Far East Journal of Mathematical Sciences (FJMS)

http://www.pphmj.com
http://dx.doi.org/10.17654/MS101040703
Volume 101, Number 4, 2017, Pages 703-719 **ISSN: 0972-0871**

SOME RESULTS ON LOCAL SPECTRAL THEORY

Jong-Kwang Yoo

Department of Liberal Arts and Science
Chodang University
380 Muanro Muangun 534-701
Chonnam, Korea

Abstract

Suppose that X is a complex Banach space and $T, S \in L(X)$. If there exists an integer $k \in \mathbb{N}$ such that $C(S, T)^k(I) = 0$, we say that T belongs to Helton class of S with order k. In this paper, we study SVEP, property (C), property (β), property (δ) and decomposability for Helton class of operators. We also show that if $f : U \to \mathbb{C}$ is an analytic function on an open neighborhood U of $\sigma(T)$, then T has SVEP at $\mu \in \sigma(T)$ if and only if $f(T)$ has SVEP at λ for which $f(\mu) = \lambda$.

1. Introduction and Preliminaries

Let X and Y denote complex Banach spaces and $L(X, Y)$ denote the Banach algebra of all bounded linear operators of X into Y. As usual, when $X = Y$, we simply write $L(X)$ for $L(X, X)$. Given $T \in L(X)$, we use $KerT$, $\sigma(T)$, $\sigma_p(T)$, $\sigma_{sur}(T)$ and $\rho(T)$ to denote the kernel, the spectrum,

Received: October 20, 2016; Accepted: December 4, 2016
2010 Mathematics Subject Classification: 47A11, 47A53.
Keywords and phrases: Bishop's property (β), decomposable, Helton class operator, local spectrum, property (δ), SVEP.

the point spectrum, surjective spectrum, and the resolvent set of T, respectively. For a T-invariant closed linear subspace Y of X, let $T|Y$ denote the operator given by the restriction of T to Y. The dual space X is denoted by X^* and the adjoint of $T \in L(X)$ by T^*.

Definition 1.1. An operator $T \in L(X)$ is said to have the *single-valued extension property* at a point $\lambda \in \mathbb{C}$ (for brevity, SVEP at λ), if for every open disc $D \subseteq \mathbb{C}$ centered at λ the only analytic function $f : D \to X$ satisfies the equation

$$(\mu I - T) f(\mu) = 0 \text{ for all } \mu \in D$$

is the constant function $f \equiv 0$ on D. Moreover, an operator $T \in L(X)$ is said to have the *SVEP* if T has the SVEP at every point $\lambda \in \mathbb{C}$.

It is easy to see from definition of localized SVEP that if $\sigma_p(T)$ does not cluster at λ, then T has the SVEP at λ. Moreover, every operator T has the SVEP at an isolated point of the spectrum. Obviously, T has the SVEP at every $\lambda \in \rho(T)$, see [1, 14, 16].

The SVEP may be characterized by means of typical tool of the local spectral theory. Obviously, an operator T has SVEP at a point λ precisely when $\lambda I - T$ has SVEP at 0. It is immediate to verify that the SVEP is inherited by the restrictions on closed invariant subspaces. It follows from [14, Proposition 1.2.16] that

$$T \text{ has SVEP} \Leftrightarrow X_T(\phi) = \{0\} \Leftrightarrow X_T(\phi) \text{ is closed.}$$

The *local resolvent set* $\rho_T(x)$ of T at the point $x \in X$ is defined as the set of all $\lambda \in \mathbb{C}$ for which there exists an analytic function $f : U \to X$ on some open neighborhood U of λ such that

$$(\mu I - T) f(\mu) = x \text{ for all } \mu \in U.$$

The *local spectrum* $\sigma_T(x)$ of T at x is the set defined by $\sigma_T(x) :=$

$\mathbb{C} \setminus \rho_T(x)$. Obviously, $\sigma_T(x)$ is a compact subset of $\sigma(T)$. For given $F \subseteq \mathbb{C}$, the *local spectral subspace* of *T* associated with *F* is the set

$$X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}.$$

It is easily seen from the definition that $X_T(F)$ is a linear subspace *T*-invariant of *X*. A variant of the local spectral subspaces, which is more useful for operators without SVEP, is given by the *glocal spectral subspace* $\mathcal{X}_T(F)$. This subspace is defined, for an operator $T \in L(X)$ and a closed subset *F* of $\mathbb{C}$, as the set of all $x \in X$ for which there exists an analytic function $f : \mathbb{C} \setminus F \to X$ which satisfies the identity

$$(\mu I - T) f(\mu) = x \text{ for all } \mu \in \mathbb{C} \setminus F.$$

In general, $\mathcal{X}_T(F) \subseteq X_T(F)$ for every closed subset $F \subseteq \mathbb{C}$, but the two concepts of glocal spectral subspace and local spectral subspace coincide if *T* has SVEP, i.e., if *T* has SVEP, then $X_T(F) = \mathcal{X}_T(F)$ for every closed subset $F \subseteq \mathbb{C}$, see [14, Proposition 3.3.2]. Note that $\mathcal{X}_T(F)$, as well as $X_T(F)$, in general is not closed.

By [1, Theorem 2.22 and Corollary 2.41], the localized SVEP may be characterized as follows.

Theorem 1.2 [1]**.** *For every operator* $T \in L(X)$ *and* $\lambda \in \mathbb{C}$, *the following assertions are equivalent*:

(a) *T has SVEP at* λ,

(b) $Ker(\lambda I - T) \cap X_T(\phi) = \{0\}$,

(c) $\mathcal{N}^{\infty}(\lambda I - T) \cap X_T(\phi) = \{0\}$,

where $\mathcal{N}^{\infty}(T) := \bigcup_{k=1}^{\infty} KerT^k$ *denotes the hyper-kernel of T.*

2. Main Results

For an arbitrary operator $T \in L(X)$ and an analytic function $f : U \to \mathbb{C}$

on an open neighborhood U of $\sigma(T)$, let $f(T) \in L(X)$ denote the operator given by the Riesz functional calculus

$$f(T) := \frac{1}{2\pi i}\int_{\Gamma} f(\lambda)(\lambda I - T)^{-1} d\lambda,$$

where Γ is a contour in U that surrounds $\sigma(T)$. By the classical spectral mapping theorem, we have $f(\sigma(T)) = \sigma(f(T))$.

Theorem 2.1. *Let $T \in L(X)$, let $U \subseteq \mathbb{C}$ be an open neighborhood of $\sigma(T)$, and let $f : U \to \mathbb{C}$ be an analytic function that is non-constant on each connected component of U. Then $f(T)$ has SVEP at λ for which $f(\mu) = \lambda$ if and only if T has SVEP at $\mu \in \sigma(T)$.*

Proof. Suppose that $f(T)$ has SVEP at $\lambda \in \mathbb{C}$, where $f(\mu) = \lambda$ for some $\mu \in \sigma(T)$. Then, by Theorem 1.2, $Ker(\lambda I - f(T)) \cap X_{\lambda I - f(T)}(\phi) = \{0\}$. It suffices to show that $Ker(\mu I - T) \cap X_{\mu I - T}(\phi) = \{0\}$. Let $x \in Ker(\mu I - T) \cap X_{\mu I - T}(\phi)$. Then $(\mu I - T)x = 0$ and $\sigma_{\mu I - T}(x) = \phi$, and hence $\sigma_T(x) = \phi$. Since $f(\mu) = \lambda$, there exists an analytic function $g : U \to \mathbb{C}$ such that

$$\lambda I - f(T) = g(T)(\mu I - T).$$

Since $g(T)$ is invertible, $x \in Ker(\lambda I - f(T))$. By Theorem 3.3.8 [14],

$$\sigma_{f(T)}(x) = f(\sigma_T(x)) = \phi,$$

and we have $x \in Ker(\lambda I - f(T)) \cap X_{\lambda I - f(T)}(\phi) = \{0\}$, it follows that

$$Ker(\mu I - T) \cap X_{\mu I - T}(\phi) = \{0\}.$$

By Theorem 1.2, T has SVEP at μ.

Conversely, let $\lambda \in \mathbb{C}$, and suppose that T has SVEP at $\mu \in \sigma(T)$ for which $f(\mu) = \lambda$. Then, by Theorem 1.2, $Ker(\mu I - T) \cap X_{\mu I - T}(\phi) = \{0\}$. It suffices to show that $Ker(\lambda I - f(T)) \cap X_{\lambda I - f(T)}(\phi) = \{0\}$. Let $x \in$

$Ker(\lambda I - f(T)) \cap X_{\lambda I - f(T)}(\phi)$. Then $(\lambda I - f(T))x = 0$ and $\sigma_{\lambda I - f(T)}(x) = \phi$. Thus, by Theorem 3.3.8 [14],

$$f(\sigma_T(x)) = \sigma_{f(T)}(x) = \phi.$$

By the classical spectral mapping theorem for the Riesz functional calculus, we may assume that $\lambda \in f(\sigma(T))$. Since f is non-constant on each connected component of U, if follows from the identity theorem for analytic functions that the function $f - \lambda$ has only finitely many zeros in $\sigma(T)$, and that all these zeros are of finite multiplicity. Hence there exists an analytic function $g : U \to \mathbb{C}$ such that $\lambda - f = gp$, and p is a polynomial of the form

$$p := (\mu_1 - Z)(\mu_2 - Z) \cdots (\mu_n - Z)$$

with not necessarily distinct elements $\mu_1, \mu_2, ..., \mu_n \in \sigma(T)$, where Z denotes the identity function. By the classical spectral mapping theorem, $g(T)$ is invertible. Since $p(T) = g(T)^{-1}(\lambda I - f(T))$ and $(\lambda I - f(T))x = 0$, we obtain $p(T)x = 0$, i.e.,

$$(\mu_1 I - T)(\mu_2 I - T) \cdots (\mu_n I - T)x = 0.$$

Let $y := h(T)x$, where $h(T) := (\mu_2 I - T)(\mu_3 I - T) \cdots (\mu_n I - T) \in L(X)$. By Proposition 1.2.17 [14],

$$\sigma_T(y) = \sigma_T(h(T)x) \subseteq \sigma_T(x) = \phi.$$

Since T has SVEP at μ_1, Theorem 1.2 implies that $y = 0$. An obvious repetition of this argument for $\mu_2, \mu_3, ..., \mu_n$ leads to the desired conclusion that $x = 0$. Hence $f(T)$ has SVEP at $\lambda = f(\mu)$. □

For given operators $T \in L(X)$ and $S \in L(Y)$, we consider the corresponding commutator $C(S, T) : L(X, Y) \to L(X, Y)$ defined by

$$C(S, T)(A) := SA - AT \text{ for all } A \in L(X, Y).$$

The iterates $C(S, T)^n$ of the commutator are defined by $C(S, T)^0(A) := A$

and

$$C(S, T)^n(A) := C(S, T)^{n-1}(SA - AT) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} S^{n-k} A T^k,$$

for all $A \in L(X, Y)$ and $n \in \mathbb{N}$. It is clear that

$$C(S, T)^{n+1}(A) = C(S, T)^n(SA - AT) = SC(S, T)^n(A) - C(S, T)^n(A)T,$$

and $C(S, T)^n(A) = (-1)^n C(\lambda I - S, \lambda I - T)^n(A)$ for all $n \in \mathbb{N}$ and for all $\lambda \in \mathbb{C}$.

If $X = Y$ and T, S and A are pairwise commuting operators on X, then

$$C(S, T)^n(A) = (S - T)^n A \text{ for all } n \in \mathbb{N}.$$

In particular, if the operators S and T commute, then $C(S, T)^k(I) = 0$ holds for some $k \in \mathbb{N}$ if and only if $S = T + N$ for some nilpotent operator N of order at most k.

In [9], Helton initiated the study of operators T which satisfy an identity of the form $C(T^*, T)^k(I) = 0$ for some integer $k \in \mathbb{N}$.

Definition 2.2. Let $S, T \in L(X)$ be operators on complex Banach space X. If there is an integer $k \geq 1$ such that an operator T satisfies

$$C(S, T)^k(I) = 0,$$

we say that T belongs to Helton class of S with order k, and we denote this by $T \in Helton_k(S)$.

It is clear that S and T are nilpotent equivalent if and only if $T \in Helton_k(S)$ and $S \in Helton_k(T)$ for some integer $k \in \mathbb{N}$. Obviously, if $T \in Helton_k(S)$, then $S^* \in Helton_k(T^*)$ and $\lambda I - T \in Helton_k(\lambda I - S)$ for every $\lambda \in \mathbb{C}$. Moreover, straightforward induction shows that if $N \in L(X)$ is nilpotent and satisfies $TN = NT$, then $T + N \in Helton_p(T)$ and $T \in$

$Helton_p(T+N)$ for some integer $p \in \mathbb{N}$. It is well known that $C(S, T)^k(I) = 0$ does not imply $C(T, S)^k(I) = 0$ in general.

Proposition 2.3. *Let* $S, T \in L(X)$. *If* $T \in Helton_k(S)$, *then* $\sigma_S(x) \subseteq \sigma_T(x)$ *for all* $x \in X$. *Moreover,* $X_T(F) \subseteq X_S(F)$ *and* $\mathcal{X}_T(F) \subseteq \mathcal{X}_S(F)$ *for every closed subset* $F \subseteq \mathbb{C}$.

Proof. Let $\lambda \notin \sigma_T(x)$. Then $\lambda \in \rho_T(x)$, so that there exist an open neighborhood U of λ and an analytic function $f : U \to X$ such that

$$(\mu I - T)f(\mu) = x \text{ for all } \mu \in U.$$

We define $g : U \to X$ by

$$g(\mu) := \sum_{m=0}^{k} (-1)^m C(S, T)^m(I) \frac{f^{(m)}(\mu)}{m!} \text{ for all } \mu \in U.$$

It is easy to check that $g(\lambda)$ converges locally uniformly on U. Hence g is analytic and it is easily seen that g satisfies the identity $(\mu I - S)g(\mu) = x$ for all $\mu \in U$. Thus $\lambda \notin \sigma_S(x)$ and hence $\sigma_S(x) \subseteq \sigma_T(x)$ for all $x \in X$. □

The *quasi-nilpotent part* $H_0(T)$ of T is defined as the set

$$H_0(T) := \{x \in X : \lim_{n\to\infty} \| T^n x \|^{\frac{1}{n}} = 0\}.$$

It is clear that $H_0(T)$ is a linear subspace of X, generally not closed. Furthermore, $Ker(T^n) \subseteq H_0(T)$ for every $n \in \mathbb{N}$ and T is quasi-nilpotent if and only if $H_0(T) = X$, see [14, Proposition 3.3.7 and 3.3.13 or 1, Theorem 1.68]. It is clear that if $T \in Helton_k(S)$ and $ST = TS$, then $T + S \in Helton_k(2S)$. The following result is an immediate consequence of Proposition 1.2.16 [14] and Proposition 2.3.

Corollary 2.4. *Let* $S, T \in L(X)$ *and let* $T \in Helton_k(S)$. *Suppose that*

S has SVEP. Then T has SVEP and $H_0(T) \subseteq H_0(S)$. *Moreover, if* $ST = TS$, *then* $T + S$ *has SVEP.*

Proposition 2.5. *Suppose that* $T \in Helton_k(S)$. *Then we have*

$$Ker(\lambda I - T) \subseteq Ker(\lambda I - S)^k \subseteq \mathcal{N}^\infty(\lambda I - S)$$

for all $\lambda \in \mathbb{C}$. *Moreover,* $Ker(\lambda I - T)^n \subseteq \mathcal{N}^\infty(\lambda I - T) \subseteq H_0(\lambda I - T)$ *for all* $n \in \mathbb{N}$.

Proof. Let $x \in Ker(\lambda I - T)$. Then $(\lambda I - T)^m x = 0$ for all positive integer $m \in \mathbb{N}$. Since

$$C(\lambda I - S, \lambda I - T)^k(I) = \sum_{m=0}^{k} (-1)^m \binom{k}{m} (\lambda I - S)^{k-m} (\lambda I - T)^m,$$

$$0 = C(S, T)^k(I)x = (-1)^k C(\lambda I - S, \lambda I - T)^k(I)x = (-1)^k (\lambda I - S)^k x.$$

Thus $x \in Ker(\lambda I - S)^k$ and hence $Ker(\lambda I - T) \subseteq Ker(\lambda I - S)^k$. It follows from Lemma 1.67 [1] that $Ker(\lambda I - T)^n \subseteq \mathcal{N}^\infty(\lambda I - T) \subseteq H_0(\lambda I - T)$ for all $n \in \mathbb{N}$. □

A bounded operator $T \in L(X)$ is said to have *property* (*Q*) if $H_0(\lambda I - T)$ is closed for every $\lambda \in \mathbb{C}$. Evidently, if *T* has SVEP, then $H_0(\lambda I - T) = X_T(\{\lambda\})$ for every $\lambda \in \mathbb{C}$. If *T* has property (*C*), then $H_0(\lambda I - T) = X_T(\{\lambda\})$ is closed for every $\lambda \in \mathbb{C}$, so that the implication hold:

$$T \text{ has property } (C) \Rightarrow T \text{ has property } (Q) \Rightarrow T \text{ has SVEP.}$$

The following result is an immediate consequence of Proposition 2.3.

Corollary 2.6. *If the operators* $S \in L(X)$ *and* $T \in L(X)$ *are nilpotent equivalent, then T has property* (*Q*) *if and only if S has property* (*Q*).

We shall be concerned with the following classical parts of the spectrum $\sigma(T)$ of the operator *T*, the *point spectrum* $\sigma_p(T) := \{\lambda \in \mathbb{C} : \lambda I - T$ is not injective}, the *surjectivity spectrum* $\sigma_{sur}(T) := \{\lambda \in \mathbb{C} : (\lambda I - T)X \neq X\}$. *The localizable spectrum* $\sigma_{loc}(T)$ of an operator $T \in L(X)$ defined as the set of all $\lambda \in \mathbb{C}$ for which $\mathcal{X}_T(\overline{V}) \neq \{0\}$ for each open neighborhood V of λ. As shown by Muller and Neumann [12], the localizable spectrum plays an important role in the theory of invariant subspaces, see more details [8, 15, 18].

Proposition 2.7. *Let* $S, T \in L(X)$ *and let* $T \in Helton_k(S)$. *Then we have the following*:

(a) $\sigma_p(T) \subseteq \sigma_p(S)$;

(b) $\sigma_{loc}(T) \subseteq \sigma_{loc}(S)$;

(c) $\sigma_{sur}(S) \subseteq \sigma_{sur}(T)$;

(d) *if S has SVEP*, *then* $\sigma(S) \subseteq \sigma(T)$.

Moreover, *if the operators* $S \in L(X)$ *and* $T \in L(X)$ *are nilpotent equivalent*, *then* $\sigma_p(T) = \sigma_p(S)$, $\sigma_{loc}(T) = \sigma_{loc}(S)$, $\sigma_{sur}(T) = \sigma_{sur}(S)$ *and* $\sigma(T) = \sigma(S)$.

Proof. (a) Suppose that $\lambda \in \sigma_p(T)$ and let $x \in X$ be an eigenvector for the eigenvalue λ of T, then $(\lambda I - T)^m x = 0$ for all $m \in \mathbb{N}$. Thus we have

$$0 = C(\lambda I - S, \lambda I - T)^k (I)x = \sum_{m=0}^{k} \binom{k}{m} (\lambda I - S)^{k-m} (\lambda I - T)^m x$$

$$= (S - \lambda I)^k x.$$

Thus $\lambda \in \sigma_p(S)$, and hence $\sigma_p(T) \subseteq \sigma_p(S)$.

(b) Proposition 2.3 guarantees that $\mathcal{X}_T(\overline{V}) \subseteq \mathcal{X}_S(\overline{V})$ for every open subset $V \subseteq \mathbb{C}$. Hence we get $\sigma_{loc}(T) \subseteq \sigma_{loc}(S)$.

(c) By Theorem 2.43 [1] and Proposition 2.3,

$$\sigma_{sur}(S) = \bigcup_{x \in X} \sigma_S(x) \subseteq \bigcup_{x \in X} \sigma_T(x) = \sigma_{sur}(T).$$

(d) Suppose that S has SVEP. Then Corollary 2.4 ensures that T has SVEP. It follows from Proposition 1.3.2 [14] that $\sigma(S) \subseteq \sigma(T)$. □

Let U be an open subset of the complex plane $\mathbb{C}$ and $H(U, X)$ be the Frécht algebra of all analytic X-valued functions on U endowed with uniform convergence on compact sets of U.

Recall that an operator $T \in L(X)$ is said to satisfy *Bishop's property* (β) at $\lambda \in \mathbb{C}$ if there exists $r > 0$ such that for every open subset $U \subset D(\lambda, r)$, open disc centered at λ with radius r, and for any sequence $(f_n)_n \subset H(U, X)$ if

$$\lim_{n \to \infty} (\mu I - T) f_n(\mu) = 0$$

in $H(U, X)$, then $\lim_{n \to \infty} f_n(\mu) = 0$ in $H(U, X)$. We denote by $\sigma_\beta(T)$ by the set where T fails to satisfy (β), i.e.,

$$\sigma_\beta(T) := \{\lambda \in \mathbb{C} : T \text{ fails to satisfy Bishop's property } (\beta) \text{ at } \lambda\}.$$

We say that $T \in L(X)$ satisfies *Bishop's property* (β) precisely when $\sigma_\beta(T) = \phi$.

The *analytic residuum* $\mathcal{S}(T)$ is the open set of points $\lambda \in \mathbb{C}$ for which there exists a non-vanishing analytic function $f : U \to X$ on some open neighborhood U of λ so that $(\mu I - T) f(\mu) = 0$ for all $\mu \in U$. Obviously, $\mathcal{S}(T)$ is a subset of the interior of the point spectrum $\sigma_p(T)$ of T. It is clear that the set $\mathcal{S}(T)$ is empty precisely when T has SVEP.

Recall that an operator $T \in L(X)$ is said to have the *decomposition property* (δ) if the adjoint operator T^* on the dual space X^* satisfies Bishop's property (β), equivalently, $X = \mathcal{X}_T(\overline{U}) + \mathcal{X}_T(\overline{V})$ for every open covering $\{U, V\}$ of $\mathbb{C}$.

An operator $T \in L(X)$ is called *decomposable* if, for every open covering $\{U, V\}$ of the complex plane $\mathbb{C}$, there are T-invariant closed linear subspaces Y and Z of X such that

$$X = Y + Z, \quad \sigma(T|Y) \subseteq U \text{ and } \sigma(T|Z) \subseteq V.$$

In [4], Albrecht et al. show that an operator $T \in L(X)$ is decomposable if and only if T has both properties (β) and (δ). Moreover, Albrecht and Eschmeier proved that the property (β) and (δ) are dual to each other in the sense that an operator $T \in L(X)$ satisfies (β) if and only if the adjoint operator T^* on the dual space X^* satisfies (δ) and that the corresponding statement remains valid if both properties are interchanged.

Theorem 2.8. *Let $T, S \in L(X)$ be operators on the complex Banach space X. If $T \in Helton_k(S)$, then $\sigma_\beta(T) \subseteq \sigma_\beta(S)$ and $\mathcal{S}(T) \subseteq \mathcal{S}(S)$.*

Proof. We only give the proof for (β), the case of the analytic residuum is clear similar. Let $\lambda \notin \sigma_\beta(S)$ and let $(f_n)_n$ be sequence of X-valued analytic functions in a neighborhood U of λ such that

$$\lim_{n\to\infty} (\mu I - T) f_n(\mu) = 0 \text{ in } H(U, X).$$

At first, we claim that $\lim_{n\to\infty} (S - \mu I)^k f_n(\mu) = 0$. Since

$$C(S, T)^k(I) = (-1)^k C(\mu I - S, \mu I - T)^k(I) = C(S - \mu I, \mu I - T)^k(I),$$

for all $n \in \mathbb{N}$ and all $\mu \in \mathbb{C}$, we have

$$\lim_{n\to\infty} [C(S, T)^k(I) - (S - \mu I)^k] f_n(\mu)$$

$$= \lim_{n\to\infty} [C(S - \mu I, \mu I - T)^k(I) - (S - \mu I)^k] f_n(\mu)$$

$$= \lim_{n\to\infty} \sum_{j=0}^{k-1} \binom{k}{j} (S - \mu I)^j (\mu I - T)^{k-j} f_n(\mu)$$

$$= \lim_{n\to\infty} \left[\sum_{j=0}^{k-1} \binom{k}{j} (S - \mu I)^j (\mu I - T)^{k-j-1} \right] (\mu I - T) f_n(\mu) = 0,$$

and hence

$$\lim_{n\to\infty} (S - \mu I)^k f_n(\mu) = 0.$$

Since $\lambda \notin \sigma_\beta(S)$, $\lim_{n\to\infty} (S - \mu I)^{k-1} f_n(\mu) = 0$. By induction, we have

$$\lim_{n\to\infty} f_n(\mu) = 0.$$

Thus T has Bishop's property (β) at λ, and hence $\lambda \notin \sigma_\beta(T)$. □

The following result is an immediate consequence of Theorem 2.8.

Corollary 2.9. *Let* $T, S \in L(X)$ *and let* $T \in Helton_k(S)$. *If* S *has Bishop's property* (β), *then* T *has Bishop's property* (β).

If $T \in L(X)$ and $S \in L(X)$ are commuting operators on X, then $C(S, T)^n(A) = (S - T)^n A$ for all $A \in L(X)$ and $n \in \mathbb{N}$.

Theorem 2.10. *Let* $T, N \in L(X)$ *be commuting operators, and suppose that* N *is quasi-nilpotent. Then we have*

(a) $T + N$ *has property* (δ) *if and only if* T *has property* (δ);

(b) $T + N$ *has property* (Q) *if and only if* T *has property* (Q).

Proof. It suffices to show that $\mathcal{X}_{T+N}(F) = \mathcal{X}_T(F)$ for every closed subset $F \subseteq \mathbb{C}$. Evidently, $C(T + N, T)^n(I) = N^n$ and $C(T, T + N)^n(I) = (-1)^n N^n$ for all $n \in \mathbb{N}$. Since N is quasi-nilpotent, we have

$$\lim_{n\to\infty} \| C(T + N, T)^n(I) \|^{\frac{1}{n}} = 0 \quad \text{and} \quad \lim_{n\to\infty} \| C(T, T + N)^n(I) \|^{\frac{1}{n}} = 0.$$

By Proposition 2.2 of [13], $\sigma_{T+N}(x) = \sigma_T(x)$ for all $x \in X$. Hence $\mathcal{X}_{T+N}(F) = \mathcal{X}_T(F)$ for every closed subset $F \subseteq \mathbb{C}$.

(a) For every open cover $\{U, V\}$ of $\mathbb{C}$,

$$\mathcal{X}_T(\overline{U}) + \mathcal{X}_T(\overline{V}) = \mathcal{X}_{T+N}(\overline{U}) + \mathcal{X}_{T+N}(\overline{V}),$$

and hence $T + N$ has property (δ) if and only if T has property (δ).

(b) By Theorem 2.20 [1], $H_0(\lambda I - T) = \mathcal{X}_T(\{\lambda\})$ for all $\lambda \in \mathbb{C}$. Thus we have $H_0(\lambda I - T - N) = H_0(\lambda I - T)$, and hence $T + N$ has property (Q) if and only if T has property (Q). □

It is well known that $T \in L(X)$ is decomposable if and only if T has both properties (β) and (δ). It is well known that $T \in L(X)$ is decomposable if and only if $T^* \in L(X^*)$ is decomposable, see more details [1, 3, 4, 14].

Corollary 2.11. *Let* $T, S \in L(X)$ *and let* $T \in Helton_k(S)$. *If* T *has property* (δ), *then* S *has property* (δ). *Moreover*, *if* T *is decomposable*, *then* S *is decomposable*.

Proof. Suppose that T has property (δ). Then, by standard duality theory, $S^* \in Helton_k(T^*)$ and T^* has property (β). By Corollary 2.9, S^* has property (β), and hence S has property (δ). □

Recall that an operator $T \in L(X)$ is said to be a *spectral operator* in the sense of Dunford if $T = S + N$, where S is a scalar type operator and N is a quasi-nilpotent operator commuting with S.

Corollary 2.12. *Let* $T, S \in L(X)$ *and let* $T \in Helton_k(S)$. *If* S *is a spectral operator with* $\sigma_T(x) \subseteq \sigma_S(x)$ *for all non-zero* $x \in X$, *then* T *is decomposable*.

Proposition 2.13. *Let* $T, S \in L(X)$ *and let* $T \in Helton_k(S)$. *If* T *is algebraic*, *then* S *is decomposable*.

Proof. Suppose that T is algebraic. Then T has SVEP. Let p be the non-zero polynomial p such that $p(T) = 0$. By the spectral mapping theorem, $p(\sigma(T)) = \sigma(p(T)) = \{0\}$. Thus $\sigma(T)$ is contained in the set of zeros of p. By

Proposition 2.7, $\sigma(S) \subseteq \sigma(T)$. Thus $\sigma(T)$ is contained in the set of zeros of p, and hence $\sigma(S)$ is a finite set. It follows that S is decomposable. □

Proposition 2.14. *Let $T, S \in L(X)$ be operators on complex Banach space X, and let $T \in Helton_k(S)$. Suppose that $\sigma_T(x) \subseteq \sigma_S(x)$ for all non-zero $x \in X$. Then*

(a) *S has SVEP if and only if T has SVEP*;

(b) *S has property* (C) *if and only if T has property* (C);

(c) *S has property* (δ) *if and only if T has property* (δ);

(d) *S is decomposable if and only if T is decomposable*.

Proof. By Proposition 2.3, $\sigma_T(x) = \sigma_S(x)$ for all $x \in X$ and hence $X_T(F) = X_S(F)$ for all closed subset $F \subseteq \mathbb{C}$.

(a) follows from Proposition 1.2.16 of [14].

(b), (c) clear.

(d) If S is decomposable, then S^* is decomposable. Since $S^* \in Helton_k(T^*)$, by Proposition 2.14, T^* is decomposable, and hence T is decomposable. □

Recall that an operator $T \in L(X)$ has *fat local spectra* if $\sigma_T(x) = \sigma(T)$ for all non-zero $x \in X$. For example, semi-shift operators and quasi-nilpotent operators have fat local spectra. Obviously, if $T \in L(X)$ has fat local spectra, then $X_T(F) = \{0\}$ for every closed subset $F \subseteq \mathbb{C}$ that does not contain $\sigma(T)$, and $X_T(F) = X$ otherwise, and hence T has property (C).

Corollary 2.15. *Let $T, S \in L(X)$ and let $T \in Helton_k(S)$. Suppose that T^* has SVEP. If S has fat local spectra, then T has fat local spectra.*

Proof. Suppose that $\sigma(S) = \sigma_S(x)$ for all non-zero $x \in X$. Then S has property (C) and hence S has SVEP. By Proposition 2.3 and Proposition

1.3.2 [14],

$$\sigma(S) = \sigma_{sur}(S) = \bigcup_{x \in X} \sigma_S(x) \subseteq \bigcup_{x \in X} \sigma_T(x) = \sigma_{sur}(T) = \sigma(T).$$

Since T^* has SVEP and $S^* \in Helton_k(T^*)$, S^* has SVEP, by Corollary 2.4. It follows from Proposition 2.7 that

$$\sigma(T^*) \subseteq \sigma(S^*) = \sigma(S)^* = \sigma_S(x)^* \subseteq \sigma_T(x)^* \text{ for all non-zero } x \in X.$$

This implies that $\sigma(T) \subseteq \sigma_T(x)$ for all non-zero $x \in X$, and hence T has fat local spectra. □

Corollary 2.16. *Suppose that $T \in L(X)$ and $S \in L(X)$ are nilpotent equivalent. Then T is decomposable if and only if S is decomposable.*

An operator $T \in L(X)$ is said to be a *Riesz operator* if, for each non-zero $\lambda \in \mathbb{C}$, the operator $\lambda I - T$ has finite-dimensional kernel and cofinite-dimensional range. By the classical Fredholm alternative, all compact operators, also quasi-nilpotent operators are Riesz operators. The spectrum of a Riesz operator is at most countable, clusters only at the origin if any where, and consists, except for zero, only of eigenvalues, see [1] and [14].

Proposition 2.17. *Let $T, S \in L(X)$ and let $T \in Helton_k(S)$. Suppose that T^* has SVEP. If S is a Riesz operator with $\sigma_T(x) \subseteq \sigma_S(x)$ for all non-zero $x \in X$, then T is a Riesz operator. Moreover, if $T \in L(X)$ and $S \in L(X)$ are nilpotent equivalent, then S is a Riesz operator if and only if T is a Riesz operator.*

Proof. Suppose that S is a Riesz operator with $\sigma_T(x) \subseteq \sigma_S(x)$ for all non-zero $x \in X$. By Theorem 1.4.7 [14], S is decomposable and $X_S(F)$ is finite-dimensional for every closed set $F \subseteq \mathbb{C}$ for which $0 \notin F$. By Proposition 2.7, T is decomposable and $\sigma_T(x) = \sigma_S(x)$ for all $x \in X$. Thus $X_T(F) = X_S(F)$ is finite-dimensional. Hence T is a Riesz operator. □

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