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THE NECESSARY AND SUFFICIENT CONDITION OF INPUT-OUTPUT GROUP DECOUPLING FOR LINEAR DESCRIPTOR SYSTEMS WITH INDEX ONE

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Abstract

In this paper, we study the problem of input-output group decoupling for regular linear descriptor systems of index at most one. The problem is handled in geometric setting. Necessary and sufficient conditions for a solution of the input-output group decoupling problem are established. Finally, we give other equivalent formulations of the input-output group decoupling problem for linear descriptor system with index one.

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1. Introduction

It is known that in multivariable control systems, the relations of input and output are very fundamental as a realization control role. In the control system, every input generally controls more than one output and every output can be controlled by more than one input. This phenomenon is called *coupled system*. In general, the coupled system is very difficult to control. It is known that not all coupled system can always be converted into decoupled system. Therefore, we need to design a control law that a coupled system may become a decoupled system in the sense that every input controls only one output and every output is controlled by only one input. Consequently, a decoupled system can be considered as consisting of a set of independent single-variable systems. This problem is called *input-output decoupling*. Thus, to simplify the relations of input and output of the system is an important problem in the control system theory. For the case of linear descriptor system, the problems of input-output decoupling have a more complex structure and properties than the problem of input-output decoupling in classical linear system. With the development and implementation of the research on descriptor systems, it is found that many practical systems, such as engineering systems (for example, power system, electrical network, aerospace engineering, chemical process), social economic system, network analysis, biological systems, time-series analysis, system modeling, and so on (Dai [5]), can be modeled as descriptor (singular) system. Therefore, discussion for input-output decoupling problem of linear descriptor system becomes very important.

Input-output decoupling is a problem that reduces a multiple input-multiple output system to a set of single input-single output systems. For classical linear systems, this issue was first introduced by Morgan [13], which investigated the decoupling synthesis via state feedback control using a state space approach. Then the problems posed by Morgan [13] can be solved by Falb and Wolovich [9], through the study of decoupling in the design and synthesis of multivariable control systems. Wonham and Morse [19] and Morse and Wonham [14] presented a condition that is more general for decoupling problem using geometric approach based on the concept of controllability subspace.

A generalization of the state feedback decoupling problem for classical linear system has been investigated by Sato and Lopresti [16], who obtained conditions for decoupling subsets of elements of the output set when the necessary conditions for system decoupling are not satisfied. Decoupling system under these conditions is referred to as partially decoupling. Furthermore, Sato and Lopresti [17] presented the concept of decoupling by state feedback which is extended from its normal interpretation of decoupling output elements to that of decoupling groups. All the results obtained by Sato and Lopresti [16, 17] are investigated using algebraic approach. Then Descusse et al. [6], block decoupling problem can be solved using static state feedback. In Dion et al. [7] and Commault et al. [4], block decoupling problem could be solved with a transfer function approach. All results studied above use only classical linear system tools for the solution of the problem.

The problem of input-output decoupling for descriptor system has been discussed by some researchers. For instance, Christodoulou [3] investigated the necessary and sufficient condition of input-output decoupling singular system using proportional plus derivative feedback. Dai [5] investigated the problem of input-output decoupling with impulse-free response. Ailon [1] presented a necessary and sufficient condition for decoupling singular system using proportional state feedback. The results on the same subject with the structure of closed loop system were also established by Paraskevopoulos and Koumboulis [15]. Chu and Hung [2] proposed the problem of row by row decoupling (RRDP) using a proportional state feedback and input transformation.

For the problem of input-output block decoupling for linear descriptor system was first presented by Koumboulis [11] through the regular static state feedback using algebraic approach. Furthermore, Liu et al. [12] proposed the problem of input-output block decoupling using state feedback for time-varying singular systems. Vaviadis and Karcanias [18] addressed the problem of block decoupling for singular systems through state feedback and input transformation using matrix fraction description approach (MFD).

Thus, it can be said that the main focus have been made by many authors is input-output decoupling problem via feedback control law using algebraic method. They derived the analytic expressions and structural properties of the transfer matrix function for closed-loop system. Wonham and Morse [19] and Morse and Wonham [14] studied only the decoupling problem for classical linear system using geometric approach. They derived more general conditions for decoupling problem by geometric approach based on the concept of controllability subspace of classical linear systems. Input-output group decoupling problem for regular linear descriptor system with index one using geometric approach through controllability subspaces is still to be studied in more detail.

In this paper, we mainly discuss the problem of input-output group decoupling for the case of regular linear descriptor system with index one using geometric approach. Here input and output can be partitioned into multiple subvectors such that every group of inputs affect only one group of outputs and does not affect other outputs. Then the necessary and sufficient condition for solution of input-output group decoupling is derived. Finally, we give some equivalent formulations of input-output group decoupling problem for regular linear descriptor system with index one.

This paper is organized as follows: In Section 2, we formalize the problem statement and summarize some basic properties about linear descriptor systems. We give the necessary and sufficient condition of input-output group decoupling for regular linear descriptor system with index one in Section 3. In Section 4, we present other equivalent formulation for the problem of input-output group decoupling. In Section 5, we illustrate our results by an example. Finally, conclusion is given in Section 6.

2. Preliminaries

Consider the following linear descriptor systems of the form:

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are, respectively, the state vector, the input vector and the output vector of system; $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are constant coefficient matrices, and E is singular. It is well known that the existence and uniqueness of solution to (1) are guaranteed if (E, A) is regular, i.e., $\det(sE - A) \neq 0$, for some $s \in \mathbb{C}$. The systems (1) are said to have an index at most one if the dimension of the largest nilpotent block in the Kronecker canonical form of (E, A) is at most one, Gerstner et al. [10].

The Kronecker canonical equivalent form for a general linear descriptor system is very complicated. However, the following theorem shows that the Kronecker form for a regular descriptor system is very simple.

Theorem 2.1 (Duan [8])**.** *Given the linear descriptor systems* (1) *with* $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, *and* (E, A) *is regular, there exist two nonsingular matrices Q and P such that*

$$(E, A, B, C) \overset{(Q, P)}{\Leftrightarrow} (\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C})$$

with

$$\begin{cases} \tilde{E} = QEP = diag(I_{n_1}, N), \\ \tilde{A} = QAP = diag(A_1, I_{n_2}), \\ \tilde{B} = QB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \\ \tilde{C} = CP = [C_1 \ C_2], \end{cases} \tag{2}$$

where $n_1 + n_2 = n$, *and the involved partitions are compatible. Furthermore, the matrix* $N \in \mathbb{R}^{n_2\times n_2}$ *is nilpotent.*

Theorem 2.1 shows that for regular linear descriptor systems (1), there exist two nonsingular matrices Q and P such that the systems (1) are a restricted system equivalent by the following systems:

$$\begin{cases} \dot{x}_1(t) = A_1 x_1(t) + B_1 u(t), \\ y_1(t) = C_1 x_1(t), \end{cases} \tag{3}$$

and

$$\begin{cases} N\dot{x}_2(t) = x_2(t) + B_2 u(t), \\ y_2(t) = C_2 x_2(t), \end{cases} \tag{4}$$

with the output of system as

$$y(t) = y_1(t) + y_2(t) = C_1 x_1(t) + C_2 x_2(t), \tag{5}$$

where

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = P^{-1}x(t), \quad x_1 \in \mathbb{R}^{n_1}, \quad x_2 \in \mathbb{R}^{n_2}; \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = QB; \quad [C_1 \ C_2] = CP.$$

In this form, subsystems (3) and (4) are called the *slow* and *fast subsystems*, respectively. The system represented by (3)-(5) is the Kronecker form for regular systems. This form is usually called the *standard decomposition form* of the linear descriptor systems (1). Furthermore, if the nilpotent matrix N in (4) has index h (i.e., $N^h = 0$ and $N^{h-1} \neq 0$), then the systems (1) are called a *system* with index h. If $N = 0$ in the fast subsystem (4), then the original systems (1) are called a *regular linear descriptor system* with index one.

The solution of regular linear descriptor systems (1) takes the form

$$x(t) = P\begin{bmatrix} I \\ 0 \end{bmatrix} x_1(t) + P\begin{bmatrix} 0 \\ I \end{bmatrix} x_2(t),$$

where

$$x_1(t) = e^{A_1 t} x_1(0) + \int_0^t e^{A_1(t-\tau)} B_1 u(\tau)\, d\tau,$$

$$x_2(t) = -\sum_{i=0}^{h-1} N^i B_2 u^{(i)}(t).$$

Here h is the degree of nilpotency of N. That is the integer h for which $N^h = 0$ and $N^{h-1} \neq 0$. The index of the systems (1) is the degree of nilpotency of N. We define the index to be zero, if E is nonsingular.

From the above formulae, it is obvious that the solution $x(t)$ will not contain derivatives of the input function u if and only if $h \leq 1$. In that case, the solution $x(t)$ is called *impulse free*. In general, the solution $x(t)$ involves derivatives of order $h - 1$ of the forcing input function u if the systems (1) have index h.

The following lemma presents a necessary and sufficient condition for the system (E, A, B) to be regular with index one.

Lemma 2.2 (Dai [5] and Gerstner et al. [10])**.** *Let* $E, A \in \mathbb{R}^{n \times n}$. *Then the following statements are equivalent*:

(i) (E, A) *is regular and of index at most one*.

(ii) $rank[E\ AS_\infty(E)] = n$, *where* $S_\infty(E)$ *denotes a matrix with orthogonal columns spanning the kernel of matrix E*.

(iii) $deg(det(sE - A)) = rank(E)$.

Since we do not want to consider derivatives of the input function in this paper, we restrict the analysis to regular index one systems here. The problem addressed in this paper is to find the necessary and sufficient condition for solution of input-output group decoupling for regular linear descriptor system with index one using geometric approach method. Then the definition of input-output group decoupling is given below.

Suppose that the systems (1) are a regular linear descriptor system with index one. Given $u_1, u_2, ..., u_m$ are input elements and $y_1, y_2, ..., y_p$ are output elements of the systems (1). Then the relationship between input and output of system can be presented in the following definition:

Definition 2.3. Given the regular linear descriptor system of the form (1). The output y_i is not controlled by the input u_j (or equivalently, the input u_j does not control the output y_i), if we have, for all $x_0 \in \mathbb{R}^n$ and all admissible inputs $u_1, u_2, ..., u_{j-1}, u_{j+1}, ..., u_m$,

$$y_i(t; x_0, u_1, ..., u_{j-1}, v, u_{j+1}, ..., u_m)$$

$$= y_i(t; x_0, u_1, ..., u_{j-1}, w, u_{j+1}, ..., u_m)$$

for all $t \in [0, T]$ and all admissible inputs v, w.

We assume that the input vector $u \in \mathbb{R}^m$ and the output vector $y \in \mathbb{R}^p$ can be partitioned into q subvectors, i.e.,

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_q \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{bmatrix}, \tag{6}$$

where $u_i \in \mathbb{R}^{l_i}$; $l_1 + l_2 + \cdots + l_q = m$ and $y_i \in \mathbb{R}^{k_i}$; $k_1 + k_2 + \cdots + k_q = p$; for $i = 1, 2, ..., q$. Obviously,

$$y_i(t) = C_i x(t); \quad i = 1, 2, ..., q \quad \text{and} \quad u(t) = \sum\nolimits_{i=1}^{q} H_i u_i(t),$$

where $C_i \in \mathbb{R}^{k_i \times n}$ and $H_i \in \mathbb{R}^{m \times l_i}$ are determined by

$$C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_q \end{bmatrix} \quad \text{and} \quad H_i = \begin{bmatrix} 0 \\ \vdots \\ I_i \\ \vdots \\ 0 \end{bmatrix}, \quad i = 1, 2, ..., q.$$

Here I_i is the $l_i \times l_i$ identity matrix. If we define the $n \times l_i$-matrix B_i by $B_i = BH_i$, $i = 1, 2, ..., q,$ then the systems (1) become

$$E\dot{x}(t) = Ax(t) + Bu(t) = Ax(t) + B\sum_{i=1}^{q} H_i u_i(t)$$

$$= Ax(t) + \sum_{i=1}^{q} BH_i u_i(t) = Ax(t) + \sum_{i=1}^{q} B_i u_i(t).$$

Systems (1) can be rewritten as

$$\begin{cases} E\dot{x}(t) = Ax(t) + \sum_{i=1}^{q} B_i u_i(t), \\ y_i(t) = C_i x(t), \quad i = 1, 2, ..., q, \end{cases} \tag{7}$$

where u_i and y_i are group of inputs and group of outputs, respectively, with

$$u_i = \begin{bmatrix} u_{i1} \\ u_{i2} \\ \vdots \\ u_{il_i} \end{bmatrix}; \quad i = 1, 2, ..., q; \quad l_1 + l_2 + \cdots + l_q = m,$$

$$y_i = \begin{bmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{ik_i} \end{bmatrix}; \quad i = 1, 2, ..., q; \quad k_1 + k_2 + \cdots + k_q = p.$$

The following definition is a slight extension of Definition 2.3:

Definition 2.4. Given a regular linear descriptor systems of the form (7), with $i, j = 1, 2, ..., q$.

(i) The group of outputs $y_i = (y_{i1}, y_{i2}, ..., y_{ik_i})^T$ is not controlled by the group of inputs $u_j = (u_{j1}, u_{j2}, ..., u_{jl_j})^T$ if for all $s = 1, 2, ..., k_i$, the output y_{is} is not controlled by any of the inputs u_{jr}, $r = 1, 2, ..., l_j$.

(ii) The group of outputs $y_i = (y_{i1}, y_{i2}, ..., y_{ik_i})^T$ is controlled by the group of inputs $u_j = (u_{j1}, u_{j2}, ..., u_{jl_j})^T$ if part (i) does not hold, i.e., there exist an $s \in \{1, 2, ..., k_i\}$ and an $r \in \{1, 2, ..., l_j\}$ such that u_{jr} controls y_{is}.

Regular linear descriptor system (7) is said to be *input-output group decoupling* if for given a set of inputs $\{u_1, u_2, ..., u_q\}$, the group of inputs u_i only controls the group of outputs y_i and does not control other output y_j for $j \neq i$. This leads to the following definition:

Definition 2.5. A regular linear descriptor system (7) is said to be *input-output group* decoupling if the following statements are true:

(i) The group y_i of output is not controlled by the group u_j of input for $i \neq j$, i, $j = 1, 2, ..., q$.

(ii) The group y_i of output is controlled by the group u_i of input for $i = 1, 2, ..., q$.

3. Input-output Group Decoupling for Linear Descriptor System with Index One

We refer to systems (1) for linear descriptor system that are regular and of index at most one can be separated into slow subsystem (3) and fast subsystems (4) with $N = 0$ of the form

$$\begin{cases} 0 = x_2(t) + B_2 u(t), \\ y_2(t) = C_2 x_2(t). \end{cases} \tag{8}$$

In order to solve linear descriptor systems (1) that are regular and of index at most one, it suffices only to find the solution of the slow and fast subsystems (3) and (8). Note that the slow subsystem (3) is an ordinary differential equation. It has a unique solution with any initial condition $x_1(0) = x_{10}$ for any piecewise continuous input function $u(t)$. This solution is given by

$$x_1(t) = e^{A_1 t} x_1(0) + \int_0^t e^{A_1(t-\tau)} B_1 u(\tau) d\tau. \tag{9}$$

Thus, the response, $x_1(t)$, of the slow subsystem (8) is completely determined by the initial value $x_1(0) = x_{10}$ and the control $u(\tau)$ $(0 \leq \tau \leq t)$.

The fast subsystem (8) has a solution

$$x_2(t) = -B_2 u(t). \tag{10}$$

By combining the solutions (9) and (10), we obtain the solution

$$\begin{aligned} x(t) &= P\begin{bmatrix} I \\ 0 \end{bmatrix} x_1(t) + P\begin{bmatrix} 0 \\ I \end{bmatrix} x_2(t) \\ &= P\begin{bmatrix} I \\ 0 \end{bmatrix}\left(e^{A_1 t} x_1(0) + \int_0^t e^{A_1 t - \tau} B_1 u(\tau)\, d\tau \right) - P\begin{bmatrix} 0 \\ I \end{bmatrix} B_2 u(t). \end{aligned} \tag{11}$$

Therefore, the output $y(t)$ can be expressed as

$$\begin{aligned} y(t) &= Cx(t) \\ &= CP\begin{bmatrix} I \\ 0 \end{bmatrix}\left(e^{A_1 t} x_1(0) + \int_0^t e^{A_1 (t-\tau)} B_1 u(\tau)\, d\tau \right) - CP\begin{bmatrix} 0 \\ I \end{bmatrix} B_2 u(t) \\ &= C_1\left(e^{A_1 t} x_1(0) + \int_0^t e^{A_1 (t-\tau)} B_1 u(\tau)\, d\tau \right) - C_2 B_2 u(t) \\ &= C_1 e^{A_1 t} x_1(0) + C_1 \int_0^t e^{A_1 (t-\tau)} B_1 u(\tau)\, d\tau - C_2 B_2 u(t). \end{aligned} \tag{12}$$

Suppose that the output $y(t) \in \mathbb{R}^p$ and the input $u(t) \in \mathbb{R}^m$ can be partitioned into q subvectors as (6). Similarly, $B_1 \in \mathbb{R}^{n_1 \times m}$ and $B_2 \in \mathbb{R}^{n_2 \times m}$ can be partitioned into q subvectors as

$$B_1 = [B_{11} \;\; B_{12} \cdots B_{1q}]; \quad B_{1j} \in \mathbb{R}^{n_1 \times l_j}; \quad j = 1, 2, \ldots, q,$$

$$B_2 = [B_{21} \;\; B_{22} \cdots B_{2q}]; \quad B_{2j} \in \mathbb{R}^{n_2 \times l_j}; \quad j = 1, 2, \ldots, q.$$

Let $C_1 \in \mathbb{R}^{p \times n_1}$ and $C_2 \in \mathbb{R}^{p \times n_2}$ be partitioned into q subvectors as

$$C_1 = \begin{bmatrix} C_{11} \\ C_{12} \\ \vdots \\ C_{1q} \end{bmatrix} \quad \text{and} \quad C_2 = \begin{bmatrix} C_{21} \\ C_{22} \\ \vdots \\ C_{2q} \end{bmatrix}, \text{ where } C_{1i} \in \mathbb{R}^{k_i \times n_1} \text{ and } C_{2i} \in \mathbb{R}^{k_i \times n_2}.$$

Then equation (12), for $i = 1, 2, ..., q,$ can be written as

$$y_i(t) = C_{1i}e^{A_1 t}x_1(0) + \int_0^t C_{1i}e^{A_1(t-\tau)}\sum_{j=1}^{q} B_{1j}u_j(\tau)d\tau$$

$$- C_{2i}\sum_{j=1}^{q} B_{2j}u_j(t). \tag{13}$$

With power series expansion of $e^{A_1(t-\tau)},$ equation (13) can be written as

$$y_i(t) = C_{1i}e^{A_1 t}x_1(0) + \int_0^t \sum_{k=0}^{\infty} C_{1i}\frac{A_1^k(t-\tau)^k}{k!}\left(\sum_{j=1}^{q} B_{1j}u_j(\tau)\right)d\tau$$

$$- \sum_{j=1}^{q} C_{2i}B_{2j}u_j(t)$$

$$= C_{1i}e^{A_1 t}x_1(0) + \int_0^t \sum_{k=0}^{\infty} \frac{1}{k!}C_{1i}A_1^k\left(\sum_{j=1}^{q} B_{1j}u_j(\tau)\right)(t-\tau)^k d\tau$$

$$- \sum_{j=1}^{q} C_{2i}B_{2j}u_j(t)$$

$$= C_{1i}e^{A_1 t}x_1(0) + \int_0^t \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{j=1}^{q} (C_{1i}A_1^k B_{1j})u_j(\tau)\right)(t-\tau)^k d\tau$$

$$- \sum_{j=1}^{q} C_{2i}B_{2j}u_j(t). \tag{14}$$

Based on these results, we obtain the theorem which is a necessary and sufficient condition for the problem of input-output group decoupling of regular linear descriptor systems with index one.

Theorem 3.1. *Consider a regular linear descriptor system with index one of the form* (7) *and let* $i, j \in \{1, 2, ..., q\}$. *Then the group of outputs* y_i *is not controlled by the group of inputs* u_j, *for* $i \neq j$ *if and only if*

$$C_{1i}A_1^k B_{1j} = 0; \quad i \neq j, \quad k = 0, 1, 2, ... \tag{15}$$

and

$$C_{2i}B_{2j} = 0, \quad \forall i \neq j. \tag{16}$$

Proof. Suppose that the output y_i is not controlled by the input u_j, for $i, j = 1, 2, ..., q.$ With Definition 2.3, for any inputs $v, w \in \mathbb{R}^{l_j}$, we have

$$y_i(t;\, x_0, u_1, u_2, ..., u_{j-1}, v, u_{j+1}, ..., u_m)$$

$$= C_{1i}e^{A_1 t}x_1(0) + \int_0^t C_{1i}e^{A_1(t-\tau)}\left(\sum_{\substack{l=1\\ l\neq j}}^{q} B_{1l}u_l(\tau) + B_{1j}v(\tau)\right)d\tau$$

$$- C_{2i}\left(\sum_{\substack{l=1\\ l\neq j}}^{q} B_{2l}u_l(\tau) + B_{2j}v(\tau)\right),$$

and the analogous one with input v replaced by input w, we obtain

$$\int_0^t C_{1i}e^{A_1(t-\tau)}B_{1j}(v(\tau) - w(\tau))d\tau - C_{2i}B_{2j}(v(t) - w(t)) = 0. \tag{17}$$

Let $z = v - w.$ Then equation (17) can be written as

$$\int_0^t C_{1i}e^{A_1(t-\tau)}B_{1j}z(\tau)d\tau - C_{2i}B_{2j}z(t) = 0 \tag{18}$$

for all $z \in \mathbb{R}^{l_j}$.

Let $z(t) \equiv z_0$. Differentiating (18) with respect to t, we have

$$C_{1i}B_{1j}z_0 + \int_0^t C_{1i}A_1e^{A_1(t-\tau)}B_{1j}z_0 d\tau = 0. \tag{19}$$

For $t = 0,$ condition (19) becomes $C_{1i}B_{1j}z_0 = 0$ for all $z_0 \in \mathbb{R}^{l_j}$, i.e., we get $C_{1i}B_{1j} = 0.$ Consequently, condition (19) becomes

$$C_{1i}A_1\int_0^t e^{A_1(t-\tau)}B_{1j}z_0 d\tau = 0 \tag{20}$$

for all $z_0 \in \mathbb{R}^{l_j}$.

Repeating the procedure, differentiating (20) with respect to t using Leibniz rule, we obtain

$$C_{1i}A_1B_{1j}z_0 + C_{1i}A_1^2\int_0^t e^{A_1(t-\tau)}B_{1j}z_0 d\tau = 0. \tag{21}$$

For $t = 0$, $C_{1i}A_1B_{1j}z_0 = 0$ for all $z_0 \in \mathbb{R}^{l_j}$, i.e., we get $C_{1i}A_1B_{1j} = 0$.

Consequently, we have

$$C_{1i}A_1^2\int_0^t e^{A_1(t-\tau)}B_{1j}z_0 d\tau = 0.$$

Similarly, repeating the procedure, we get

$$C_{1i}A_1^kB_{1j} = 0; \quad i \neq j, \quad k = 0, 1, 2, \ldots.$$

Next, we will prove that $C_{2i}B_{2j} = 0$, for every $i \neq j$. From equation (18) for $t = 0$, we get $C_{2i}B_{2j}z_0 = 0$. Consequently, we have $C_{2i}B_{2j} = 0$, for $i \neq j$.

Conversely, suppose that (15) and (16) are satisfied, for $i \neq j$. Then group y_i of output can be written as

$$y_i(t) = C_{1i}e^{A_1t}x_1(0) + \int_0^t C_{1i}e^{A_1(t-\tau)}\sum_{l=1}^q B_{1l}u_l(\tau)d\tau$$

$$- C_{2i}\sum_{l=1}^q B_{2l}u_l(t). \tag{22}$$

From (15)-(16) and the power series expansion of $e^{A_1(t-\tau)}$, equation (22) becomes

$$y_i(t) = C_{1i}e^{A_1t}x_1(0) + \int_0^t C_{1i}\sum_{k=0}^{\infty}\frac{A_1^k(t-\tau)^k}{k!}\sum_{l=1}^{q} B_{1l}u_l(\tau)d\tau$$

$$- C_{2i}\sum_{l=1}^q B_{2l}u_l(t)$$

$$= C_{1i}e^{A_1 t}x_1(0) + \int_0^t \sum_{k=0}^{\infty} \frac{1}{k!} C_{1i}A_1^k \left(\sum_{l=1}^{q} B_{1l}u_l(\tau)\right)(t-\tau)^k d\tau$$

$$- \sum_{l=1}^{q} C_{2i}B_{2l}u_l(t)$$

$$= C_{1i}e^{A_1 t}x_1(0) + \int_0^t \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{l=1}^{q} (C_{1i}A_1^k B_{1l})u_l(\tau)\right)(t-\tau)^k d\tau$$

$$- \sum_{l=1}^{q} C_{2i}B_{2l}u_l(t).$$

Because $C_{1i}A_1^k B_{1j} = 0$ and $C_{2i}B_{2j} = 0,$ for $i \neq j,$ $k = 0, 1, 2, \ldots,$ $y_i(t)$ can be written as

$$y_i(t) = C_{1i}e^{A_1 t}x_1(0) + \int_0^t \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{\substack{l=1 \\ l \neq j}}^{q} (C_{1i}A_1^k B_{1j})u_l(\tau)\right)(t-\tau)^k d\tau$$

$$- \sum_{\substack{l=1 \\ l \neq j}}^{q} (C_{2i}B_{2l})u_l(\tau).$$

Consequently, the group y_i of output is not controlled by the group u_j of input, for $i \neq j,$ with $i,$ $j \in \{1, 2, \ldots, q\}.$ □

4. Equivalent Formulation of Input-output Group Decoupling

In this section, we give some equivalent formulations of conditions (15) and (16). If M is a matrix of order $n \times m,$ then we shall denote by $\mathfrak{M}$ the image of M. In other words, $\mathfrak{M}$ is the subspace of $\mathbb{R}^n$ generated by the columns of M. With this notation, first we consider the subspace $\bigcap_{k=0}^{n_1-1} Ker C_{1i}A_1^k.$

Lemma 4.1. *Subspace* $\bigcap_{k=0}^{n_1-1} Ker C_{1i}A_1^k$ *is invariant under* $A_1.$

Proof. To prove that $\bigcap_{k=0}^{n_1-1} Ker C_{1i}A_1^k$ is invariant under $A_1,$ it will be shown that

$$A_1\left(\bigcap_{k=0}^{n_1-1} KerC_{1i}A_1^k\right) \subset \bigcap_{k=0}^{n_1-1} KerC_{1i}A_1^k.$$

For $z \in \bigcap_{k=0}^{n_1-1} KerC_{1i}A_1^k$, it will be shown that $A_1 z \in \bigcap_{k=0}^{n_1-1} KerC_{1i}A_1^k$. From $z \in \bigcap_{k=0}^{n_1-1} KerC_{1i}A_1^k$, we infer that $C_{1i}A_1^k z = 0$, $k = 0, 1, 2, ..., n_1 - 1$. Because $C_{1i}A_1^k(A_1 z) = C_{1i}A_1^{k+1} z = C_{1i}A_1^l z = 0$, for every $l = 1, 2, ..., n$, we obtain $A_1 z \in KerC_{1i}A_1^k$. Consequently, $A_1 z \in \bigcap_{k=0}^{n_1-1} KerC_{1i}A_1^k$, $k = 0, 1, ..., n-1$. This proves that the subspace $\bigcap_{k=0}^{n_1-1} KerC_{1i}A_1^k$ is invariant under A_1. □

Next, we give an equivalent formulation of conditions (15) and (16) according to subspaces $\bigcap_{k=0}^{n_1-1} KerC_{1i}A_1^k$ and $\bigcap_{i=0}^{q} KerC_{2i}$. This is stated in the following theorem:

Theorem 4.2. *Consider linear descriptor system* (7) *and let* $i, j \in \{1, 2, ..., q\}$. *Then the group of outputs is not controlled by the group of inputs* u_j, *for* $i, j = 1, 2, ..., q$, $\forall i \neq j$ *if and only if the following conditions are satisfied*:

$$\mathfrak{B}_{1j} \subset \bigcap_{k=0}^{n_1-1} KerC_{1i}A_1^k \tag{23}$$

and

$$\mathfrak{B}_{2j} \subset \bigcap_{i=0}^{q} KerC_{2i}. \tag{24}$$

Proof. It follows from Theorem 3.1 that $C_{1i}A_1^k B_{1j} = 0$. This implies that $B_{1j} \in KerC_{1i}A_1^k$, for $k = 0, 1, 2, ..., n_1 - 1$. Thus, $\mathfrak{B}_{1j}$ is a subset of

$KerC_{1i}A_1^k$. Consequently, we have $\mathfrak{B}_{1j} \subset \bigcap_{k=0}^{n_1-1} KerC_{1i}A_1^k$. Then, from $C_{2i}B_{2j} = 0$, it follows that $B_{2j} \in KerC_{2i}$. Therefore, $\mathfrak{B}_{2j}$ is a subset of $KerC_{2i}$, i.e., $\mathfrak{B}_{2j} \subset KerC_{2i}$. Consequently, $\mathfrak{B}_{2j} \subset \bigcap_{i=0}^{q} KerC_{2i}$.

On the other hand, suppose that (23) and (24) are satisfied. It will be shown that the output y_i is not controlled by the input u_j, for every $i \neq j$ with $i, j = 1, 2, ..., q$. From $\mathfrak{B}_{1j} \subset \bigcap_{k=0}^{n_1-1} KerC_{1i}A_1^k$, we have $B_{1j}z \in KerC_{1i}A_1^k$, for every $z \in \mathbb{R}^{l_j}$. It implies that $C_{1i}A_1^k B_{1j}z = 0$. Consequently, we have $C_{1i}A_1^k B_{1j} = 0$.

Then from $\mathfrak{B}_{2j} \subset \bigcap_{i=0}^{q} KerC_{2i}$, we have $B_{2j}z \in KerC_{2i}$, for every $z \in \mathbb{R}^{l_j}$. This implies that $C_{2i}B_{2j}z = 0$. As a consequence, $C_{2i}B_{2j} = 0$. Using Theorem 3.1, we conclude that the output y_i is not affected by the input u_j, for $i \neq j$ with $i, j = 1, 2, ..., q$. □

Furthermore, we consider the subspaces $\sum_{k=0}^{n_1-1} ImA_1^k B_{1j}$ and ImB_{2j}, for $j = 1, 2, ..., q$. We have the following lemma:

Lemma 4.3. *The subspace* $\sum_{k=0}^{n_1-1} ImA_1^k B_{1j}$ *is the smallest subspace which contains* $\mathfrak{B}_{1j}$ *and is invariant under* A_1.

Proof. First, it will be shown that $\sum_{k=0}^{n_1-1} ImA_1^k B_{1j}$ contains $\mathfrak{B}_{1j}$ (where $\mathfrak{B}_{1j} = ImB_{1j}$). Since $\sum_{k=0}^{n_1-1} ImA_1^k B_{1j} = ImB_{1j} + \cdots + ImA_1^{n_1-1}B_{1j}$, $ImB_{1j} \subset \sum_{k=0}^{n_1-1} ImA_1^k B_{1j}$. Further, it will be shown that $\sum_{k=0}^{n_1-1} ImA_1^k B_{1j}$ is invariant under A_1. From $A_1 \sum_{k=0}^{n_1-1} ImA_1^k B_{1j} = \sum_{k=0}^{n_1-1} ImA_1^{k+1} B_{1j}$ and Cayley-Hamilton theorem, $A_1^{n_1} B_{1j}$ can be expressed as a linear combination of the column vectors of B_{1j}, A_1B_{1j}, ..., $A_1^{n_1-1}B_{1j}$. Therefore, we have

$$A_1\sum_{k=0}^{n_1-1} ImA_1^k B_{1j} = \sum_{k=0}^{n_1-1} ImA_1^{k+1}B_{1j} \subset \sum_{k=0}^{n_1-1} ImA_1^k B_{1j}.$$

It remains to show that $\sum_{k=0}^{n_1-1} ImA_1^k B_{1j}$ is the smallest subspace invariant under A_1 that contains $\mathfrak{B}_{1j}$. Let V be a linear subspace which is invariant under A_1 and contains $\mathfrak{B}_{1j}$. It will be shown that $\sum_{k=0}^{n_1-1} ImA_1^k B_{1j} \subset V$. Since $ImB_{1j} \subset V$ and $A_1V \subset V$, the following holds:

$$ImA_1B_{1j} = A_1(ImB_{1j}) \subset A_1V \subset V$$

$$ImA_1^2B_{1j} = A_1(ImA_1B_{1j}) \subset A_1V \subset V$$

$$\vdots$$

$$ImA_1^{n_1-1}B_{1j} = A_1(ImA_1^{n_1-2}B_{1j}) \subset A_1V \subset V.$$

Therefore, $ImB_{1j} + ImA_1B_{1j} + ImA_1^2B_{1j} + \cdots + ImA_1^{n_1-1}B_{1j} \subset V$. We have $\sum_{k=0}^{n_1-1} ImA_1^k B_{1j} \subset V$. Thus, the subspace $\sum_{k=0}^{n_1-1} ImA_1^k B_{1j}$ is the smallest subspace which contains $\mathfrak{B}_{1j}$. □

Further, we have an equivalent formulation of conditions (15) and (16) according to the subspaces $\sum_{k=0}^{n_1-1} ImA_1^k B_{1j}$ and ImB_{2j}. This result is expressed as follows:

Theorem 4.4. *Consider linear descriptor system* (7) *with index one and let* $i, j \in \{1, 2, ..., q\}$. *Then the group of outputs* y_i *is not controlled by the group of inputs* u_j *if and only if one of the following conditions is satisfied*:

(i) $\sum_{k=0}^{n_1-1} ImA_1^k B_{1j} \subset \bigcap_{k=0}^{n_1-1} KerC_{1i}A_1^k$ *and* $\mathfrak{B}_{2j} \subset \bigcap_{i=0}^{q} KerC_{2i}$.

(ii) $\sum_{k=0}^{n_1-1} ImA_1^k B_{1j} \subset KerC_{1i}$ *and* $\mathfrak{B}_{2j} \subset KerC_{2i}$.

Proof. It follows from Theorem 4.2 that $\mathfrak{B}_{1j} \subset \bigcap_{k=0}^{n_1-1} KerC_{1i}A_1^k$ and by Lemma 4.3, that the subspace $\sum_{k=0}^{n_1-1} ImA_1^k B_{1j}$ is the smallest subspace that contains $\mathfrak{B}_{1j}$. Thus, it satisfies the relation $\mathfrak{B}_{1j} \subset \sum_{k=0}^{n_1-1} ImA_1^k B_{1j} \subset \bigcap_{k=0}^{n_1-1} KerC_{1i}A_1^k$. Therefore, the first term of condition (i) is proven, i.e.,

$$\sum_{k=0}^{n_1-1} ImA_1^k B_{1j} \subset \bigcap_{k=0}^{n_1-1} KerC_{1i}A_1^k. \tag{25}$$

It is clear that

$$\bigcap_{k=0}^{n_1-1} KerC_{1i}A_1^k \subset KerC_{1i}. \tag{26}$$

Thus, from (25) and (26), we get the relation

$$\sum_{k=0}^{n_1-1} ImA_1^k B_{1j} \subset \bigcap_{k=0}^{n_1-1} KerC_{1i}A_1^k \subset KerC_{1i}.$$

This implies that

$$\sum_{k=0}^{n_1-1} ImA_1^k B_{1j} \subset KerC_{1i}. \tag{27}$$

Thus, the first term of condition (ii) is proven. For $ImB_{2j} \subset KerC_{2i}$ which implies $ImB_{2j} \subset \bigcap_{i=0}^{q} KerC_{2i}$ has been proved in Theorem 4.2.

Furthermore, to show the reverse implication of this statement, we consider the system (ii). This implies $A_1^k \mathfrak{B}_{1j} = ImA_1^k B_{1j} \subset KerC_{1i}$, for $k = 0, 1, 2, ..., n_1 - 1$. Therefore, we get $C_{1i}A_1^k \mathfrak{B}_{1j} = 0$, for $k = 0, 1, 2, ..., n_1 - 1$. Consequently, this implies $\mathfrak{B}_{1j} \subset \bigcap_{k=0}^{n_1-1} KerC_{1i}A_1^k$. Since $\sum_{k=0}^{n_1-1} ImA_1^k B_{1j}$ is the smallest subspace that contains $\mathfrak{B}_{1j}$, we have the following relationship:

$$\mathfrak{B}_{1j} \subset \sum_{k=0}^{n_1-1} Im A_1^k B_{1j} \subset \bigcap_{k=0}^{n_1-1} Ker C_{1i} A_1^k.$$

It follows from Theorem 4.2 that the group of outputs y_i is not controlled by the group of inputs u_j, for $i, j = 1, 2, ..., q$ with $i \neq j$. $\square$

Let $\mathcal{V}_1$ be a subspace of $\mathbb{R}^{n_1}$ which is invariant under A_1 and satisfies

$$\mathfrak{B}_{1j} \subset \mathcal{V}_1 \subset Ker C_{1i}, \tag{28}$$

and let $\mathcal{V}_2$ be a subspace of $\mathbb{R}^{n_2}$ that satisfies

$$\mathfrak{B}_{2j} \subset \mathcal{V}_2 \subset Ker C_{2i}. \tag{29}$$

This means that condition (ii) of Theorem 4.4 holds.

According to subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$ in (28) and (29), we have another equivalent formulation of conditions (15) and (16). This result is presented in the following theorem:

Theorem 4.5. *Consider linear descriptor system* (7) *with index one and let* $i, j \in \{1, 2, ..., q\}$. *Then the group of outputs* y_i *is not controlled by the group of inputs* u_j, *for* $i \neq j$, *if and only if there exist subspaces* $\mathcal{V}_1 \subset \mathbb{R}^{n_1}$ *and* $\mathcal{V}_2 \subset \mathbb{R}^{n_2}$ *with the following properties*:

(i) $\mathcal{V}_1$ *is invariant under* A_1.

(ii) $\mathfrak{B}_{1j} \subset \mathcal{V}_1 \subset Ker C_{1i}$.

(iii) $\mathfrak{B}_{2j} \subset \mathcal{V}_2 \subset Ker C_{2i}$.

Proof. Let the output y_i be not controlled by the input u_j, for every $i \neq j$, with $i, j = 1, 2, ..., q$ and suppose that $\mathcal{V}_1$ is a subspace of $\mathbb{R}^{n_1}$. By

Lemma 4.3 and Theorem 4.4, we have the following relation:

$$\mathfrak{B}_{1j} \subset \sum_{k=0}^{n_1-1} ImA_1^k B_{1j} \subset \bigcap_{k=0}^{n_1-1} KerC_{1i}A_1^k \subset KerC_{1i}.$$

Therefore, the subspace $\mathcal{V}_1 = \bigcap_{k=0}^{n_1-1} KerC_{1i}A_1^k$ is invariant under A_1 (by Lemma 4.1) and satisfies $\mathfrak{B}_{1j} \subset \mathcal{V}_1 \subset KerC_{1i}$. Thus, conditions (i) and (ii) are established. Further, to prove (iii), it follows from Theorem 4.4 that $\mathfrak{B}_{2j} \subset \bigcap_{i=0}^{q} KerC_{2i} \subset KerC_{2i}$. Thus, $\mathcal{V}_2 = \bigcap_{i=0}^{q} KerC_{2i}$ satisfies $\mathfrak{B}_{2j} \subset \mathcal{V}_2 \subset KerC_{2i}$. This proves (iii).

Conversely, let conditions (i), (ii) and (iii) hold. Then there exist subspaces $\mathcal{V}_1 \subset \mathbb{R}^{n_1}$ and $\mathcal{V}_2 \subset \mathbb{R}^{n_2}$ such that $\mathfrak{B}_{1j} \subset \mathcal{V}_1 \subset KerC_{1i}$ and $\mathfrak{B}_{2j} \subset \mathcal{V}_2 \subset KerC_{2i}$. By Lemma 4.3, $\sum_{k=0}^{n_1-1} ImA_1^k B_{1j}$ is the smallest subspace containing $\mathfrak{B}_{1j}$. Thus, $\mathfrak{B}_{1j} \subset \sum_{k=0}^{n_1-1} ImA_1^k B_{1j}$. Therefore, we get

$$\mathfrak{B}_{1j} \subset \sum_{k=0}^{n_1-1} ImA_1^k B_{1j} \subset \bigcap_{k=0}^{n_1-1} KerC_{1i}A_1^k \subset KerC_{1i}$$

and $\mathfrak{B}_{2j} \subset \bigcap_{i=0}^{q} KerC_{2i} \subset KerC_{2i}$. Based on the equivalence of conditions (i) and (ii) of Theorem 4.4, we have the conclusion that the output y_i is not controlled by the input u_j, for every $i \neq j$ with $i, j = 1, 2, ..., q$. □

By Theorem 4.4(ii) and Definition 2.4(ii), it is clear how to characterize the fact that the group of outputs y_i is controlled by the group of inputs u_j for $i, j = 1, 2, ..., q$. A necessary and sufficient condition for this condition is

$$C_{1i} \sum_{k=0}^{n_1-1} ImA_1^k B_{1j} \neq \{0\} \quad \text{and} \quad C_{2i} ImB_{2j} \neq \{0\}.$$

Thus, we have the following theorem:

Theorem 4.6. *The linear descriptor system* (7) *with index one is an input-output group decoupled in the sense of Definition* 2.5 *if and only if the following statements are true*:

(i) *For any* $i, j \in \{1, 2, ..., q\}$, *with* $i \neq j$, *one of the conditions of Theorem* 4.4 *is satisfied*.

(ii) *For all* $i = 1, 2, ..., q$,

$$C_{1i} \sum_{k=0}^{n_1 - 1} ImA_1^k B_{1i} \neq \{0\} \tag{30}$$

and

$$C_{2i} ImB_{2i} \neq \{0\}. \tag{31}$$

If subsystem (3) *is controllable*, *then the condition* (30) *can be replaced by*

$$C_{1i} \sum_{k=0}^{n_1 - 1} ImA_1^k B_{1j} = ImC_{1i}, \tag{32}$$

and if subsystem (8) *is controllable*, *then the condition* (31) *can be replaced by*

$$C_{2i} ImB_{2i} = ImC_{2i}. \tag{33}$$

Proof. Condition (i) has been proved in Theorem 4.4. Thus, we need only to prove (ii) that (30) implies (32). By using condition (ii) of Theorem 4.4, for $i \neq j$ and controllability of (A_1, B_1), we have

$$\sum_{j=1}^{q} \sum_{k=0}^{n_1 - 1} ImA_1^k B_{1j} = \mathbb{R}^{n_1}.$$

Thus, we get

$$ImC_{1i} = C_{1i}\mathbb{R}^{n_1} = C_{1i} \sum_{j=1}^{q} \sum_{k=0}^{n_1 - 1} ImA_1^k B_{1j}$$

$$= C_{1i}\left(\sum_{k=0}^{n_1-1} ImA_1^k B_{11} + \cdots + \sum_{k=0}^{n_1-1} ImA_i^k B_{1i} + \cdots + \sum_{k=0}^{n_1-1} ImA_1^k B_{1q}\right)$$

$$= C_{1i}\sum\nolimits_{k=0}^{n_1-1} ImA_i^k B_{1i}.$$

This establishes condition (31).

Further, we need to prove (31) implies (33). By controllability properties of subsystem (8), we have $\sum_{j=1}^{q} ImB_{2i} = \mathbb{R}^{n_2}$. Thus, we get

$$ImC_{2i} = C_{2i}\mathbb{R}^{n_2} = C_{2i}\sum_{j=1}^{q} ImB_{2i}$$

$$= C_{2i}(ImB_{21} + \cdots + ImB_{2i} + \cdots + ImB_{2q})$$

$$= C_{2i}ImB_{2i}.$$

This shows that condition (33) is proved. □

5. A Numerical Example

In this section, a numerical example is provided to illustrate the results obtained in this paper. Consider a linear descriptor system with index one

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}u(t),$$

$$y(t) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}x(t).$$

There exist two nonsingular matrices

$$Q = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}.$$

Then

$$QEP = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad QAP = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$QB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad CP = [C_1 \quad C_2] = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \\ 1 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}.$$

It will become a regular linear descriptor system with index one of the form

$$\begin{cases} \dot{x}_1(t) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x_1(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} u(t), \\ 0 = x_2(t) + [0 \quad 0] u(t), \end{cases}$$

$$y(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} x_1(t) + \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} x_2(t).$$

We obtain

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_2 = [0 \; 0], \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix},$$

and we have partition

$$B_{11} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B_{21} = 0, \quad B_{22} = 0,$$

$$C_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad C_{12} = [0 \quad 0], \quad C_{21} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}, \quad C_{22} = \begin{bmatrix} -\frac{1}{2} \end{bmatrix}.$$

We can calculate that

$$C_{11}B_{12} = 0, \quad C_{11}A_1B_{12} = 0, \ldots, C_{11}A_1^kB_{12} = 0,$$

$$C_{12}B_{11} = 0, \quad C_{12}A_1B_{11} = 0, \ldots, C_{12}A_1^kB_{11} = 0,$$

$$C_{21}B_{22} = 0, \quad C_{22}B_{21} = 0.$$

Hence, conditions (15) and (16) are satisfied. Therefore, the group of outputs y_i is not controlled by the group of inputs u_j, for $i \neq j$ with $i, j = 1, 2$.

6. Conclusion

The problem of input-output group decoupling for regular linear descriptor system with index at most one using geometric approach has been solved. The input and output of system can be partitioned into multiple subvectors such that every group of input controls only one group of outputs and does not control other outputs. The necessary and sufficient condition of input-output group decoupling problem has been derived. Furthermore, other equivalent formulations of input-output group decoupling problem for regular linear descriptor system with index one have been presented.

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