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ON UNIQUENESS OF A SOLUTION TO THE PROBLEM OF RECONSTRUCTION OF HOMOGENEOUS DIELECTRIC LAYER PARAMETERS

D. Tumakov

Institute of Computer Mathematics and Information Technologies
Kazan Federal University
18 Kremlyovskaya St., Kazan 420008
Russian Federation

Abstract

Uniqueness of a solution to the problem of reconstruction of the dielectric layer parameters arising during analysis of a scattered field is investigated. The study is concerned with two cases, which include determining refractive index of a dielectric material filling the layer, and determining thickness of the layer. It is concluded that two measurements, conducted at different and "properly chosen" frequencies, are sufficient to provide uniqueness of a solution to the reconstruction problem.

1. Introduction

In development of new technologies for creating layered coatings and during testing of the layers in the course of manufacturing as well as in many other similar applications encountered in optics and electrodynamics, one often requires reconstructing refractive indices of the layered structures.

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The problem of reconstruction of a one-dimensional profile of the layer's refractive index is an inverse problem that represents the basis in this particular field. A search for an unknown profile can be conducted with the use of various datasets including data for coefficients of the reflected or transmitted fields, input impedance, scattered electric or magnetic fields.

In case of solving the problems of the profile's reconstruction, harmonic oscillations are often used. By using a similar approach, one can determine the required parameters through measuring data at different frequencies in case of numerous incidence angles and different waves polarizations. In the present study, we consider the case of normally falling plane waves at various frequencies. The error of reconstruction of the parameters depends on selecting the frequency range [1, 2]. However, exact criteria for selecting the frequency ranges are not always easy to predetermine in advance.

The problem of determining uniqueness of the solution is essential when working with the inverse problems. The first concern is about uniqueness of a solution to the inverse problem itself. In certain cases, diffracted fields for various layers coincide with each other in a rather wide range of frequencies [3, 4]. Such situations are crucial when, for example, neural networks are to be trained [5], since, to various inputs to the network, there may correspond many absolutely identical outputs.

In the present work, we investigate uniqueness of solutions to the two problems: a problem of determining the layer's thickness in case of the known refractive index and a problem of reconstruction of refractive index in case of the known layer's thickness. We show that the problems may have more than one solution. For the case of two experiments, we indicate the conditions, under which the solution is unique.

2. Problem Statement

Let a harmonic electromagnetic wave u_0 fall normally from half-plane $\{x < 0\}$ on a homogeneous dielectric layer (see Figure 1). As a result, a reflected wave u_1 and a transmitted half-plane $\{x > L\}$ wave u_3 appear.

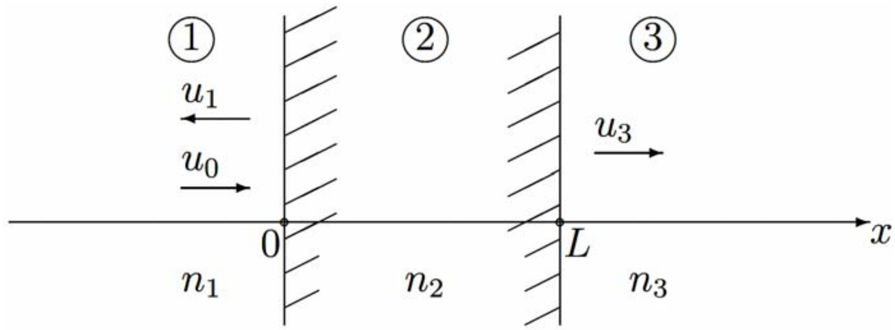

Figure 1. Geometry of the problem.

The inverse problem is such that one needs to determine the layer parameters by using some known components of the diffracted field. For the known components of the field, we choose an incident wave of the form

$$u_0(x, t) = A_0 \exp\{-ik_0 n_1 x + i\omega t\},$$

where A_0 is amplitude of the wave; $k_0 = \omega/c$ is wave number; n_1 is refractive index of the material filling the half-plane $\{x < 0\}$; ω is frequency of the wave's propagation. In addition, the known components of the field include the incident wave u_3 and the reflected wave u_1. The layer has two parameters including refractive index n_2 and layer thickness L.

Let us determine the conditions, under which the inverse problem has a unique solution. We consider two cases: (1) we seek L for the known n_2; (2) we seek n_2 for the known L.

We need to determine whether the stated problems have unique solutions. For the case of non-uniqueness of the solutions, we specify the algorithms, which ensure uniqueness of the solutions to the problems.

3. Direct Problem of Diffraction

Before passing on to studying uniqueness of a solution to the inverse problem, we consider a direct problem, i.e., the problem of diffraction of an electromagnetic wave by the dielectric layer. We solve the direct problem for a harmonic wave having frequency ω [6]. In this section, we neglect the variable ω, considering it as a parameter. Let the plane electromagnetic wave of type $u_0(x) = A_0 \exp\{-ik_0 n_1 x\}$ fall on a layer of thickness L, which has a known refractive index n_2, from a homogeneous isotropic medium (see Figure 1). It is required to find a diffracted field or, more precisely, the reflected wave u_1 and transmitted wave u_3.

The diffraction problem is reduced to an ordinary differential equation [6]

$$u_2''(x) + k_0^2 n_2^2 u_2(x) = 0, \quad 0 < x < L, \tag{1}$$

with boundary conditions

$$u_2'(0) - ik_1 u_2(0) = -2ik_1 A_0, \quad u_2'(L) + ik_3 u_2(L) = 0, \tag{2}$$

where $k_j = k_0 n_j$ are wave numbers of the media. We solved the direct problem analytically.

As a result, the reflected wave u_1 has amplitude of the form

$$A_2 = 2A_0 n_2 \frac{(n_2 - n_3)e^{-iLk_0 n_2} - (n_2 + n_3)e^{iLk_0 n_2}}{(n_1 - n_2)(n_2 - n_3)e^{-iLk_0 n_2} + (n_1 + n_2)(n_2 + n_3)e^{-iLk_0 n_2}}, \tag{3}$$

and transmitted wave u_3 has amplitude of the form

$$B_1 = \frac{4A_0 n_1 n_2}{(n_1 - n_2)(n_2 - n_3)e^{-iLk_0 n_2} + (n_1 + n_2)(n_2 + n_3)e^{iLk_0 n_2}}. \tag{4}$$

We assume that one of the amplitudes of the diffracted waves is known. In the next sections, we tackle questions of uniqueness of solutions to the problems of reconstruction of one of the layer's parameters for the case of the amplitude B_1 being known.

4. Uniqueness of a Solution to the Problem of Determining the Layer Thickness

Let us consider the problem, in which the refractive index n_2 is known, and the layer thickness L needs to be found. From formulas (3) and (4), it follows that since L is introduced into the trigonometric functions in the form of a part of the argument, and the argument Lk_0n_2 itself has a period $2\pi m$ (m is an integer number), the inverse problem of reconstruction of the layer thickness has an infinite number of solutions of the form

$$L_m = L + \frac{2\pi m}{k_0 n_2} = L + \frac{\lambda m}{n_2}, \quad m = 0, \pm 1, \pm 2, \ldots. \tag{5}$$

Theorem 1. *Let the refractive index* n_2 *be known. In this case, one measurement does not ensure uniqueness of determination of* L, *and solutions to the inverse problem become related to each other via formula* (5). *For unique determination of* L, *it is sufficient to conduct two measurements at different frequencies* (*wavelengths*). *Ratio of the frequencies* (*wavelengths*) *of the two measurements must be an irrational number*.

Proof. The fact that the L_m values are related to each other via formula (5) was shown above. Now we assume that one conducted two measurements. For unique determination of L, it is required that, for different measurements (different values $\lambda \neq \lambda'$, where λ' is wavelength corresponding to the second experiment), the L_m values must be different. Namely, it is required that

$$L + \frac{\lambda m}{n_2} \neq L + \frac{\lambda' m'}{n_2}$$

for any integers m and m'.

From this, it follows that $\lambda'/\lambda \neq m'/m$. Hence, ratio of wavelengths should not be a rational number. By virtue of the fact that $\omega = c/\lambda$, where c is speed of light, the same statement also holds true for ratio of frequencies.

Thus, the theorem is proved. □

It is worth noting that the layer thickness L can be also expressed through amplitude of the transmitted wave B_1. For that purpose, equation (4) can be presented in the form

$$B_1 = \frac{4A_0 n_1 n_2 e^{iLk_0 n_2}}{(n_1 - n_2)(n_2 - n_3) + (n_1 + n_2)(n_2 + n_3)e^{2iLk_0 n_2}}$$

and the equation can be converted into a quadratic equation with respect to $\exp\{iLk_0 n_2\}$. The equation has following solution:

$$e^{iLk_0 n_2} = \frac{2A_0 n_1 n_2 \pm \sqrt{4A_0^2 n_1^2 n_2^2 - B_1^2 (n_1^2 - n_2^2)(n_2^2 - n_3^2)}}{B_1 (n_1 + n_2)(n_2 + n_3)}.$$

From the equation, one can obtain L having period $\lambda m/n_2$. In a similar way, one can express L via A_2 having the same period.

5. Uniqueness of a Solution to the Problem of Determining Refractive Index

Now we investigate the problem of reconstruction of the layer's refractive index n_2 based on the transmitted wave. Let the transmitted wave's amplitude B_1 be known. We assume that the media (filling the half-planes) before and after the layer are identical; thus, $n_3 = n_1$. In this case, amplitude of the transmitted wave takes the following form:

$$B_1 = \frac{4A_0 n_1 n_2}{(n_1 + n_2)^2 e^{iLk_0 n_2} - (n_1 - n_2)^2 e^{-iLk_0 n_2}}.$$

Theorem 2. *Let the layer thickness* L *be fixed. Amplitudes of the transmitted waves coincide with each other at different positive values of* $n_2^{(1)}$ *and* $n_2^{(2)}$, *when*

$$n_2^{(1)} = \frac{\sqrt{(Lk_0n_1)^2 + m^2\pi^2} - m\pi}{Lk_0}, \quad n_2^{(2)} = \frac{\sqrt{(Lk_0n_1)^2 + m^2\pi^2} + m\pi}{Lk_0}, \quad (6)$$

for the same $m = 1, 2, \ldots$.

Proof. We denote $L_0 = Lk_0$. Then, from equation (5), it follows that it is sufficient to investigate the following expression:

$$\frac{n_2}{(n_1 + n_2)^2 e^{iL_0n_2} - (n_1 - n_2)^2 e^{-iL_0n_2}}. \quad (7)$$

Expression (7) can be converted to the form

$$\frac{L_0}{2} \frac{x}{2Rx\cos x + i(R^2 + x^2)\sin x},$$

where $x = L_0n_2$; $R = L_0n_1$.

The obtained expression has the same form for various x_j if

$$\begin{cases} \cos x_1 = \cos x_2, \\ \dfrac{x_1^2 + R^2}{x_1}\sin x_1 = \dfrac{x_2^2 + R^2}{x_2}\sin x_2. \end{cases} \quad (8)$$

From equation (8), it follows that $x_2 = x_1 + 2m\pi$, $m = \pm1, \pm2, \ldots$. Then, from the second equation, it follows that

$$\frac{x_1^2 + R^2}{x_1} = \frac{(x_1 + 2m\pi)^2 + R^2}{x_1 + 2m\pi}.$$

The last equation has two solutions

$$x_1 = -m\pi \pm \sqrt{m^2\pi^2 + R^2}.$$

Since n_2 is positive, for $m = 1, 2, \ldots,$ it follows that

$$x_1 = \sqrt{m^2\pi^2 + R^2} - m\pi,$$

$$x_2 = \sqrt{m^2\pi^2 + R^2} + m\pi.$$

This proves the theorem. □

Corollary 3. *Values of refractive indices* $n_2^{(1)}$ *and* $n_2^{(2)}$, *which correspond to the same transmitted wave, must satisfy the condition*

$$n_2^{(1)} < n_1 < n_2^{(2)}.$$

At that,

$$\frac{L}{\lambda} = \frac{mn_2^{(1)}}{n_1^2 - (n_2^{(1)})^2} = \frac{mn_2^{(2)}}{(n_2^{(2)})^2 - n_1^2},$$

where λ *is wavelength in vacuum*, $m = 1, 2, \ldots.$

Proof. We solve equation (6) with respect to Lk_0 for various n_2 and obtain

$$Lk_0 = \frac{2m\pi n_2^{(1)}}{n_1^2 - (n_2^{(1)})^2}, \quad Lk_0 = \frac{2m\pi n_2^{(2)}}{(n_2^{(2)})^2 - n_1^2}.$$

From formula $k_0 = 2\pi/\lambda,$ the corollary's statement is proved. □

Despite the fact that there exist two solutions, for practical applications $n_2 > n_1,$ one can find the sought parameter n_2 from Corollary 3.

Corollary 4. *Two measurements of amplitude of the transmitted wave conducted at different frequencies are sufficient for unique determination of the layer's refractive index, if ratio of the frequencies is an irrational number*.

Proof. Let the condition (6) be satisfied for a certain frequency. After rearrangement with respect to k_0, we obtain

$$k_0 = \pm \frac{2\pi m n_2}{L(n_2^2 - n_1^2)}.$$

From this equation, it follows that if for the two different frequencies, at which the measurements of amplitude of the transmitted wave are conducted, the ratio k_0 is an irrational number, then it is impossible to pick the number m such that the ratio (6) is satisfied. □

6. Conclusions

The problem of determination of the dielectric layer's thickness L has an infinitely large number of solutions. However, by conducting only two measurements at different frequencies ω_1 and ω_2, such that their ratio is an irrational number, it is possible to achieve unique determination of the thickness.

For some dielectric materials, there can be two solutions for refractive index of the substance n_2, filling the layer. This scenario can occur, if refractive index of the layer is equal to

$$n_2 = \frac{\sqrt{(Lk_0 n_1)^2 + m^2\pi^2} \pm m\pi}{Lk_0}$$

for any integer m. Here k_0 is wave number of vacuum; n_1 is refractive index of a medium surrounding the layer. Two measurements at different frequencies, such that their ratio is an irrational number, provide unique determination of n_2.

Thus, two measurements guarantee uniqueness of a solution to the problem of reconstruction of the homogeneous dielectric layer parameters.

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