frac{5k_2\Delta t}{h} z_2 - \frac{5k_2\Delta t}{2h} z_1, \qquad A_7 = \frac{65k_2\Delta t}{h} z_2 - \frac{25k_2\Delta t}{h} z_1,$$

$$A_8 = \frac{165k_2\Delta t}{h} z_2, \qquad A_9 = \frac{65k_2\Delta t}{h} z_2 + \frac{25k_2\Delta t}{h} z_1,$$

$$A_{10} = \frac{5k_2\Delta t}{2h} z_2 + \frac{5k_2\Delta t}{2h} z_1, \qquad A_{11} = 1 + \frac{10\Delta t}{h^2},$$

$$A_{12} = 26 + \frac{20\Delta t}{h^2}, \qquad A_{13} = 66 - \frac{60\Delta t}{h^2},$$

$$B_1 = 1 - \frac{10\Delta t}{h^2} - \frac{5k_1\Delta t}{2h} z_3 + \frac{5k_1\Delta t}{2h} z_4 - \frac{5k_3\Delta t}{2h} z_1 + \frac{5k_3\Delta t}{2h} z_2,$$

$$B_2 = 26 - \frac{20\Delta t}{h^2} - \frac{25k_1\Delta t}{h} z_3 + \frac{65k_1\Delta t}{h} z_4 - \frac{25k_3\Delta t}{h} z_1 + \frac{65k_3\Delta t}{h} z_2,$$

$$B_3 = 66 + \frac{60\Delta t}{h^2} - \frac{165k_1\Delta t}{h} z_4 + \frac{165k_3\Delta t}{h} z_2,$$

$$B_4 = 26 - \frac{20\Delta t}{h^2} + \frac{25k_1\Delta t}{h} z_3 + \frac{65k_1\Delta t}{h} z_4 + \frac{25k_3\Delta t}{h} z_1 + \frac{65k_3\Delta t}{h} z_2,$$

$$B_5 = 1 - \frac{10\Delta t}{h^2} + \frac{5k_1\Delta t}{2h} z_3 + \frac{5k_1\Delta t}{2h} z_4 + \frac{5k_3\Delta t}{2h} z_1 + \frac{5k_3\Delta t}{2h} z_2,$$

$$B_6 = \frac{5k_3\Delta t}{2h} z_4 - \frac{5k_3\Delta t}{2h} z_3, \qquad B_7 = \frac{65k_3\Delta t}{h} z_4 - \frac{25k_3\Delta t}{h} z_3,$$

$$B_8 = \frac{165k_3\Delta t}{h} z_4, \qquad B_9 = \frac{65k_3\Delta t}{h} z_4 + \frac{25k_3\Delta t}{h} z_3,$$

$$B_{10} = \frac{5k_3\Delta t}{2h} z_4 + \frac{5k_3\Delta t}{2h} z_3,$$

$$z_1 = c_{j-2} + 26c_{j-1} + 66c_j + 26c_{j+1} + c_{j+2},$$

$$z_2 = c_{j+2} + 10c_{j+1} - 10c_{j-1} - c_{j-2},$$

$$z_3 = \delta_{j-2} + 26\delta_{j-1} + 66\delta_j + 26\delta_{j+1} + \delta_{j+2},$$

$$z_4 = \delta_{j+2} + 10\delta_{j+1} - 10\delta_{j-1} - \delta_{j-2}.$$

The system thus obtained on simplifying equations (11) and (12) consists of $(2N+2)$ linear equations in the $(2N+10)$ unknowns $(c_{-2}, c_{-1}, c_0, \ldots, c_{N+1}, c_{N+2}),\ (\delta_{-2}, \delta_{-1}, \delta_0, \ldots, \delta_N, \delta_{N+1}, \delta_{N+2})^T$. To obtain a unique solution to the resulting system four additional constraints are required. These are obtained by imposing boundary conditions. Eliminating $c_{-2}, c_{-1}, c_{N+1}, c_{N+2}$ and $\delta_{-2}, \delta_{-1}, \delta_{N+1}, \delta_{N+2}$, the system gets reduced to a matrix system of dimension $(2N+2)\times(2N+2)$ which is the penta-diagonal system that can be solved by any algorithm.

4. Initial Values

At a particular time-level, the approximate solutions $U(x, t)$ and $V(x, t)$ can be determined repeatedly by solving the recurrence relation, once the initial vectors have been computed from the initial and boundary conditions. From the initial condition $u(x_j, 0) = f(x_j)$, we get $(N+1)$ linear equations in the $(N+5)$ unknowns. The four unknowns c_{-2}, c_{-1}, c_{N+1} and c_{N+2} can be obtained from the relations $u_x(x_0, 0) = f'(x_0)$, $u_x(x_N, 0) = f'(x_N)$, $u_{xx}(x_0, 0) = f''(x_0)$, $u_{xx}(x_N, 0) = f''(x_N)$, at the knots. It leads to system of $(N+1)$ linear equations in the $(N+1)$ unknowns. Which can be solved by any algorithm. Similarly, using initial condition $v(x_j, 0) = g(x_j)$, the initial vectors for v can be computed.

5. Stability Analysis of the Method

The stability analysis of nonlinear partial differential equations is not easy task to undertake. Most researchers copy with the problem by linearizing the partial differential equation. Our stability analysis will be based on the von Neumann concept in which the growth factor of a typical Fourier mode is defined as

$$c_j^n = A\zeta^n \exp(ij\phi),$$

$$\delta_j^n = B\zeta^n \exp(ij\phi),$$

$$g = \frac{\zeta^{n+1}}{\zeta^n}, \tag{13}$$

where A and B are the harmonics amplitude, $\phi = kh$, k is the mode number, $i = \sqrt{-1}$ and g is the amplification factor of the schemes. We applied the stability of the quintic schemes by assuming the nonlinear term as constants λ_1, λ_2. This is equivalent to assuming that all the c_j^n and δ_j^n as local constants λ_1, λ_2, respectively. At $x = x_j$ system (11) can be written as

$$a_1 c_{j-2}^{n+1} + a_2 c_{j-1}^{n+1} + a_3 c_j^{n+1} + a_4 c_{j+2}^{n+1} + a_5 c_{j+2}^{n+1} + a_6 \delta_{j-2}^{n+1} + a_7 \delta_{j-1}^{n+1}$$

$$+ a_8 c_{j+1}^{n+1} + a_9 \delta_{j+2}^{n+1} = a_{10} c_{j-2}^{n} + a_{11} c_{j-1}^{n} + a_{12} c_j^{n} + a_{13} c_{j+1}^{n} + a_{14} c_{j+2}^{n}$$

$$- a_6 \delta_{j-2}^{n} - a_7 \delta_{j-1}^{n} - a_8 \delta_{j+1}^{n} - a_9 \delta_{j+2}^{n}, \tag{14}$$

where

$$a_1 = 1 - \frac{10\Delta t}{h^2} - \frac{5k_1\Delta t}{2h}\lambda_1 - \frac{5k_2\Delta t}{2h}\lambda_2,$$

$$a_2 = 26 - \frac{20\Delta t}{h^2} - \frac{25k_1\Delta t}{h}\lambda_1 - \frac{25k_2\Delta t}{h}\lambda_2,$$

$$a_3 = 66 + \frac{60\Delta t}{h^2}, \qquad a_4 = 26 - \frac{20\Delta t}{h^2} + \frac{25k_1\Delta t}{h}\lambda_1 + \frac{25k_2\Delta t}{h}\lambda_2,$$

$$a_5 = 1 - \frac{10\Delta t}{h^2} + \frac{5k_1\Delta t}{2h}\lambda_1 + \frac{5k_2\Delta t}{2h}\lambda_2, \qquad a_6 = -\frac{5k_2\Delta t}{2h}\lambda_1,$$

$$a_7 = -\frac{25k_2\Delta t}{h}\lambda_1, \qquad a_8 = \frac{5k_2\Delta t}{2h}\lambda_1,$$

$$a_9 = \frac{25k_2\Delta t}{h}\lambda_1, \qquad a_{10} = 1 - \frac{10\Delta t}{h^2} + \frac{5k_1\Delta t}{2h}\lambda_1 + \frac{5k_2\Delta t}{2h}\lambda_2,$$

$$a_{11} = 26 - \frac{20\Delta t}{h} + \frac{25k_1\Delta t}{h}\lambda_1 + \frac{25k_2\Delta t}{h}\lambda_2, \qquad a_{12} = 66 - \frac{60\Delta t}{h^2},$$

$$a_{13} = 26 - \frac{20\Delta t}{h} - \frac{25k_1\Delta t}{h}\lambda_1 - \frac{25k_2\Delta t}{h}\lambda_2,$$

$$a_{14} = 1 - \frac{10\Delta t}{h^2} - \frac{5k_1\Delta t}{2h}\lambda_1 - \frac{5k_2\Delta t}{2h}\lambda_2.$$

Substituting (13) into the difference (14), we get

$$\zeta^{n+1}\left[\begin{array}{l} A\left[2\left(1-\frac{10\Delta t}{h^2}\right)\cos 2\phi + 2\left(26-\frac{20\Delta t}{h^2}\right)\cos\phi + \left(66+\frac{60\Delta t}{h^2}\right)\right] \\ + i\left[\begin{array}{l} \sin 2\varphi\left(A\left(\frac{5k_2\Delta t}{h}\lambda_2 + \frac{5k_1\Delta t}{h}\lambda_1\right) + B\left(\frac{5k_2\Delta t}{h}\lambda_1\right)\right) \\ + \sin\varphi\left(A\left(\frac{50k_2\Delta t}{h}\lambda_2 + \frac{50k_1\Delta t}{h}\lambda_1\right) + B\left(\frac{50k_2\Delta t}{h}\lambda_1\right)\right) \end{array}\right] \end{array}\right]$$

$$= \zeta^{n}\left[\begin{array}{l} A\left[2\left(1+\frac{10\Delta t}{h^2}\right)\cos 2\phi + 2\left(26+\frac{20\Delta t}{h^2}\right)\cos\phi + \left(66+\frac{60\Delta t}{h^2}\right)\right] \\ - i\left[\begin{array}{l} \sin 2\varphi\left(A\left(\frac{5k_2\Delta t}{h}\lambda_2 + \frac{5k_1\Delta t}{h}\lambda_1\right) + B\left(\frac{5k_2\Delta t}{h}\lambda_1\right)\right) \\ + \sin\varphi\left(A\left(\frac{50k_2\Delta t}{h}\lambda_2 + \frac{50k_1\Delta t}{h}\lambda_1\right) + B\left(\frac{50k_2\Delta t}{h}\lambda_1\right)\right) \end{array}\right] \end{array}\right],$$

we get

$$g = \frac{X_2 + iY}{X_1 - iY}, \tag{15}$$

where

$$X_1 = A\left[2\left(1-\frac{10\Delta t}{h^2}\right)\cos 2\phi + 2\left(26-\frac{20\Delta t}{h^2}\right)\cos\phi + \left(66+\frac{60\Delta t}{h^2}\right)\right],$$

$$X_2 = A\left[2\left(1+\frac{10\Delta t}{h^2}\right)\cos 2\phi + 2\left(26+\frac{20\Delta t}{h^2}\right)\cos\phi + \left(66+\frac{60\Delta t}{h^2}\right)\right]$$

and

$$Y = \left[\begin{array}{l} \sin 2\varphi\left(A\left(\frac{5k_2\Delta t}{h}\lambda_2 + \frac{5k_1\Delta t}{h}\lambda_1\right) + B\left(\frac{5k_2\Delta t}{h}\lambda_1\right)\right) \\ + \sin\varphi\left(A\left(\frac{50k_2\Delta t}{h}\lambda_2 + \frac{50k_1\Delta t}{h}\lambda_1\right) + B\left(\frac{50k_2\Delta t}{h}\lambda_1\right)\right) \end{array}\right].$$

From (15) we get $|g| \leq 1$, hence the scheme is unconditionally stable. It means that there is no restriction on the grid size, i.e., on h and Δt, but we should choose them in such a way that the accuracy of the scheme is not degraded.

Similar results can be obtained from the difference (12), due to symmetric u and v.

6. Numerical Tests and Results of Coupled Burgers' Equations

In this section, we present some numerical examples to test validity of our scheme for solving coupled Burgers' equations.

The norms L_2-norm and L_∞-norm are used to compare the numerical solution with the analytical solution [25],

$$L_2 = \| u^E - u^N \| = \sqrt{h \sum_{i=0}^{N} (u_j^E - u_j^N)^2},$$

$$L_\infty = \max_j | u_j^E - u_j^N |, \quad j = 0, 1, \ldots, N, \tag{16}$$

where u^E is the exact solution u and u^N is the approximate solution U_N. Now, we consider two test problems.

Test problem (1)

Consider the coupled Burgers' equations (1) and (2) with the following initial and boundary conditions:

$$u(x, 0) = v(x, 0) = \sin(x), \quad -\pi \leq x \leq \pi,$$

and

$$u(-\pi, t) = u(\pi, t) = 0, \quad 0 \leq t \leq T,$$

$$v(-\pi, t) = v(\pi, t) = 0, \quad 0 \leq t \leq T.$$

The exact solution is

$$u(x, t) = v(x, t) = e^{-t} \sin(x), \quad -\pi \leq x \leq \pi, \quad 0 \leq t \leq T.$$

We compute the numerical solutions using the selected values $k_1 = -2$, $k_2 = 1$ and $k_3 = 1$ with different values of time step length Δt. In our first computation, we compute L_2-norm and L_∞-norm at $t = 0.1$, $k = 0.001$ while the number of partition N changes. The corresponding results are presented in Table 2. In our second computation, we compute L_2-norm and L_∞-norm at time level $t = 1$ for the same parameters in first computation with different decreasing values of Δt. The corresponding results are reported in Table 3. In both computations, the results are same for $u(x, t)$ and $v(x, t)$ because of symmetric initial and boundary conditions. Also, we make comparison of our numerical results of the problem (1) with the results obtained from [15] and [5] for $N = 50$, $k = 0.01$, $k_1 = -2$, $k_2 = k_3 = 1$ with different time t. The corresponding results are presented in Table 4.

Table 2. L_2-norm and L_∞-norm for $t = 0.1$, $k = 0.001$ at different N

	$u(x, t)$		$v(x, t)$		[7]
N	L_2-norm	L_∞-norm	L_2-norm	L_∞-norm	L_∞-norm
50	3.36761E-6	4.47952E-5	3.36761E-6	4.47952E-5	-
100	3.23312E-6	5.94996E-6	3.23312E-6	5.94996E-6	-
128	2.85406E-6	5.15038E-6	2.85406E-6	5.15038E-6	1.8178E-5
200	2.03096E-6	3.62184E-6	2.03096E-6	3.62184E-6	-

Table 3. L_2-norm and L_∞-norm for $t = 1$, $k = 0.01, 0.001$ at different $N = 200$

	$u(x, t)$		$v(x, t)$		[7]
k	L_2-norm	L_∞-norm	L_∞-norm	L_∞-norm	L_∞-norm
$k = 0.01$	2.22605E-5	5.95755E-5	2.22605E-5	5.95755E-5	-
$k = 0.001$	2.36146E-6	5.95761E-6	2.36146E-6	5.95761E-6	3.00E-5

Table 4. Comparison of numerical results of the problem (1) with the results obtained from [15] and [5] for the variables *u* and *v* with $N = 50$, $k = 0.01$

t	$u(x, t)$		$v(x, t)$		[15]	[7]
	L_2-norm	L_∞-norm	L_2-norm	L_∞-norm	L_∞-norm	L_∞-norm
$t = 0.5$	1.1066E-4	1.48333E-4	1.1066E-4	1.48333E-4	1.10308E-4	-
$t = 1$	1.3621E-4	2.38302E-4	1.3621E-4	2.38302E-4	1.33688E-4	1.84705E-3

In Table 4, we show that our results are related with the results in [5] and [15].

The corresponding graphical illustrations are presented in Figure 1 showing computed solutions of $u(x, t)$ and $v(x, t)$ for $k_1 = -2$, $k_2 = 1$, $k_3 = 1$, $N = 200$ and $\Delta t = k = 0.001$ at $t = 0, 0.5, 1$. In Figure 2, computed solutions of $u(x, t)$ and $v(x, t)$ for $k_1 = -2$, $k_2 = 1$, $k_3 = 1$, $N = 200$ and $\Delta t = k = 0.001$ at $t = 0, 0.05, 0.1$. In Figure 3, computed solutions (exact and approximate) of $u(x, t)$ and $v(x, t)$ for $k_1 = -2$, $k_2 = 1$, $k_3 = 1$, $N = 200$ and $\Delta t = k = 0.001$ at $t = 0.1$. In Figure 4-6, computed solutions of $u(x, t)$ and $v(x, t)$ at $t = 0.1$, $\Delta t = k = 0.001$ and $N = 200$ for k_1, k_2, k_1, k_3 and k_2, k_3 fixed, respectively.

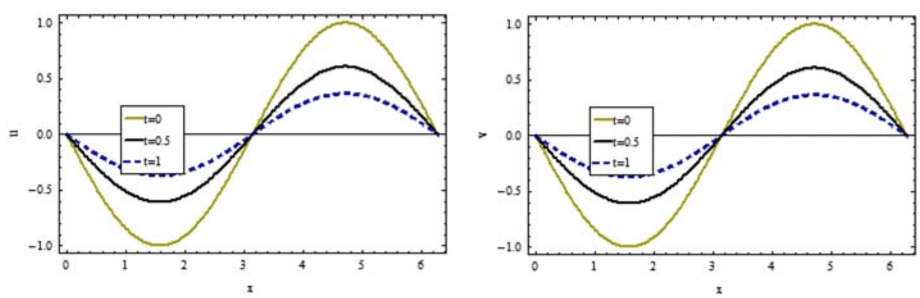

Figure 1. Computed approximate solutions of *u* and *v* for $k_1 = -2$, $k_2 = 1$, $k_3 = 1$, $N = 200$ and $\Delta t = k = 0.001$ at $t = 0, 0.5, 1$.

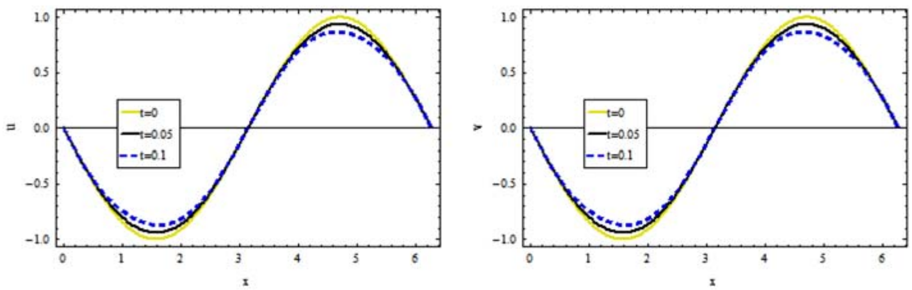

Figure 2. Computed approximate solutions of u and v for $k_1 = -2,\ k_2 = 1,\ k_3 = 1,\ N = 200$ and $\Delta t = k = 0.001$ at $t = 0, 0.05, 0.1$.

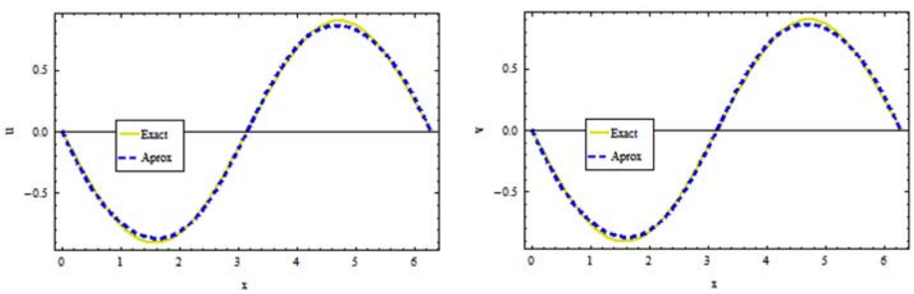

Figure 3. Computed solutions (exact and approximate) of u and v for $k_1 = -2,\ k_2 = 1,\ k_3 = 1,\ N = 200$ and $\Delta t = k = 0.001$ at $t = 0.1$.

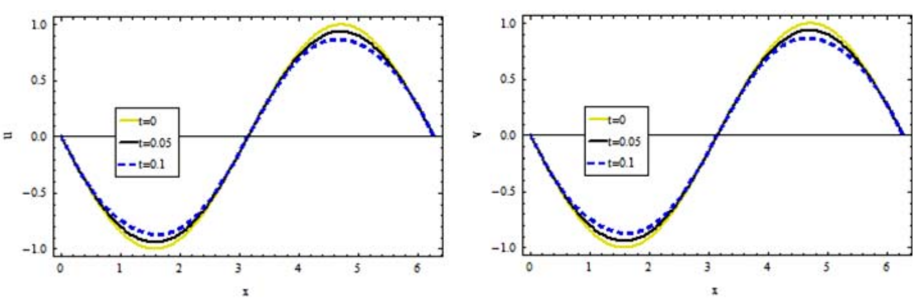

Figure 4. Computed approximate solutions of u and v for $k_1 = -2,\ k_2 = 1,\ k_3 = 8,\ N = 200$ and $\Delta t = k = 0.001$ at $t = 0, 0.05, 0.1$.

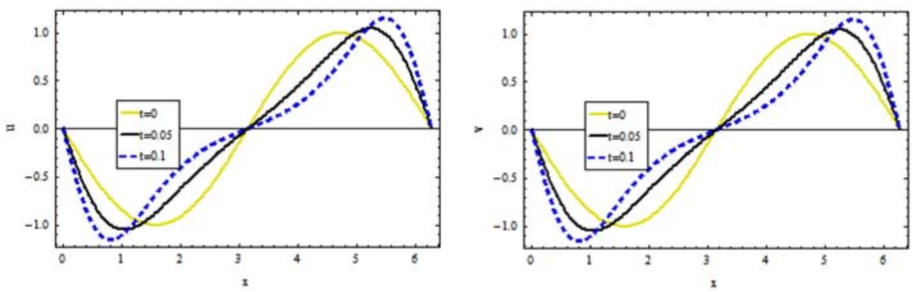

Figure 5. Computed approximate solutions of u and v for $k_1 = -2,\ k_2 = 8,$ $k_3 = 1,\ N = 200$ and $\Delta t = k = 0.001$ at $t = 0,\ 0.05,\ 0.1.$

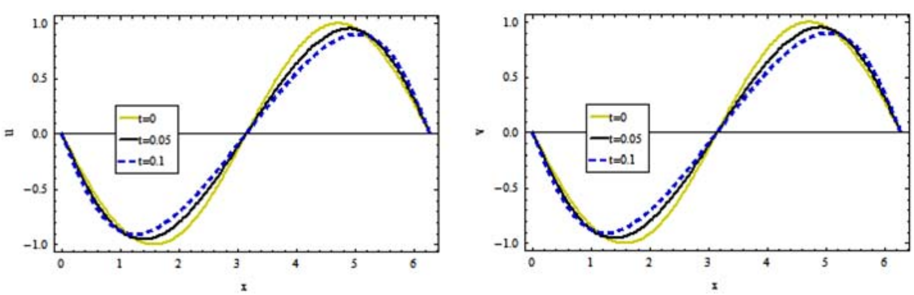

Figure 6. Computed approximate solutions of u and v for $k_1 = 2,\ k_2 = 1,$ $k_3 = 1,\ N = 200$ and $\Delta t = k = 0.001$ at $t = 0,\ 0.05,\ 0.1.$

Test problem (2)

Numerical solutions of considered coupled Burgers' equations are obtained for $k_1 = 2$ with different values of k_2 and k_3 at different time levels. In this situation, the exact solution is

$$u(x,\ t) = a_0 - 2A\left[\frac{2k_2 - 1}{4k_2k_3 - 1}\right]\tanh(A(x - 2At)),$$

$$v(x,\ t) = a_0\left[\frac{2k_3 - 1}{2k_1 - 1}\right] - 2A\left[\frac{2k_2 - 1}{4k_2k_3 - 1}\right]\tanh(A(x - 2At)).$$

Thus, the initial and boundary conditions taken from the exact solution are

$$u(x, 0) = a_0 - 2A\left[\frac{2k_2 - 1}{4k_2k_3 - 1}\right]\tanh(A(x)),$$

$$v(x, 0) = a_0\left[\frac{2k_3 - 1}{2k_2 - 1}\right] - 2A\left[\frac{2k_2 - 1}{4k_2k_3 - 1}\right]\tanh(A(x)).$$

Thus, the initial and boundary conditions are extracted from the exact solution, where $a_0 = 0.05$ and $A = \frac{1}{2}\left[\frac{a_0(4k_2k_3 - 1)}{2k_2 - 1}\right]$. The numerical solutions for $u(x, t)$ and $v(x, t)$ have been computed for the domain $x \in [-10, 10]$, $k = 0.01$ and number of partitions $N = 10$, $N = 100$ and $N = 200$. L_2-norm and L_∞-norm have been computed in Table 5 for $t = 1$, $k_1 = 2$, $k_2 = 0.1$ and $k_3 = 0.3$. In Tables 6 and 7, we make comparison of our numerical results of the problem (2) with the results obtained from [3] and [5] for the variables $u(x, t)$ and $v(x, t)$ with $a_0 = 0.05$, $N = 16$, $k = 0.01$ at different time *t* and different values of k_2, k_3. In Tables 8 and 9, we make comparison of our numerical results of the problem (2) with the results obtained from [10] and [7] for the variables $u(x, t)$ and $v(x, t)$ with $a_0 = 0.05$, $N = 21$, $k = 0.01$ at different time *t* and different values of k_2, k_3.

Table 5. L_2-norm and L_∞-norm for $t = 1$, $k = 0.01$ at different values of N, $k_1 = 2$, $k_2 = 0.1$ and $k_3 = 0.3$

N	$u(x, t)$		$v(x, t)$	
	L_2-norm	L_∞-norm	L_2-norm	L_∞-norm
10	2.92177E-4	9.02098E-5	1.110479E-4	4.54119E-5
50	3.00538E-4	8.22513E-5	1.114941E-4	4.18722E-5
100	3.01301E-4	8.23741E-5	1.115306E-4	4.19293E-5
200	3.01678E-4	8.19936E-5	1.155112E-4	4.20344E-5

Table 6. Comparison of numerical results of the problem (2) with the results obtained from [3] and [5] for the variable u with $a_0 = 0.05,\ N = 16,\ k = 0.01$

			$u(x, t)$		[3]	[5]
t	k_2	k_3	L_2-norm	L_∞-norm	L_∞-norm	L_∞-norm
0.5	0.1	0.30	1.50682E-4	4.43465E-5	1.44E-3	9.619E-4
	0.3	0.30	2.06357E-4	6.41587E-5	-	-
1.0	0.1	0.30	2.96741E-4	8.44084E-5	1.27E-3	1.153E-3
	0.3	0.30	4.06829E-4	1.19154E-4	-	-

Table 7. Comparison of numerical results of the problem (2) with the results obtained from [3] and [5] for the variable v with $a_0 = 0.05,\ N = 16,\ k = 0.01$

			$v(x, t)$		[3]	[5]
t	k_2	k_3	L_2-norm	L_∞-norm	L_∞-norm	L_∞-norm
0.5	0.1	0.30	5.77357E-4	2.34474E-5	5.42E-4	3.332E-4
	0.3	0.30	2.06357E-4	6.41587E-5	-	-
1.0	0.1	0.30	1.12928E-4	4.42146E-5	1.29E-3	1.162E-3
	0.3	0.30	4.06829E-4	1.19154E-4	-	-

In Tables 6 and 7, we show that our results are related with the results in [3] and [5].

Table 8. Comparison of numerical results of the problem (2) with the results obtained from [10] and [7] for the variable u with $a_0 = 0.05,\ N = 21,\ k_1 = 2,\ k = 0.01$

			$u(x, t)$		[10]	[7]
t	k_2	k_3	L_2-norm	L_∞-norm	L_∞-norm	L_∞-norm
0.5	0.1	0.30	1.51522E-4	4.33232E-5	4.173E-5	4.167E-5
	0.3	0.30	2.07396E-4	6.10213E-5	-	-
1.0	0.1	0.30	2.98215E-4	8.16821E-5	8.275E-5	8.258E-5
	0.3	0.30	4.08648E-4	1.17123E-4	-	-

Table 9. Comparison of numerical results of the problem (2) with the results obtained from [10] and [7] for the variable v with $a_0 = 0.05$, $N = 21$, $k = 0.01$

t	k_2	k_3	$v(x, t)$		[10]	[7]
			L_2-norm	L_∞-norm	L_∞-norm	L_∞-norm
0.5	0.1	0.30	5.81893E-5	2.31878E-5	5.418E-5	1.480E-4
	0.3	0.30	2.07396E-4	6.10213E-5	-	-
1.0	0.1	0.30	1.13699E-4	4.15257E-5	1.074E-4	4.770E-4
	0.3	0.30	4.08648E-4	1.17123E-4	-	-

In Tables 8 and 9, we show that our results are related with the results in [10] and [7].

Now we take the test problem (2) at the domain $x \in [0, 1]$, $k = 0.01$ and $k_1 = 2$, $k_2 = 0.1$, $k_3 = 0.3$. L_2-norm and L_∞-norm have been computed, see Table 10 for $t = 1$ with different values of N.

Table 10. L_2-norm and L_∞-norm for $t = 1$, $k = 0.01$ at different values of N, $k_1 = 2$, $k_2 = 0.1$ and $k_3 = 0.3$

N	$u(x, t)$		$v(x, t)$	
	L_2-norm	L_∞-norm	L_2-norm	L_∞-norm
10	5.44676E-6	7.73063E-6	1.53623E-6	2.23523E-6
50	6.01254E-6	8.36307E-6	1.69601E-6	2.41866E-6
100	6.08574E-6	8.13723E-6	1.71844E-6	2.35492E-6
200	6.12261E-6	8.46841E-6	1.72986E-6	2.44898E-6

The corresponding graphical illustrations are presented in Figure 7: computed approximate solutions of $u(x, t)$ and $v(x, t)$ for $k_1 = 2$, $k_2 = 0.1$, $k_3 = 0.3$, $N = 200$ and $\Delta t = k = 0.01$ at $t = 0, 0.5, 1$, $x \in [0, 1]$.

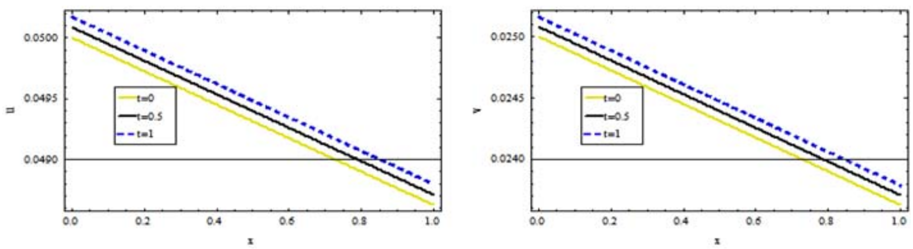

Figure 7. Computed approximate solutions of u and v for $k_1 = 2$, $k_2 = 1$, $k_3 = 0.3$, $N = 200$ and $\Delta t = k = 0.01$ at $t = 0, 0.5, 1$.

7. Conclusions

In this paper, a numerical treatment for the nonlinear coupled Burgers' equations is proposed using a collection method with the quintic *B*-splines. The stability analysis of the method is shown to be unconditionally stable. We make linearization for the nonlinear term. We tested our schemes through two test problems. Accuracy was shown by calculating error norms L_2 and L_∞. The obtained approximate numerical solutions maintain good accuracy compared with the exact solutions.

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