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ON CERTAIN HYPERALGEBRAIC STRUCTURES

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Abstract

We introduce the notion of hypervector space graphs as hypergraphs endowed with an appropriate structure of hypervector spaces. We study some of their more elementary properties.

1. Introduction

The concept of hyperstructure was first introduced by Marty [4] in 1934. Some other structures such as hyperrings, hypergroups, hypermodules, hyperfields, and hypervector spaces were introduced later. These notions found applications in various disciplines such as geometry, combinatorics, cryptography, and discrete mathematics. For more information about these concepts, we refer to [1-3, 5, 7, 8].

In the present paper, we introduce the notion of a hypervector space graph as a hypergraph endowed with an appropriate structure of a hypervector space, and also that of a hyperfield graph. We study some of their elementary properties.

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2. Notation and Earlier Results

Let H be a set. A map $\circ : H \times H \to \rho^{\star}(H)$ is called a *hyperoperation* or *join operation*, where $\rho^{\star}(H)$ is the set of non-empty subsets of H. The join operation is extended to subsets of H in a natural way, so that $A \circ B$ is given by $A \circ B = \bigcup\{a \circ b \mid a \in A, b \in B\}$. The notations $a \circ A$ and $A \circ a$ are used for $\{a\} \circ A$ and $A \circ \{a\}$, respectively.

Definition 2.1. Let X be a set, and $V(X)$ be a subset of X. A (directed) *hypergraph* is the power set $\rho^{\star}(X) = \rho(X) \setminus \varnothing$ together with power subset $\rho^{\star}(V(X))$ and two hypermaps $\circ^{\star}, t^{\star} : \rho^{\star}(X) \to \rho^{\star}(V(X))$ such that $\bar{v} \in \circ^{\star}(\bar{v}) \bigcap t^{\star}(\bar{v}), \forall \bar{v} \in \rho^{\star}(V(X))$.

The set $\rho^{\star}(V(X))$ represents the set of *hypervertices*, while $E(\rho^{\star}(X)) = \rho^{\star}(X) \setminus \rho^{\star}(V(X))$ represents the set of *hyperedges*. The symbol $\circ^{\star}$ stands for the *original hypermap* and $t^{\star}$ for the *terminal hypermap*.

Definition 2.2. A hypergraph $(\rho^{\star}(X), \rho^{\star}(V(X)), \circ^{\star}, t^{\star})$ is *hyperdiscrete*, when $\rho^{\star}(X) = \rho^{\star}(V(X))$.

Definition 2.3. Let K be a field and $V(K)$ be a subfield. Then the two hypermaps $\circ^{\star}, t^{\star} : \rho^{\star}(K) \to \rho^{\star}(V(K))$ are *hyperhomomorphisms* if the following hold:

(i) $\circ^{\star}(\bar{x} + \bar{y}) \subset \circ^{\star}(\bar{x}) + \circ^{\star}(\bar{y})$, where $\bar{x} + \bar{y} = \bigcup w + z$ for all $w \in \bar{x}$ and $z \in \bar{y}$.

(ii) $\circ^{\star}(\bar{x}.\bar{y}) \subset \circ^{\star}(\bar{x}). \circ^{\star}(\bar{y})$, where $\bar{x}.\bar{y} = \bigcup w.z$ for all $w \in \bar{x}$ and $z \in \bar{y}$.

(i)′ $t^{\star}(\bar{x} + \bar{y}) \subset t^{\star}(\bar{x}) + t^{\star}(\bar{y})$ and

(ii)′ $t^{\star}(\bar{x}.\bar{y}) \subset t^{\star}(\bar{x}).t^{\star}(\bar{y})$.

Definition 2.4. A *hyperfield graph* is a quadraplet $(\rho^{\star}(X), \rho^{\star}(V(X)), \circ^{\star}, t^{\star})$.

Definition 2.5. Let $\circ^{\star}$ be the hyperhomomorphism $\circ^{\star} : \rho^{\star}(K) \to \rho^{\star}(V(K))$. Then $Ker \circ^{\star}$ is defined as $Ker \circ^{\star} = \{\bar{x} \in \rho^{\star}(K) | \{0\} \in \circ^{\star}(\bar{x})\}$.

Proposition 2.1. *Every hyperfield graph is hyperdiscrete.*

Proof. $\{1\} \in \rho^{\star}(V(K)), \circ^{\star}$ is not the zero hyperhomomorphism (since $\{1\} \in \circ^{\star}\{1\}$). $\{0\} \in \circ^{\star}(\bar{x} - \circ^{\star}(\bar{x}))$ as $\bar{x} \in \circ^{\star}(\bar{x})$, $\bar{x} \in \circ^{\star}(\circ^{\star}(\bar{x}))$ and $\circ^{\star}(\bar{x} - \circ^{\star}(\bar{x})) \subset \circ^{\star}(\bar{x}) - \circ^{\star}(\circ^{\star}(\bar{x}))$ for any $\bar{x} \in \rho^{\star}(K)$. It follows that $\bar{x} - \circ^{\star}(\bar{x}) \in Ker \circ^{\star}$. Hence, the claim.

3. Hypervector Space Graphs

Definition 3.1 [6]**.** Let K be a field and $(V, +)$ be an abelian group. Then a *hypervector space* V over K is a quadraplet $(V, +, \circ, K)$, where "$\circ$" is a mapping $\circ : K \times V \to \rho^{\star}(V)$ such that the following conditions hold:

(i) $\forall a \in K, \ \forall x, y \in V, \ a \circ (x + y) \subseteq a \circ x + a \circ y$ (right distributive law).

(ii) $\forall a, b \in K, \ \forall x \in V, \ a \circ (b \circ x) = (a \circ b) \circ x$ (associative law).

(iii) $\forall a, b \in K, \ \forall x \in V, \ (a + b) \circ x \subseteq a \circ x + b \circ x$ (left distributive law).

(iv) $\forall a \in K, \ \forall x \in V, \ a \circ (-x) = (-a) \circ x = -(a \circ x)$.

(v) $\forall x \in V, \ x \in 1 \circ x$.

Remark 3.1. Let $(V, +, \circ, K)$ be a hypervector space over a field K. Let $\Omega = 0 \circ \underline{0}$, where 0 is the zero element of K and $\underline{0}$ is the zero vector of $(V, +)$. Then Ω is a subgroup of $(V, +)$ [4].

Note 3.1. In (ii), $a \circ (b \circ x)$ means the set theoretical union $a \circ (b \circ x) = \bigcup_{y \in b \circ x} a \circ y.$

Definition 3.2. Let $(V, +, \circ, K)$ be a hypervector space over K. A nonempty subset W of V is called a *hypersubspace* of V if the following conditions hold:

(i) $W \neq \varnothing$,

(ii) $\forall x,\ y \in W,\ x - y \in W$,

(iii) $\forall a \in K, \forall x \in W,\ a \circ x \subseteq W$.

In this case, we write $W \mathrel{\hat{<}} V$.

Definition 3.3. A hypervector space graph $\Gamma = (L, V(L), \hat{\circ}, \hat{t})$ is called *hypercombinatorial* provided for $x, x' \in L$ and $\hat{\circ}(x) = \hat{\circ}(x'),\ \hat{t}(x) = \hat{t}(x')$, $x - x' \in \Omega$.

Definition 3.4 [6]**.** Let $(V, +, \circ, K)$ and $(V', +', \circ', K')$ be two hypervector spaces over K. A hyperhomomorphism between V and V' is a mapping $f : V \to V'$ satisfying

$$\forall x,\ y \in V,\ f(x + y) = f(x) +' f(y),$$

$$\forall a \in K, \forall x \in V,\ f(a \circ x) \subseteq a \circ' f(x).$$

Definition 3.5. Let K be a field. Then a *directed hypervector space graph* over K is a quadraplet $(L, V(L), \hat{\circ}, \hat{t})$, where L is a hypervector space graph over K and $V(L)$ is a hypersubspace of L, and two hyperhomomorphisms $\hat{\circ}, \hat{t} : L \to V(L)$ are such that $\hat{\circ}(v) = \hat{t}(v) = v,\ \forall v \in V(L)$.

We call $V(L)$ to be the *set of vertices*, and $E(L) = L \setminus V(L)$ to be the *set of edges*.

For the definition of a vectorspace graph over a field, see [5].

Definition 3.6. Let $(L, +, \circ, K)$ be a hypervector space and $(L, V(L), \hat{\circ}, \hat{t})$ be a hypervector space graph over a field K. Then $Ker(\hat{\circ})$ and $Ker(\hat{t})$ are defined by $Ker(\hat{\circ}) = \{x \in L \,|\, \hat{\circ}(x) \in \Omega\}$ and $Ker(\hat{t}) = \{x \in L \,|\, \hat{t}(x) \in \Omega\}$.

Proposition 3.1. *Let $(L, V(L), \hat{\circ}, \hat{t})$ be a hypervector space graph over a field K. Then $Ker(\hat{\circ})$ and $Ker(\hat{t})$ are hypersubspaces of L.*

Proof. It is straightforward.

Proposition 3.2. *Let $(L, V(L), \hat{\circ}, \hat{t})$ be a hypervector space graph. Then $L \subset V(L) + Ker(\hat{\circ})$ and $L \subset V(L) + Ker(\hat{t})$.*

Proof. If $x \in L$, then $x = (x - \circ(x)) + \circ(x)$ and $\hat{\circ}(x - \hat{\circ}(x)) = \hat{\circ}(x) - \hat{\circ}(\hat{\circ}(x)) = \hat{\circ}(x) - \hat{\circ}(x) = 0$ as $\hat{\circ}(\hat{\circ}(x)) = \hat{\circ}(x)$. Since $(0.1) \circ \underline{0} = 0 \circ (1 \circ \underline{0})$ and $\underline{0} \in 1 \circ \underline{0}$, by Definition 3.1, it follows that $\underline{0} \in 0 \circ \underline{0} = \Omega$ and thus $x - \hat{\circ}(x) \in Ker(\hat{\circ})$. From $\hat{\circ}(x) \in V(L)$, it follows that $L \subset V(L) + Ker(\hat{\circ})$, and similarly $L \subset V(L) + Ker(\hat{t})$, proving the assertion.

Remark 3.2. If $(L, V(L), \circ, t)$ is a vectorspace graph in the sense of Ribenboim [5], $L = V(L) \oplus Ker(\circ) = V(L) \oplus Ker(t)$, and $L/V(L) \cong Ker(\circ) \cong Ker(t)$.

Theorem 3.1. *Let $(L, V(L), \hat{\circ}, \hat{t})$ be a hypervector space graph over a field K. If for any two vertices v, v', there exists an edge x such that $v = \hat{\circ}(x)$ and $v' = \hat{t}(x)$, then $\hat{t}(Ker(\hat{\circ})) = V(L)$ and $\hat{\circ}(Ker(\hat{t})) = V(L)$.*

Proof. Let $v \in V(L)$, $v \neq \underline{0}$, so there exists an edge x such that $\hat{\circ}(x) \in \Omega$, i.e., $x \in Ker(\hat{\circ})$ and $\hat{t}(x) = v$. Hence, $v \in \hat{t}(Ker(\hat{\circ}))$ which implies that $V(L) \subseteq \hat{t}(Ker(\hat{\circ}))$. Conversely, assume that $v \in V(L)$. As $\underline{0} \in V(L)$, there exists an edge $x \in L \setminus V(L)$ such that $\hat{\circ}(x) = \underline{0}$ and $\hat{t}(x) = v$ which means that $x \in Ker(\hat{\circ})$ and $v \in t(Ker(\hat{\circ}))$. Hence, the claim.

Theorem 3.2. *Let* $\Gamma = (L, V(L), \hat{\circ}, \hat{t})$ *be a hypervector space graph over a field K. Then* Γ *is hypercombinatorial if and only if* $Ker(\hat{\circ}) \bigcap \ker(\hat{t}) \subseteq \Omega$.

Proof. Let Γ be hypercombinatorial and $x \in Ker(\hat{o}) \bigcap \ker(\hat{t})$. Then $\hat{o}(x) \in \Omega$ and $\hat{t}(x) \in \Omega$. Hence, $x \in \Omega$. Thus $Ker(\hat{o}) \bigcap \ker(\hat{t}) \subseteq \Omega$.

Conversely, let $x, x' \in L$ be such that $\hat{o}(x) = \hat{o}(x')$ and $\hat{t}(x) = \hat{t}(x')$. Then $\hat{o}(x - x') \in \Omega$ and $\hat{t}(x - x') \in \Omega$. It follows that $x - x' \in Ker(\hat{o}) \bigcap \ker(\hat{t}) \subseteq \Omega$. Hence, $x - x' \subseteq \Omega$. Thus Γ is hypercombinatorial.

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References

[1] Y. Alkhezi and M. Bani-Ata, On hypervector spaces over a hyperring, Far East J. Math. Sci. (FJMS) 100(8) (2016), 1195-1203.

[2] P. Corsini, Prolegomena of Hypergroup Theory, Aviani Editore, 1993.

[3] P. Corsini and V. Leoreanu, Applications of hyperstructure theory, Advances in Mathematics, Kluwer Academic Publishers, Dordrecht, 2003.

[4] F. Marty, Sur une generalization de la notion de groupe, 8th Congress of Scand. Math., Stockholm, p. 1934.

[5] P. Ribenboim, Vector space graphs, Nanta Mathematica 12(2) (1978), 125-132.

[6] M. S. Tallini, Hypervector spaces, Proceedings of the Fourth International Congress on Algebraic Hyperstructures and Applications, Xanthi, Greece, 1990, pp. 167-174.

[7] T. Vougiouklis, The fundamental relation in hyperrings, The General Hyper-field. Algebraic Hyperstructures and Applications, Xanthi, 1990, World Sci. Publishing, Teaneck, NJ, 1991, pp. 203-211.

[8] T. Vougiouklis, Hyperstructures and Their Representations, Hadronic Press, 1994.