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SYMMETRY ANALYSIS AND EXACT SOLUTIONS OF THE DAMPED WAVE EQUATION ON THE SURFACE OF THE SPHERE

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Abstract

In this paper, the damped wave equation, $u_{tt} + \alpha u_t = \Delta u,$ on the surface of the sphere is considered. Using Lie symmetry approach, similarity transformations are obtained and certain interesting exact solutions of the damped wave equation are presented.

1. Introduction

Studies of differential equations arising in mathematical physics constitute an active area of research [1-4]. Of these, wave equations, in particular, have attracted most attention for the central role they play in understanding physical systems which involve propagation phenomena. Due to their descriptive power, wave equations have much wider applications

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in many areas such as acoustics, hydrodynamics [1], optics [2], electromagnetism, general relativity, and classical and quantum mechanics [3], etc.

The significance of wave phenomena was realized at early stages when Pythagoras studied the connection between pitch and length of string in musical instruments [4]. Later, advances in these phenomena were made by Benedetti, Beekman and Galileo [5]. Nevertheless, the most fundamental breakthrough in the study of wave phenomena was made in the nineteenth century by Maxwell who founded the electromagnetic field theory [6].

From mathematical stand point, obtaining exact solutions of wave equations is a major area of interest. In this regard, D'Alembert obtained the first exact solution to the linear wave equation [7, 8]. Apart from exact solutions, methods are also developed to obtain different numeric and approximate solutions of the nonlinear wave equations, see for example [9-13].

Since the descriptive power of the classical wave equations is limited when studying physical systems involving the propagation of waves in dissipative media, the damped wave equation has been introduced. This equation arises from coupling a damping term to the usual wave equation where the extra term plays the role of controlling the speed of oscillation.

In fact, different approaches have been implemented to study both classical and damped wave equations. One of the powerful approaches in this context is to exploit the symmetry properties of these equations and using them to find exact solutions. This approach of solving differential equations has attracted the attention of many researchers [14-16]. In particular, exploiting Lie symmetries to study wave equation in flat background metric received considerable attention and a lot of work has been published in this context [17-22]. This kind of work has also been extended to non-flat background geometries in order to understand the effect of curvature on the solutions of the wave equation. In this context, Azad and Mustafa published an interesting work and gave a complete symmetry analysis of the wave equation on sphere [23].

In this note, we extend the investigation of the wave equation in non-flat background metric by incorporating a damping term. In particular, the symmetry algebra of the damped wave equation on the surface of the sphere will be constructed. Then we will implement the obtained symmetries to perform different similarity reductions of the studied equation. For each case of reduction, the considered wave equation will be reduced into an ODE where an exact invariant solution will be derived.

2. The Symmetry Algebra of the Damped Wave Equation

The metric of the surface of the sphere is given by a second rank symmetric metric tensor

$$g_{ij} = (1, \sin^2\theta). \tag{2.1}$$

In the light of this metric, the term Δu takes the form

$$\Delta u = \cot\theta u_\theta + u_{\theta\theta} + \csc^2\theta u_{\phi\phi}. \tag{2.2}$$

Thus, in the light of above, the damped wave equation on the surface of the sphere takes the form

$$u_{tt} + \alpha u_t = \cot\theta u_\theta + u_{\theta\theta} + \csc^2\theta u_{\phi\phi}, \tag{2.3}$$

where u is a function of t, θ and ϕ.

The procedure of obtaining Lie symmetries of a differential equation is described in details in several references, e.g., [24-26]. In order to obtain the symmetry algebra of equation (2.3), we construct the infinitesimal generator of the form

$$X = \xi(\theta, \phi, t, u)\frac{\partial}{\partial\theta} + \eta(\theta, \phi, t, u)\frac{\partial}{\partial\phi} + \tau(\theta, \phi, t, u)\frac{\partial}{\partial t} + \psi(\theta, \phi, t, u)\frac{\partial}{\partial u}. \tag{2.4}$$

Using the invariance criterion [24], i.e., extending the symmetry generator (2.4) to the jet space, and requiring invariance criterion on equation

(2.3), yields the following system of determining equations:

$$\psi_{uu} = 0 = \tau_u = 0 = \eta_u = 0 = \xi_u = 0, \tag{2.5}$$

$$\xi_t - \tau_\theta = 0, \tag{2.6}$$

$$\tau_t - \xi_\theta = 0, \tag{2.7}$$

$$\xi_\phi + \sin^2(\theta)\eta_\theta = 0, \tag{2.8}$$

$$\eta_t - \csc^2(\theta)\tau_\phi = 0, \tag{2.9}$$

$$\cot\theta\xi + \eta_\phi - \xi_\theta = 0, \tag{2.10}$$

$$\alpha\tau_\theta + 2\psi_{\theta u} + \xi_{\theta\theta} = 0, \tag{2.11}$$

$$\alpha\psi_t + \psi_{tt} - \csc^2\theta\psi_{\phi\phi} - \cot\theta\psi_\theta - \psi_{\theta\theta} = 0, \tag{2.12}$$

$$2\psi_{tu} + \csc^2\theta\tau_{\phi\phi} + \alpha\xi_\theta + \cot\theta\tau_\theta - \xi_{t,\theta} + \tau_{\theta\theta} = 0, \tag{2.13}$$

$$\alpha\tau_\phi + 2\psi_{\phi u} - \eta_{\phi\phi} - \cos\theta\sin\theta\eta_\theta + \xi_{\theta\phi} - \sin^2\theta\eta_{\theta\theta} = 0. \tag{2.14}$$

Solving the determining system gives rise to the following infinitesimals:

$$\tau = c_1,\ \xi = c_4\cos\phi + c_5\sin\phi,\ \eta = c_2 - c_4\cot\theta\sin\phi + c_5\cos\phi\cot\theta,$$

$$\psi = c_3u + f(t, r, \theta), \tag{2.15}$$

where f is any function satisfying equation (2.3). The six infinitesimal symmetry generators associated with infinitesimal given in equation (2.15) are:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial\phi}, \quad X_3 = u\frac{\partial}{\partial u}, \quad X_4 = \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi},$$

$$X_5 = \sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}, \quad X_\infty = f(t, r, \theta)\frac{\partial}{\partial u}.$$

At this stage, we construct the commutator table for the derived symmetries. Notice that the commutation relation $[X_i, X_j]$ for two symmetries X_i, X_j is defined by

$$[X_i, X_j] = X_i(X_j(f)) - X_j(X_i(f)).$$

The commutation relation is shown in Table 1:

Table 1. Commutator table for the symmetries of equation (2.3)

	X_1	X_2	X_3	X_4	X_5
X_1	0	0	0	0	0
X_2	0	0	0	$-X_5$	X_4
X_3	0	0	0	0	0
X_4	0	X_5	0	0	$-X_2$
X_5	0	$-X_4$	0	X_2	0

From above table, it is clear that one can make different choices of two dimensional subalgebras to perform double reduction. In particular, we choose the following four subalgebras χ_i given by:

$$\Big\{\langle X_5, X_2 + X_4\rangle, \Big\langle -\frac{2}{\alpha} X_1 + X_3, X_4\Big\rangle, \Big\langle -\frac{2}{\alpha} X_1 + X_3, X_5\Big\rangle,$$

$$\langle aX_1 + bX_2 + cX_3, dX_1 + eX_2 + fX_3\rangle\Big\}. \tag{2.16}$$

3. Reductions and Exact Solutions of the Damped Wave Equation

We can use symmetries to reduce PDEs into simpler forms by introducing new similarity variables. These variables are invariant functions that can be utilized to reduce the number of variables in the considered equation by one. The invariant functions are obtained by solving the characteristic system produced by the equation $X(I) = 0$, where X is a symmetry generator of the equation [14, 15, 18].

The procedure for performing the reduction of a given PDE using its symmetries is explained in many references, e.g., [14, 15, 18, 27]. In this section, we will perform the reductions corresponding to the four subalgebras that have been mentioned in the previous section.

3.1. Reduction under the subalgebra $\chi_1 = \langle X_5, X_2 + X_4 \rangle$

We can convert equation (2.3) into an ODE by performing double reduction. The first level of reduction is performed using the symmetry X_5. This symmetry gives rise to the following characteristic system:

$$\frac{dt}{0} = \frac{d\theta}{\sin\phi} = \frac{d\phi}{\cos\phi\cot\theta} = \frac{du}{0}. \tag{3.1}$$

Integrating equation (3.1) leads to the following new similarity variables:

$$\xi_1(t, \theta, \phi, u) = t, \quad \xi_2(t, \theta, \phi, u) = \sin\theta\cos\phi, \quad W(\xi_1, \xi_2) = u.$$

Now substituting the new variables in (2.3) reduces the equation to

$$W_{\xi_1\xi_1} + \alpha W_{\xi_1} = -2\xi_2 W_{\xi_2} + (1 - \xi_2^2) W_{\xi_2\xi_2}. \tag{3.2}$$

At this stage, we perform second reduction by using a linear combination of symmetries, $X = X_2 + X_4$. In this case, $X(\xi_1) = X(W) = 0$, and $X(\xi_2) = \cos\theta - \sin\theta\sin\phi$. Therefore, the symmetry

$$X = 0\frac{\partial}{\partial\xi_1} + (\cos\theta - \sin\theta\sin\phi)\frac{\partial}{\partial\xi_2} + 0\frac{\partial}{\partial W} \tag{3.3}$$

is inherited by equation (3.2). Utilizing above symmetry in reducing equation (3.2) gives rise to the following similarity variables:

$$y(\xi_1, \xi_2) = \xi_1, \quad V(W) = W. \tag{3.4}$$

Substituting the new variables in equation (3.2) reduces the equation to

$$V_{yy} + \alpha V_y = 0. \tag{3.5}$$

Solving equation (3.5) yields

$$V = -\frac{K_1}{\alpha} e^{-\alpha\xi_1} + K_2, \tag{3.6}$$

where K_1 and K_2 are constants. Finally, we transform the similarity variables into the original variables of equation (2.3) by substituting for $y = t$ and $V = u$ in equation (3.6). This substitution gives rise to the following non-static exact solution of the studied wave equation (2.3)

$$u(t, \theta, \phi) = -\frac{K_1}{\alpha} e^{-\alpha t} + K_2. \tag{3.7}$$

The obvious solution suggests that the amplitude of the wave decreases drops with time.

3.2. Reduction under the subalgebra $\chi_2 = \left\langle -\frac{2}{\alpha} X_1 + X_3, X_4 \right\rangle$

To solve equation (2.3), we use the following linear combination to two symmetries

$$X = -\frac{2}{\alpha} X_1 + X_3 = -\frac{2}{\alpha}\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}. \tag{3.8}$$

The characteristic system corresponding to above symmetry leads to the following similarity variables:

$$\xi_1(t, \theta, \phi, u) = \theta, \quad \xi_2(t, \theta, \phi, u) = \phi, \quad W(\xi_1, \xi_2) = ue^{\frac{\alpha}{2}t}. \tag{3.9}$$

The above variables reduce equation (2.3) to

$$-\frac{\alpha^2}{4}\sin\xi_1 W = \cos\xi_1 W_{\xi_1} + \sin\xi_1 W_{\xi_1\xi_1} + \csc\xi_1 W_{\xi_2\xi_2}. \tag{3.10}$$

To perform the second reduction via the rotational symmetry X_4, it is noted that equation (3.10) inherits the symmetry:

$$X = \cos\xi_2\frac{\partial}{\partial\xi_1} - \cot\xi_1\sin\xi_2\frac{\partial}{\partial\xi_2} + 0\frac{\partial}{\partial W}. \tag{3.11}$$

Utilizing this inherited symmetry reduces equation (3.10) to following Legendre equation:

$$(1 - y^2)V_{yy} - 2yV_y + \frac{\alpha^2}{4}V = 0, \tag{3.12}$$

where y and V are the new similarity variables given by

$$y(\xi_1, \xi_2) = \sin \xi_1 \sin \xi_2, \quad V(y) = W. \tag{3.13}$$

Obviously, the solution of the Legendre equation, equation (3.12), is given by the following Legendre function:

$$V = c_1 \text{ Legendre} P[0.5(\sqrt{\alpha^2 + 1} - 1), \ y]$$

$$+ c_2 \text{ Legendre} Q[0.5(\sqrt{\alpha^2 + 1} - 1), \ y]. \tag{3.14}$$

Substituting for $y = \sin\theta \sin\phi$ and $v = u$ yields the following Legendre type solution:

$$u(t, \theta, \phi) = c_1 e^{-\frac{\alpha}{2}t} \text{Legendre} P[0.5(\sqrt{\alpha^2 + 1} - 1), \sin\theta \sin\phi]$$

$$+ c_2 e^{-\frac{\alpha}{2}t} \text{Legendre} Q[0.5(\sqrt{\alpha^2 + 1} - 1), \sin\theta \sin\phi]. \tag{3.15}$$

Again, it is evident from the Legendre solution equation (3.15) that the amplitude of the wave vanishes for large time.

3.3. Reduction under the subalgebra $\chi_3 = \left\langle -\frac{2}{\alpha} X_1 + X_3, \ X_5 \right\rangle$

The first reduction in this case is already performed in the previous case which led into equation (3.2). Therefore, we deal only with second reduction via the rotational symmetry, X_5. It turns out that X_5 reduces equation (3.2) again into Legendre equation (3.12) but with different similarity variables given by

$$y(\xi_1, \xi_2) = \sin \xi_1 \cos \xi_2, \quad V(y) = W. \tag{3.16}$$

Similarly, substituting the new similarity variables in equation (3.10) leads again to Legendre equation (3.12) which ultimately leads to the following exact solution of equation (2.3):

$$u(t,\ \theta,\ \phi) = c_1 e^{-\frac{\alpha}{2}t}\,\text{Legendre}\,P[0.5(\sqrt{\alpha^2+1}-1),\ \sin\theta\cos\phi]$$

$$+\ c_2 e^{-\frac{\alpha}{2}t}\,\text{Legendre}\,Q[0.5(\sqrt{\alpha^2+1}-1),\ \sin\theta\cos\phi]. \quad (3.17)$$

3.4. Reduction under the subalgebra $\chi_4 = \langle aX_1 + bX_2 + cX_3,\ dX_1 + eX_2 + fX_3\rangle$

In this case, the constants a, b, c, d, e and f are parameters that will be fixed later. The first level of reduction will be performed using the symmetry $X = a\frac{\partial}{\partial t} + b\frac{\partial}{\partial\phi} + cu\frac{\partial}{\partial u}$. The characteristic system corresponding to this symmetry gives rise to the following similarity variables:

$$\xi_1(t,\ \theta,\ \phi,\ u) = \theta,\ \xi_2(t,\ \theta,\ \phi,\ u) = bt - a\phi,\ V(\xi_1,\ \xi_2,\ u) = ct - a\ln|u|. \quad (3.18)$$

Using the above variables, equation (2.3) can be reduced to the following expression:

$$\left(a\csc^2\xi_1 - \frac{b^2}{a}\right)V_{\xi_2\xi_2} + \left(\frac{c}{a} - \frac{b}{a}V_{\xi_2}\right)^2 + \alpha\left(\frac{c}{a} - \frac{b}{a}V_{\xi_2}\right)$$

$$+\frac{\cot\xi_1}{a}V_{\xi_1} + \frac{1}{a}V_{\xi_1\xi_1} - \frac{1}{a^2}V_{\xi_1}^2 = \csc^2\xi_1 V_{\xi_2}^2. \quad (3.19)$$

In order to reduce equation (3.19), we consider the second level of reduction performed via the symmetry, $X = d\frac{\partial}{\partial t} + e\frac{\partial}{\partial\phi} + fu\frac{\partial}{\partial u}$. Consequently, equation (3.19) is reduced to

$$\left(\frac{c}{a} + \frac{b(cd-af)}{a(ae-bd)}\right)^2 + \alpha\left(\frac{c}{a} + \frac{b(cd-af)}{a(ae-bd)}\right)$$

$$= -\frac{\cot x}{a(ae-bd)}W_x - \frac{W_{xx}}{a(ae-bd)} + \frac{W_x^2}{a^2(ae-bd)^2} + \csc^2 x\,\frac{(cd-af)^2}{(ae-bd)^2},$$

$$(3.20)$$

where x and W are the new similarity variables given by

$$x(\xi_1, \xi_2) = \xi_1, \quad W(x) = (cd - af)\xi_2 + (ae - bd)V. \tag{3.21}$$

In order to further simplify equation (3.20), we fix the parameters as follows:

$$b = d = 0, \quad a = e = f = 1, \quad c = -\alpha.$$

Notice that our choice of the values of the parameters is valid since the considered subalgebra χ_4 in this case becomes

$$\chi_4 = \left\langle \frac{\partial}{\partial t} - \alpha u \frac{\partial}{\partial u}, \frac{\partial}{\partial \phi} + u \frac{\partial}{\partial u} \right\rangle.$$

It is clear that the commutation relation for this two dimensional subalgebra in this case is given by

$$\left[\frac{\partial}{\partial t} - \alpha u \frac{\partial}{\partial u}, \frac{\partial}{\partial \phi} + u \frac{\partial}{\partial u} \right] = 0.$$

Hence, substituting the chosen values of the parameters a, b, c, d, e and f in equation (3.20) yields

$$\cot x W_x + W_{xx} - W_x^2 - \csc^2 x = 0. \tag{3.22}$$

We can solve equation (3.22) by making certain assumption. In particular, assume that

$$W_x = \csc x h(x). \tag{3.23}$$

This reduces equation (3.22) to

$$h^2(x) - \sin x h'(x) + 1 = 0, \tag{3.24}$$

whose solution is given by

$$h(x) = \tan\left(c_1 + \ln\left| \tan \frac{x}{2} \right| \right). \tag{3.25}$$

Using equations (3.25) and (3.23) yields

$$W_x = \csc x \tan\left(c_1 + \ln\left|\tan\frac{x}{2}\right|\right) \tag{3.26}$$

which on integrating gives

$$W = -\ln\left|\cos\left(c_1 + \ln\left|\tan\frac{x}{2}\right|\right)\right| + c_2. \tag{3.27}$$

After some simplifications, it is easy to see that the solution, equation (3.27), is reduced to the form:

$$u(t, \theta, \phi) = Ke^{\phi-\alpha t}\cos\left(c_1 + \ln\left|\tan\frac{\theta}{2}\right|\right). \tag{3.28}$$

The exact solutions corresponding to the remaining subalgebras are listed in the following table:

Table 2. Exact solutions of the damped wave equation

Algebra	Exact solution
$\langle X_5, X_2 + X_4\rangle$	$u(t, \theta, \phi) = -\frac{K_1}{\alpha}e^{-\alpha t} + K_2$
$\left\langle -\frac{2}{\alpha}X_1 + X_3, X_4\right\rangle$	$u(t, \theta, \phi) = c_1 e^{-\frac{\alpha}{2}t}\text{Legendre}\,P[0.5(\sqrt{\alpha^2+1}-1), \sin\theta\sin\phi]$ $+ c_2 e^{-\frac{\alpha}{2}t}\text{Legendre}\,Q[0.5(\sqrt{\alpha^2+1}-1), \sin\theta\sin\phi]$
$\left\langle -\frac{2}{\alpha}X_1 + X_3, X_5\right\rangle$	$u(t, \theta, \phi) = c_1 e^{-\frac{\alpha}{2}t}\text{Legendre}\,P[0.5(\sqrt{\alpha^2+1}-1), \sin\theta\cos\phi]$ $+ c_2 e^{-\frac{\alpha}{2}t}\text{Legendre}\,Q[0.5(\sqrt{\alpha^2+1}-1), \sin\theta\cos\phi]$
$\langle X_1 - \alpha X_3, X_2 + X_3\rangle$	$u(t, \theta, \phi) = Ke^{\phi-\alpha t}\cos\left(c_1 + \ln\left|\tan\frac{\theta}{2}\right|\right)$

4. Conclusion

In this note, it has been shown that the damped wave equation on the surface of the sphere is spanned by five finite dimensional symmetry algebra which is the same algebra that spans the classical wave equation on the sphere. This means that coupling the wave equation with a damping term has no effect on the symmetry properties of the wave equation in the case of constant curvature of the sphere. Furthermore, four similarity reductions have been performed. In each case of reduction, an exact solution has been obtained. The common feature of all obtained solutions is that the amplitude of the wave decreases as time evolves.

The study presented in this paper paves the way for further investigation regarding the properties and solutions of the damped wave equations on non-flat background metric. In particular, the damped wave equation on nonconstant curvature metric could be considered.

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