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# RIESZ ALMOST LACUNARY MULTIPLE TRIPLE SEQUENCE SPACES OF $\Gamma^3$ DEFINED BY AN ORLICZ FUNCTION

**K. Balasubramanian and N. Subramanian**

Department of Mathematics
Srinivasa Ramanujan Centre
SASTRA University
Kumbakonam 612 001
India

Department of Mathematics
SASTRA University
Thanjavur 613 401
India

**Abstract**

In this paper, we introduce a new concept for Riesz almost lacunary $\Gamma^3$ sequence spaces strong $P$-convergent to zero with respect to an Orlicz function and examine some properties of the resulting sequence spaces. We also introduce and study statistical convergence of Riesz almost lacunary $\Gamma^3$ sequence spaces and also some inclusion theorems are discussed.

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## 1. Introduction

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers, respectively. The new classes of difference sequence spaces and vector valued sequence spaces investigated by Mursaleen et al. [8-10]. The different types of notions of triple sequence was introduced and investigated at the initial by Sahiner et al. [12, 13], Esi et al. [1-3], Datta et al. [4], Subramanian and Esi [14], Debnath et al. [5] and many others.

A triple sequence $x = (x_{mnk})$ is said to be *triple analytic* if

$$\sup_{m, n, k} | x_{mnk} |^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple analytic sequences are usually denoted by $\Lambda^3$. A triple sequence $x = (x_{mnk})$ is called *triple entire sequence* if

$$| x_{mnk} |^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty.$$

The space of all triple entire sequences are usually denoted by $\Gamma^3$.

## 2. Definitions and Preliminaries

**Definition 2.1.** An Orlicz function (see [6, 11]) is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function $M$ is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called *modulus function*.

Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to construct Orlicz sequence space.

**Definition 2.2.** The four dimensional matrix $A$ is said to be *RH-regular* if it maps every bounded $P$-convergent sequence into a $P$-convergent sequence with the same $P$-limit. The assumption of boundedness was made because a triple sequence spaces which is $P$-convergent is not necessarily bounded.

**Definition 2.3.** A triple sequence $x = (x_{mnk})$ of real numbers is called *almost P-convergent* to a limit 0 if

$$P - \lim_{p,q,u\to\infty} \sup_{r,s,t\geq 0} \frac{1}{pqu} \sum_{m=r}^{r+p-1} \sum_{n=s}^{s+q-1} \sum_{k=t}^{t+u-1} |x_{mnk}|^{1/m+n+k}$$

$$\to 0,$$

that is, the average value of $(x_{mnk})$ taken over any rectangle

$$\{(m, n, k) : r \leq m \leq r + p - 1, s \leq n \leq s + q - 1, t \leq k \leq t + u - 1\}$$

tends to 0 as both $p$, $q$ and $u$ to $\infty$, and this $P$-convergence is uniform in $i$, $\ell$ and $j$. Let denote the set of sequences with this property as $[\widehat{\Gamma^3}]$.

**Definition 2.4.** Let $(q_{rst})$, $(\overline{q_{rst}})$, $(\overline{\overline{q_{rst}}})$ be sequences of positive numbers and

$$Q_r = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1s} & 0\cdots \\ q_{21} & q_{22} & \cdots & q_{2s} & 0\cdots \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ q_{r1} & q_{r2} & \cdots & q_{rs} & 0\cdots \\ 0 & 0 & \cdots 0 & 0 & 0\cdots \end{bmatrix} = q_{11} + q_{12} + \cdots + q_{rs} \neq 0,$$

$$\overline{Q_s} = \begin{bmatrix} \overline{q}_{11} & \overline{q}_{12} & \cdots & \overline{q}_{1s} & 0\cdots \\ \overline{q}_{21} & \overline{q}_{22} & \cdots & \overline{q}_{2s} & 0\cdots \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \overline{q}_{r1} & \overline{q}_{r2} & \cdots & \overline{q}_{rs} & 0\cdots \\ 0 & 0 & \cdots 0 & 0 & 0\cdots \end{bmatrix} = \overline{q}_{11} + \overline{q}_{12} + \cdots + \overline{q}_{rs} \neq 0,$$

$$\overline{\overline{Q_t}} = \begin{bmatrix} \overline{\overline{q}}_{11} & \overline{\overline{q}}_{12} & \cdots & \overline{\overline{q}}_{1s} & 0\cdots \\ \overline{\overline{q}}_{21} & \overline{\overline{q}}_{22} & \cdots & \overline{\overline{q}}_{2s} & 0\cdots \\ . & & & & \\ . & & & & \\ . & & & & \\ \overline{\overline{q}}_{r1} & \overline{\overline{q}}_{r2} & \cdots & \overline{\overline{q}}_{rs} & 0\cdots \\ 0 & 0 & \cdots 0 & 0 & 0\cdots \end{bmatrix} = \overline{\overline{q}}_{11} + \overline{\overline{q}}_{12} + \cdots + \overline{\overline{q}}_{rs} \neq 0.$$

Then the transformation is given by

$$T_{rst} = \frac{1}{Q_r \overline{Q_s} \overline{\overline{Q_t}}} \sum_{m=1}^{r} \sum_{n=1}^{n} \sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k | x_{mnk} |^{1/m+n+k}$$

is called the *Riesz mean* of triple sequence $x = (x_{mnk})$. If $P - \lim_{rst} T_{rst}(x) = 0$, $0 \in \mathbb{R}$, then the sequence $x = (x_{mnk})$ is said to be *Riesz convergent* to 0. If $x = (x_{mnk})$ is Riesz convergent to 0, then we write $P_R - \lim x = 0$.

**Definition 2.5.** The triple sequence $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ is called *triple lacunary* if there exist three increasing sequences of integers such that

$$m_0 = 0,\ h_i = m_i - m_{r-1} \to \infty \text{ as } i \to \infty \text{ and}$$

$$n_0 = 0,\ \overline{h_\ell} = n_\ell - n_{\ell-1} \to \infty \text{ as } \ell \to \infty,$$

$$k_0 = 0,\ \overline{h_j} = k_j - k_{j-1} \to \infty \text{ as } j \to \infty.$$

Let $m_{i,\ell,j} = m_i n_\ell k_j$, $h_{i,\ell,j} = h_i \overline{h_\ell h_j}$, and $\theta_{i,\ell,j}$ is determined by

$$I_{i,\ell,j} = \{(m, n, k) : m_{i-1} < m < m_i \text{ and } n_{\ell-1} < n \le n_\ell \text{ and } k_{j-1} < k \le k_j\},$$

$$q_k = \frac{m_k}{m_{k-1}},\ \overline{q_\ell} = \frac{n_\ell}{n_{\ell-1}},\ \overline{q_j} = \frac{k_j}{k_{j-1}}.$$

We use the notations of lacunary sequence and Riesz mean for triple sequences. $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ be a triple lacunary sequence and $q_m \overline{q}_n \overline{\overline{q}}_k$ be sequences of positive real numbers such that $Q_{m_i} = \sum_{m \in (0, m_i]} p_{m_i}$, $Q_{n_\ell} = \sum_{n \in (0, n_\ell]} p_{n_\ell}$, $Q_{n_j} = \sum_{k \in (0, k_j]} p_{k_j}$ and $H_i = \sum_{m \in (0, m_i]} p_{m_i}$, $\overline{H} =$

$\sum_{n \in (0, n_\ell]} p_{n_\ell}$, $\overline{\overline{H}} = \sum_{k \in (0, k_j]} p_{k_j}$. Clearly, $H_i = Q_{m_i} - Q_{m_{i-1}}$, $\overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}}$, $\overline{\overline{H}}_j = Q_{kj} - Q_{kj-1}$. If the Riesz transformation of triple sequences is RH-regular, and $H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty$ as $i \to \infty$, $\overline{H} = \sum_{n \in (0, n_\ell]} p_{n_\ell} \to \infty$ as $\ell \to \infty$, $\overline{\overline{H}} = \sum_{k \in (0, k_j]} p_{k_j} \to \infty$ as $j \to \infty$, then $\theta'_{i, \ell, j} = \{(m_i, n_\ell, k_j)\} = \{(Q_{m_i} Q_{n_j} Q_{k_k})\}$ is a triple lacunary sequence. If the assumptions $Q_r \to \infty$ as $r \to \infty$, $\overline{Q}_s \to \infty$ as $s \to \infty$ and $\overline{\overline{Q}}_t \to \infty$ as $t \to \infty$ may be not enough to obtain the conditions $H_i \to \infty$ as $i \to \infty$, $\overline{H}_\ell \to \infty$ as $\ell \to \infty$ and $\overline{\overline{H}}_j \to \infty$ as $j \to \infty$, respectively. For any lacunary sequences $(m_i)$, $(n_\ell)$ and $(k_j)$ are integers. Throughout the paper, we assume that $Q_r = q_{11} + q_{12} + \cdots + q_{rs} \to \infty (r \to \infty)$, $\overline{Q}_s = \overline{q}_{11} + \overline{q}_{12} + \cdots + \overline{q}_{rs} \to \infty (s \to \infty)$, $\overline{\overline{Q}}_t = \overline{\overline{q}}_{11} + \overline{\overline{q}}_{12} + \cdots + \overline{\overline{q}}_{rs} \to \infty (t \to \infty)$, such that $H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty$ as $i \to \infty$, $\overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}} \to \infty$ as $\ell \to \infty$ and $\overline{\overline{H}}_j = Q_{k_j} - Q_{k_{j-1}} \to \infty$ as $j \to \infty$.

Let $Q_{m_i, n_\ell, k_j} = Q_{m_i} \overline{Q}_{n_\ell} \overline{\overline{Q}}_{k_j}$, $H_{i\ell j} = H_i \overline{H}_\ell \overline{\overline{H}}_j$,

$$I'_{i\ell j} = \{(m, n, k) : Q_{m_{i-1}} < m < Q_{m_i}, \overline{Q}_{n_{\ell-1}} < n < Q_{n_\ell}$$
$$\text{and } \overline{Q}_{k_{j-1}} < k \leq \overline{Q}_{k_j}\},$$

$$V_i = \frac{Q_{m_i}}{Q_{m_{i-1}}}, \overline{V}_\ell = \frac{Q_{n_\ell}}{Q_{n_{\ell-1}}} \text{ and } \overline{\overline{V}}_j = \frac{Q_{k_j}}{Q_{k_{j-1}}} \text{ and } V_{i\ell j} = V_i \overline{V}_\ell \overline{\overline{V}}_j.$$

If we take $q_m = 1$, $\overline{q}_n = 1$ and $\overline{\overline{q}}_k = 1$ for all $m$, $n$ and $k$, then $H_{i\ell j}$, $Q_{i\ell j}$, $V_{i\ell j}$ and $I'_{i\ell j}$ reduce to $h_{i\ell j}$, $q_{i\ell j}$, $v_{i\ell j}$ and $I_{i\ell j}$.

Let $f$ be an Orlicz function and $p = (p_{mnk})$ be any factorable triple sequence of strictly positive real numbers, we define the following sequence spaces:

$$[\Gamma_R^3, \theta_{i\ell j}, q, f, p] = \Big\{ P - \lim_{i,\ell,j \to \infty} \frac{1}{H_{i,\ell j}}$$

$$\times \sum_{i \in I_{i\ell j}} \sum_{\ell \in I_{i\ell j}} \sum_{j \in I_{i\ell j}} q_m \overline{q}_n \overline{\overline{q}}_k [f(|x_{m+i, n+\ell, k+j}|)^{p_{mnk}}] = 0 \Big\},$$

uniformly in $i, \ell$ and $j$,

$$[\Lambda_R^3, \theta_{i\ell j}, q, f, p] = \Big\{ x = (x_{mnk}) : P - \sup_{i,\ell,j} \frac{1}{H_{i,\ell j}}$$

$$\times \sum_{i \in I_{i\ell j}} \sum_{\ell \in I_{i\ell j}} \sum_{j \in I_{i\ell j}} q_m \overline{q}_n \overline{\overline{q}}_k [f|x_{m+i, n+\ell, k+j}|^{p_{mnk}}] < \infty \Big\},$$

uniformly in $i, \ell$ and $j$.

Let $f$ be an Orlicz function, $p = p_{mnk}$ be any factorable double sequence of strictly positive real numbers and $q_m, \overline{q}_n$ and $\overline{\overline{q}}_k$ be sequences of positive numbers and $Q_r = q_{11} + \cdots q_{rs}$, $\overline{Q}_s = \overline{q}_{11} \cdots \overline{q}_{rs}$ and $\overline{\overline{Q}}_t = \overline{\overline{q}}_{11} \cdots \overline{\overline{q}}_{rs}$.

If we choose $q_m = 1$, $\overline{q}_n = 1$ and $\overline{\overline{q}}_k = 1$ for all $m$, $n$ and $k$, then we obtain the following sequence spaces:

$$[\Gamma_R^3, q, f, p] = \Big\{ P - \lim_{r,s,t \to \infty} \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t}$$

$$\times \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k [f(|x_{m+i, n+\ell, k+j}|)^{p_{mnk}}] = 0 \Big\},$$

uniformly in $i, \ell$ and $j$,

$$[\Lambda_R^3, q, f, p] = \Big\{ P - \sup_{r,s,t} \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t}$$

$$\times \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k [f(|x_{m+i, n+\ell, k+j}|)^{p_{mnk}}] < \infty \Big\},$$

uniformly in $i, \ell$ and $j$.

**Definition 2.6.** Let $f$ be an Orlicz function and $p = (p_{mnk})$ be any factorable triple sequence of strictly positive real numbers, we define the following sequence space:

$\theta_{i, \ell, j} = \{(m_i, n_\ell, k_j)\}$ be a triple lacunary sequence

$$\Gamma_f^3[AC_{\theta_{i, \ell, j}}, p] = \Bigg\{P - \lim_{i, \ell, j} \frac{1}{h_{i\ell j}}$$

$$\times \sum_{i \in I_{i, \ell, j}} \sum_{n \in I_{i, \ell, j}} \sum_{k \in I_{i, \ell, j}} [f(|x_{m+i, n+\ell, k+j}|)^{1/m+n+k}]^{p_{mnk}} = 0\Bigg\},$$

uniformly in $i$, $\ell$ and $j$.

We shall denote $\Gamma_f^3[AC_{\theta_{i, \ell, j}}, p]$ as $\Gamma^3[AC_{\theta_{i, \ell, j}}, p]$, respectively, when $p_{mnk} = 1$ for all $m$, $n$ and $k$. If $x$ is in $\Gamma^3[AC_{\theta_{i, \ell, j}}, p]$, we shall say that $x$ is *almost lacunary* $\Gamma^3$ *strongly p-convergent* with respect to the Orlicz function $f$. Also note if $f(x) = x$, $p_{mnk} = 1$ for all $m$, $n$ and $k$, then $\Gamma_f^3[AC_{\theta_{i, \ell, j}}, p]$ $= \Gamma^3[AC_{\theta_{i, \ell, j}}]$ which are defined as follows:

$$\Gamma^3[AC_{\theta_{i, \ell, j}}] = \Bigg\{P - \lim_{i, \ell, j} \frac{1}{h_{i\ell j}}$$

$$\times \sum_{m \in I_{i, \ell, j}} \sum_{n \in I_{i, \ell, j}} \sum_{k \in I_{i, \ell, j}} [f(|x_{m+i, n+\ell, k+j}|)^{1/m+n+k}] = 0\Bigg\},$$

uniformly in $i$, $\ell$ and $j$.

Again note if $p_{mnk} = 1$ for all $m$, $n$ and $k$, then

$$\Gamma_f^3[AC_{\theta_{i, \ell, j}}, p] = \Gamma_f^3[AC_{\theta_{i, \ell, j}}].$$

We define

$$\Gamma_f^3[AC_{\theta_{i,\ell,j}}, p] = \Big\{P - \lim_{i,\ell,j} \frac{1}{h_{i\ell j}}$$

$$\times \sum_{m \in I_{k,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} [f(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k}]^{p_{mnk}} = 0\Big\},$$

uniformly in $i$, $\ell$ and $j$.

**Definition 2.7.** Let $f$ be an Orlicz function $p = (p_{mnk})$ be any factorable triple sequence of strictly positive real numbers, we define the following sequence space:

$$\Gamma_f^3[p] = \Big\{P - \lim_{r,s,t \to \infty} \frac{1}{rst}$$

$$\times \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} [f(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k}]^{p_{mnk}} = 0\Big\},$$

uniformly in $i$, $\ell$ and $j$.

If we take $f(x) = x$, $p_{mnk} = 1$ for all $m$, $n$ and $k$, then $\Gamma_f^3[p] = \Gamma^3$.

**Definition 2.8.** Let $\theta_{i,\ell,j}$ be a triple lacunary sequence; the triple number sequence $x$ is $\widehat{S_{\theta_{i,\ell,j}}} - p$-convergent to 0. Then

$$P - \lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} \max_{i,\ell,j}$$

$$\times |\{(m, n, k) \in I_{i,\ell,j} : f(|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k}\}| = 0.$$

In this case, we write $\widehat{S_{\theta_{i,\ell,j}}} - \lim(f|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k} = 0$.

## 3. Main Results

**Theorem 3.1.** *If $f$ be any Orlicz function and a bounded factorable positive triple number sequence $p_{mnk}$, then $\Gamma_f^3[AC_{\theta_{i,\ell,j}}, P]$ is linear space.*

**Proof.** The proof is easy. Therefore we omit the proof.

**Theorem 3.2.** *For any Orlicz function f, we have*

$$\Gamma^3[AC_{\theta_{i,\ell,j}}] \subset \Gamma_f^3[AC_{\theta_{i,\ell,j}}].$$

**Proof.** Let $x \in \Gamma^3[AC_{\theta_{i,\ell,j}}]$ so that for each $i, \ell$ and $j$,

$$\Gamma^3[AC_{\theta_{i,\ell,j}}] = \Bigg\{\lim_{i,\ell,j} \frac{1}{h_{i\ell j}}$$

$$\times \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} [(|x_{m+i, n+\ell, k+j}|)^{1/m+n+k}] = 0\Bigg\}.$$

Since $f$ is continuous at zero, for $\varepsilon > 0$ and choose $\delta$ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for every $t$ with $0 \le t \le \delta.$ We obtain the following:

$$\frac{1}{h_{i\ell j}}(h_{i\ell j}\varepsilon) + \frac{1}{h_{i\ell j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}}$$

$$\times \sum_{k \in I_{i,\ell,j} \text{ and } |x_{m+i,n+\ell,k+j} - 0| > \delta} f[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k}]$$

$$\times \frac{1}{h_{i\ell j}}(h_{i\ell j}\varepsilon) + \frac{1}{h_{i\ell j}} K\delta^{-1} f(2) h_{i\ell j} \Gamma^3[AC_{\theta_{i,\ell,j}}].$$

Hence $i, \ell$ and $j$ goes to infinity, we are granted $x \in \Gamma_f^3[AC_{\theta_{i,\ell,j}}].$

**Theorem 3.3.** *Let* $\theta_{i,\ell,j} = \{m_i, n_\ell, k_j\}$ *be a triple lacunary sequence with* $f_i q_i > 1,$ $\liminf_\ell \overline{q_\ell} > 1$ *and* $\liminf_j q_j > 1.$ *Then for any Orlicz function* $f$, $\Gamma_f^3(P) \subset \Gamma_f^3(AC_{\theta_{i,\ell,j}}, P).$

**Proof.** Suppose $\liminf_i q_i > 1,$ $\liminf_\ell \overline{q_\ell} > 1$ and $\liminf_j q_j > 1$ then there exists $\delta > 0$ such that $q_i > 1 + \delta,$ $\overline{q_\ell} > 1 + \delta$ and $q_j > 1 + \delta.$ This implies $\frac{h_i}{m_i} \geq \frac{\delta}{1+\delta}$, $\frac{h_\ell}{n_\ell} \geq \frac{\delta}{1+\delta}$ and $\frac{h_j}{k_j} \geq \frac{\delta}{1+\delta}$. Then for $x \in \Gamma_f^3(P)$, we can write for each $r$, $s$ and $u$,

$$B_{i,\ell,j}$$

$$= \frac{1}{h_{i\ell j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} f[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k}]^{p_{mnk}}$$

$$= \frac{1}{h_{i\ell j}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} f[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k}]^{p_{mnk}}$$

$$- \frac{1}{h_{i\ell j}} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{i-1}} f[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k}]^{p_{mnk}}$$

$$- \frac{1}{h_{i\ell j}} \sum_{m=m_{i-1}+1}^{m_i} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{j-1}} f[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k}]^{p_{mnk}}$$

$$- \frac{1}{h_{i\ell j}} \sum_{k=k_j+1}^{k_j} \sum_{n=n_{\ell-1}+1}^{n_\ell} \sum_{k=1}^{m_{k-1}} f[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k}]^{p_{mnk}}$$

$$= \frac{m_i n_\ell k_j}{h_{i\ell j}} \left( \frac{1}{m_i n_\ell k_j} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} f[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k}]^{p_{mnk}} \right)$$

$$- \frac{m_{k-1} n_{\ell-1} k_{j-1}}{h_{i\ell j}} \left( \frac{1}{m_{i-1} n_{\ell-1} kj - 1} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \right.$$

$$\left. \times \sum_{k=1}^{k_{j-1}} f[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k}]^{p_{mnk}} \right)$$

$$- \frac{k_{j-1}}{h_{i\ell j}} \left( \frac{1}{k_{j-1}} \sum_{m=m_{i-1}+1}^{m_i} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_j} f[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k}]^{p_{mnk}} \right)$$

$$- \frac{n_{j-1}}{h_{i\ell j}} \left( \frac{1}{n_{\ell-1}} \sum_{m=m_{k-1}+1}^{m_k} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_j} f[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k}]^{p_{mnk}} \right)$$

$$- \frac{m_{k-1}}{h_{i\ell j}} \left( \frac{1}{m_{k-1}} \sum_{k=1}^{k_j} \sum_{n=n_{\ell-1}+1}^{n_\ell} \right.$$

$$\left. \times \sum_{m=1}^{m_{k-1}} f[(|x_{m+i, n+\ell, k+j}|)^{1/m+n+k}]^{p_{mnk}} \right).$$

Since $x \in \Gamma_f^3(P)$ the last three terms tend to zero uniformly in $m$, $n$, $k$ in the sense, thus, for each $r$, $s$ and $u$,

$$B_{i, \ell, j}$$

$$= \frac{m_i n_\ell k_j}{h_{i\ell j}} \left( \frac{1}{m_i n_\ell k_j} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} f[(|x_{m+i, n+\ell, k+j}|)^{1/m+n+k}]^{p_{mnk}} \right)$$

$$- \frac{m_{i-1} n_{\ell-1} k_{j-1}}{h_{i\ell j}} \left( \frac{1}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \right.$$

$$\left. \times \sum_{k=1}^{k_{j-1}} f[(|x_{m+i, n+\ell, k+j}|)^{1/m+n+k}]^{p_{mnk}} \right) + O(1).$$

Since $h_{i\ell j} = m_i n_\ell k_j - m_{i-1} n_{\ell-1} k_{j-1}$ we are granted for each $i$, $\ell$ and $j$ the following:

$$\frac{m_i n_\ell kj}{h_{i\ell j}} \leq \frac{1+\delta}{\delta} \text{ and } \frac{m_{i-1} n_{\ell-1} k_{j-1}}{h_{i\ell j}} \leq \frac{1}{\delta}.$$

The terms

$$\left( \frac{1}{m_i n_\ell k_j} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} f[(|x_{m+i, n+\ell, k+j}|)^{1/m+n+k}]^{p_{mnk}} \right)$$

and

$$\left( \frac{1}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{j-1}} f[(|x_{m+i, n+\ell, k+j}|)^{1/m+n+k}]^{p_{mnk}} \right)$$

are both chi sequences for all $i$, $\ell$ and $j$. Thus $B_{i\ell j}$ is a chi sequence for each $i$, $\ell$ and $j$. Hence $x \in \Gamma_f^3(AC_{\theta_{i, \ell, j}}, P)$.

**Theorem 3.4.** *Let* $\theta_{i,\ell,j} = \{m, n, k\}$ *be a triple lacunary sequence with* $\limsup_{\eta} q_{\eta} < \infty$ *and* $\limsup_{i} \bar{q}_i < \infty$. *Then for any Orlicz function f,* $\Gamma_f^3(AC_{\theta_{i,\ell,j}}, P) \subset \Gamma_f^3(p)$.

**Proof.** Since $\limsup_i q_i < \infty$ and $\limsup_i \bar{q}_i < \infty$ there exists $H > 0$ such that $q_i < H$, $\bar{q}_\ell < H$ and $q_j < H$ for all $i$, $\ell$ and $j$. Let $x \in \Gamma_f^3(AC_{\theta_{i,\ell,j}}, P)$. Also there exist $i_0 > 0$, $\ell_0 > 0$ and $j_0 > 0$ such that for every $a \geq i_0$, $b \geq \ell_0$ and $c \geq j_0$ and $i$, $\ell$ and $j$,

$$A'_{abc} = \frac{1}{h_{abc}} \sum_{m \in I_{a,b,c}} \sum_{n \in I_{a,b,c}}$$

$$\times \sum_{k \in I_{a,b,c}} f[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k}]^{p_{mnk}} \to 0 \text{ as } m, n, k \to \infty.$$

Let $G' = \max\{A'_{a,b,c} : 1 \leq a \leq i_0, 1 \leq b \leq \ell_0 \text{ and } 1 \leq c \leq j_0\}$ and $p$, $q$ and $t$ be such that $m_{i-1} < p \leq m_i$, $n_{\ell-1} < q \leq n_\ell$ and $m_{j-1} < t \leq m_j$. Thus we obtain the following:

$$\frac{1}{pqt} \sum_{m=1}^{p} \sum_{n=1}^{q} \sum_{k=1}^{t} [(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k}]^{p_{mnk}}$$

$$\leq \frac{1}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} [(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k}]^{p_{mnk}}$$

$$\leq \frac{1}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{a=1}^{i} \sum_{b=1}^{\ell} \sum_{c=1}^{j}$$

$$\times \left( \sum_{m \in I_{a,b,c}} \sum_{n \in I_{a,b,c}} \sum_{k \in I_{a,b,c}} [(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k}]^{p_{mnk}} \right)$$

$$= \frac{1}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{a=1}^{i_0} \sum_{b=1}^{\ell_0} \sum_{c=1}^{j_0} h_{a,b,c} A'_{a,b,c}$$

$$+ \frac{1}{m_{i-1}n_{\ell-1}k_{j-1}} \sum_{(i_0 < a \leq i) \cup (\ell_0 < b \leq \ell) \cup (j_0 < c \leq j)} h_{a,b,c} A'_{a,b,c}$$

$$\leq \frac{G'}{m_{i-1}n_{\ell-1}k_{j-1}} \sum_{a=1}^{i_0} \sum_{b=1}^{\ell_0} \sum_{c=1}^{j_0} h_{a,b,c}$$

$$+ \frac{1}{m_{i-1}n_{\ell-1}k_{j-1}} \sum_{(i_0 < a \leq i) \cup (\ell_0 < b \leq \ell) \cup (j_0 < c \leq j)} h_{a,b,c} A'_{a,b,c}$$

$$\leq \frac{G' m_{i_0} n_{\ell_0} k_{j_0} i_0 \ell_0 j_0}{m_{i-1}n_{\ell-1}k_{i-1}} + \frac{1}{m_{i-1}n_{\ell-1}k_{j-1}}$$

$$\times \sum_{(i_0 < a \leq i) \cup (\ell_0 < b \leq \ell) \cup (j_0 < c \leq j)} h_{a,b,c} A'_{a,b,c}$$

$$\leq \frac{G' m_{i_0} n_{\ell_0} k_{j_0} i_0 \ell_0 j_0}{m_{i-1}n_{\ell-1}k_{i-1}} + \left(\sup_{a \geq i_0 \cup b \geq \ell_0 \cup c \geq j_0} A'_{a,b,c}\right) \frac{1}{m_{i-1}n_{\ell-1}k_{j-1}}$$

$$\times \sum_{(i_0 < a \leq i) \cup (\ell_0 < b \leq \ell) \cup (j_0 < c \leq j)} h_{a,b,c}$$

$$\leq \frac{G' m_{i_0} n_{\ell_0} k_{j_0} i_0 \ell_0 j_0}{m_{i-1}n_{\ell-1}k_{i-1}} + \frac{\varepsilon}{m_{i-1}n_{\ell-1}k_{j-1}} \sum_{(i_0 < a \leq i) \cup (\ell_0 < b \leq \ell) \cup (j_0 < c \leq j)} h_{a,b,c}$$

$$\leq \frac{G' m_{i_0} n_{\ell_0} k_{j_0} i_0 \ell_0 j_0}{m_{i-1}n_{\ell-1}k_{i-1}} + \varepsilon H^3.$$

Since $m_i$, $n_\ell$ and $k_j$ both approaches infinity as both $p$, $q$ and $t$ approaches infinity, it follows that

$$\frac{1}{pqt} \sum_{m=1}^{p} \sum_{n=1}^{q} \sum_{k=1}^{t} f\left[\left(\left| x_{m+i,n+\ell,k+j} \right|\right)^{1/m+n+k}\right]^{p_{mnk}} = 0,$$

uniformly in $i$, $\ell$ and $j$.

Hence $x \in \Gamma_f^3(P)$.

**Theorem 3.5.** *Let* $\theta_{i,\ell,j}$ *be a triple lacunary sequence. Then*

(i) $(x_{mnk}) \xrightarrow{P} \Gamma^3(\widehat{S_{\theta_{i,\ell,j}}})$,

(ii) $(AC_{\theta_{i,\ell,j}})$ *is a proper subset of* $(\widehat{S_{\theta_{i,\ell,j}}})$,

(iii) *If* $x \in \Lambda^3$ *and* $(x_{mnk}) \xrightarrow{P} \Gamma^3(\widehat{S_{\theta_{i,\ell,j}}})$, *then* $(x_{mnk}) \xrightarrow{P} \Gamma^3(AC_{\theta_{i,\ell,j}})$,

(iv) $\Gamma^3(\widehat{S_{\theta_{i,\ell,j}}}) \cap \Lambda^3 = \Gamma^3[AC_{\theta_{i,\ell,j}}] \cap \Lambda^3$.

**Proof.** (i) Since for all $i$, $\ell$ and $j$,

$$|\{(m, n, k) \in I_{i,\ell,j} : (|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k}\} = 0|$$

$$\leq \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}}$$

$$\times \sum_{k \in I_{i,\ell,j} \text{ and } |x_{m+i,n+\ell,k+j}|=0} (|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k}$$

$$\leq \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} (|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k},$$

for all $i$, $\ell$ and $j$,

$$P - \lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}}$$

$$\times \sum_{k \in I_{i,\ell,j}} (|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k} = 0.$$

This implies that for all $i$, $\ell$ and $j$,

$$P - \lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} |\{(m, n, k) \in I_{i,\ell,j} : (|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k} = 0\}|$$

$$= 0,$$

(ii) let $x = (x_{mnk})$ be defined as follows:

$$(x_{mnk}) = \begin{bmatrix} 1 & 2 & 3 & \cdots\frac{[\sqrt[4]{h_{i,\ell,j}}]^{m+n+k}}{(1)!} & 0 & \cdots \\ 1 & 2 & 3 & \cdots\frac{[\sqrt[4]{h_{i,\ell,j}}]^{m+n+k}}{(1)!} & 0 & \cdots \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 1 & 2 & 3 & \cdots\frac{[\sqrt[4]{h_{i,\ell,j}}]^{m+n+k}}{(1)!} & 0 & \cdots \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & 0 & \cdots 0 & 0 & \cdots \\ . & & & & & \\ . & & & & & \\ . & & & & & \end{bmatrix}.$$

Here $x$ is a trible sequence and for all $i, \ell$ and $j$,

$$P - \lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} |\{(m, n, k) \in I_{i,\ell,j} : (|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k} = 0\}|$$

$$= P - \lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} \left( \frac{[\sqrt[4]{h_{i,\ell,j}}]^{m+n+k}}{(1)!} \right)^{1/m+n+k} = 0.$$

Therefore $(x_{mnk}) \xrightarrow{P} \Gamma^3(\widehat{S_{\theta_{i,\ell,j}}})$. Also

$$P - \lim_{i,\ell,j} \frac{1}{h_{i\ell j}} \sum_{m\in I_{i,\ell,j}} \sum_{n\in I_{i,\ell,j}} \sum_{k\in I_{i,\ell,j}} (|x_{m+i,n+\ell,k+j}|)^{1/m+n+k}$$

$= P -$

$$\frac{1}{2}\left( \lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} \left( \frac{[\sqrt[4]{h_{i,\ell,j}}]^{m+n+k}[\sqrt[4]{h_{i,\ell,j}}]^{m+n+k}[\sqrt[4]{h_{i,\ell,j}}]^{m+n+k}}{(1)!} \right)^{1/m+n+k} +1 \right)$$

$$= \frac{1}{2}.$$

Therefore $(x_{mnk}) \overset{P}{\nrightarrow} \Gamma^3(AC_{\theta, \ell, j})$.

(iii) If $x \in \Lambda^3$ and $(x_{mnk}) \overset{P}{\rightarrow} \Gamma^3(\widehat{S_{\theta_{i, \ell, j}}})$, then $(x_{mnk}) \overset{P}{\rightarrow} \Gamma^3(AC_{\theta, \ell, j})$.

Suppose $x \in \Lambda^3$ then for all $i, \ell$ and $j$, $(|x_{m+i, n+\ell, k+j} - 0|)^{1/m+n+k} \leq M$ for all $m, n, k$.

Also for given $\varepsilon > 0$ and $i, \ell$ and $j$ large for all $i, \ell$ and $j$ we obtain the following:

$$\frac{1}{h_{i\ell j}} \sum_{m \in I_{i, \ell, j}} \sum_{n \in I_{i, \ell, j}} \sum_{k \in I_{i, \ell, j}} (|x_{m+i, n+\ell, k+j} - 0|)^{1/m+n+k}$$

$$= \frac{1}{h_{i\ell j}} \sum_{m \in I_{k, \ell}} \sum_{n \in I_{i, \ell, j}}$$

$$\times \sum_{k \in I_{k, \ell, j} \text{ and } |x_{m+i, n+\ell, k+j}| \geq 0} (|x_{m+i, n+\ell, k+j} - 0|)^{1/m+n+k}$$

$$+ \frac{1}{h_{i\ell j}} \sum_{m \in I_{i, \ell, j}} \sum_{n \in I_{i, \ell, j}}$$

$$\times \sum_{k \in I_{i, \ell, j} \text{ and } |x_{m+i, n+\ell, k+j}| \leq 0} (|x_{m+i, n+\ell, k+j} - 0|)^{1/m+n+k}$$

$$\leq \frac{M}{h_{i\ell j}} \left| \{(m, n, k) \in I_{i, \ell, j} : (|x_{m+i, n+\ell, k+j} - 0|)^{1/m+n+k}\} = 0 \right| + \varepsilon.$$

Therefore $x \in \Lambda^3$ and $(x_{mnk}) \overset{P}{\rightarrow} \Gamma^3(\widehat{S_{\theta_{i, \ell, j}}})$ then

$$(x_{mnk}) \overset{P}{\rightarrow} \Gamma^3(AC_{\theta, \ell, j}).$$

(iv) $\Gamma^3(\widehat{S_{\theta_{i, \ell, j}}}) \cap \Lambda^3 = \Gamma^3[AC_{\theta_{i, \ell, j}}] \cap \Lambda^3$ follows from (i), (ii) and (iii).

**Theorem 3.6.** *If f be any Orlicz function, then*

$$\Gamma_f^3[AC_{\theta_{i,\ell,j}}] \not\subseteq \Gamma^3(\widehat{S_{\theta_{i,\ell,j}}}).$$

**Proof.** Let $x \in \Gamma_f^3[AC_{\theta_{i,\ell,j}}]$, for all $i$, $\ell$ and $j$.

Therefore we have

$$\frac{1}{h_{i\ell j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} f[(|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k}]$$

$$\geq \frac{1}{h_{i\ell j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j} \text{ and} |x_{m+i,n+\ell,k+j}|=0}$$

$$f[(|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k}]$$

$$> \frac{1}{h_{i\ell j}} f(0) |\{(m, n, k) \in I_{i,\ell,j} : (|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k}\} = 0|.$$

Hence $x \notin \Gamma^3(\widehat{S_{\theta_{i,\ell,j}}})$.


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