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SYNCHRONIZATION OF THE FRACTIONAL-ORDER CHEN SYSTEM AND ITS CIRCUIT IMPLEMENTATION

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Abstract

In this paper, fractional-order Chen system is studied. A predictor-corrector method is applied to design the controller. This method is used because the dynamical characteristics of fractional-order Chen system are obtained using multistep approximation. The designed controller is also applied to synchronize chaotic fractional-order systems in drive-response structure. Numerical simulation and circuit design of the systems demonstrate the effectiveness and feasibility of the proposed scheme.

1. Introduction

The subject of chaos synchronization has received a great attention since 1990 and grown rapidly theoretically and experimentally. The great reason of this growing is its broad and considerable applications in secure communication, automatic control, neural networks, etc. Synchronization means that the state of respond systems eventually approaches that of a driving system in the course of times. Under certain conditions, the respond systems may also evolve into the same orbit of the driving systems meant for complete synchronization. In the literature, there are several other types of synchronization such as generalized synchronization, phase synchronization and lag synchronization. There are many control strategies to synchronize chaotic systems such as impulsive control, sliding mode control, linear coupling for which the implementation is rather easy, nonlinear control and adaptive control [1, 5]. Most of these control strategies are based on the exact knowledge of the system and parameters. However, in practice, there are very few cases where the systems are exactly the same. Moreover, these parameters may change with time and become chaotic because of the chaotic disturbance [6]. For uncertain parameters, a lot of works have proceeded to solve this problem by adaptive synchronization [7, 13]. There has been an increasing interest in exploiting chaotic dynamics in engineering applications, where some attention has been focused on effectively creating chaos via simple physical systems such as electronic circuits [14, 19]. The pursuit of designing circuits to produce chaotic attractors has become a focal point for electronic engineers, not only because of their theoretical interest but also due to their potential real-world applications [21] in various chaos-based technologies and information systems [22, 29].

This paper addresses the numerical simulation and circuit realization of synchronization of chaotic Chen system through nonlinear feedback method. A synchronous controller is designed to convert the nonlinear error system of chaotic synchronization into a linear error system, and the synchronization of the fractional-order Chen system is realized according to the stability theory of a fractional-order linear system.

2. Synchronization of Fractional-order Chen System

2.1. Mathematical model of system and problem description

The mathematical model of fractional-order chaotic system can be described as follows:

$$\frac{d^q x}{dt^q} = Ax + f(x), \tag{1}$$

where $0 < q < 1$, $A \in R^{n\times n}$ is the matrix of system parameters, $x = f(x_1, x_2, ..., x_n)^T \in R^n$ is the state vector of system (1), $f(x) = (f_1(x), f_2(x), ..., f_n(x))^T \in R^{n\times 1}$ is the nonlinear term. System (1) is considered to be the drive system, and the response system is as follows:

$$\frac{d^q y}{dt^q} = By + g(y) + u, \tag{2}$$

where $0 < q < 1$, $B \in R^{n\times n}$ is the matrix of system parameters, $y = f(y_1, y_2, ..., y_n)^T \in R^n$ is the state vector of system (2), $g(y) = (g_1(y), g_2(y), ..., g_n(y))^T \in R^{n\times 1}$ is the nonlinear term, and $u = (u_1, u_2, ..., u_n)^T$ is the controller.

From the drive system (1) and the response system (2), the state error vector is obtained which can be expressed as $e = y - x = (e_1, e_2, ..., e_n)^T$, and we should design the appropriate controller u to make $\lim e_{t\to\infty} = 0$.

According to [12], we investigate the commensurate fractional-order Chen system in this paper and its mathematical model is as follows:

$$\begin{cases} \dfrac{d^q x_1}{dt^q} = a(x_2 - x_1), \\ \dfrac{d^q x_2}{dt^q} = dx_1 - x_1x_3 + cx_2, \\ \dfrac{d^q x_3}{dt^q} = x_1x_2 - bx_3, \end{cases} \tag{3}$$

where $0 < q < 1$, a, b, c and d are constants, $a > 0$, $b > 0$, $c > 0$ and $d < 0$.

If the system (3) is considered as the drive system, then the relevant response system is as follows:

$$\begin{cases} \dfrac{d^q y_1}{dt^q} = a(y_2 - y_1) + u_1, \\ \dfrac{d^q y_2}{dt^q} = dy_1 - y_1y_3 + cy_2 + u_2, \\ \dfrac{d^q y_3}{dt^q} = y_1y_2 - by_3 + u_3. \end{cases} \tag{4}$$

In order to synchronise the chaotic systems that are in different initial conditions, the target is to choose the appropriate controllers u_1, u_2 and u_3.

2.2. Design of synchronization controller

Lemma 1 [13]**.** *Consider the following fractional-order linear system*: $d^q x/dt^q = Cx$, *where* $0 < q < 1$, $x = (x_1, x_2, ..., x_n)^T \in R^n$, *and* $C \in R^{n\times n}$. *If* $|\arg(\lambda_i)| > q\pi/2$, *where* λ_i $(i = 1, 2, 3)$ *are eigenvalues of* C, *then the* *asymptotical stability of the origin of the fractional-order system, that is,* $\lim x_{t\to\infty} = 0,$ *is reached.*

The driven system (3) is written in the form of system (1) as follows:

$$\frac{d^q x}{dt^q} = \begin{pmatrix} -a & a & 0 \\ d & c & 0 \\ 0 & 0 & -b \end{pmatrix} x + \begin{pmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{pmatrix}. \tag{5}$$

Then the response system (4) can be described as system (2) in the form of

$$\frac{d^q y}{dt^q} = \begin{pmatrix} -a & a & 0 \\ d & c & 0 \\ 0 & 0 & -b \end{pmatrix} y + \begin{pmatrix} 0 \\ -y_1y_3 \\ y_1y_2 \end{pmatrix} + u. \tag{6}$$

Here, we adopt the linearization by feedback method to realize the synchronization of the fractional-order Chen system by selecting $u = f(x) - g(y) - Ke$ as the controller

$$u = f(x) - g(y) - Ke = \begin{pmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{pmatrix} - \begin{pmatrix} 0 \\ -y_1y_3 \\ y_1y_2 \end{pmatrix} - Ke,$$

where $e = y - x = (e_1, e_2, e_3)^T$.

From equations (5), (6) and u, we can obtain the error system as follows:

$$\frac{d^q e}{dt^q} = \begin{pmatrix} -a & a & 0 \\ d & c & 0 \\ 0 & 0 & -b \end{pmatrix} e - Ke. \tag{7}$$

Let

$$K = \begin{pmatrix} 1-a & 2a & 0 \\ d & 2c & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so the error system will be $\frac{d^q e}{dt^q} = \begin{pmatrix} -1 & -a & 0 \\ 0 & -c & 0 \\ 0 & 0 & -b \end{pmatrix} e = Ce.$

By calculating the equation of the error system, the eigenvalues of C are $\lambda_1 = -1$, $\lambda_2 = -c$ and $\lambda_3 = -b$ because $c > 0$ and $b > 0$, all eigenvalues λ_i $(i = 1, 2, 3)$ are negative, namely: $|\arg(\lambda_i)| = \pi > q\pi/2$. According to Lemma 1, the asymptotical stability of the origin of the error system, that is, $\lim e_{t\to\infty} = 0$, is reached, achieving the goal of synchronizing the fractional-order Chen system.

3. Model and Phase Portrait Analysis

As is well known, the mathematical model of the commensurate fractional-order Chen system is as follows:

$$\begin{cases} \frac{d^q x_1}{dt^q} = a(x_2 - x_1), \\ \frac{d^q x_2}{dt^q} = dx_1 - x_1 x_3 + cx_2, \\ \frac{d^q x_3}{dt^q} = x_1 x_2 - bx_3, \end{cases} \tag{8}$$

where $0 < q < 1$, a, b, c and d are constants, $a > 0$, $b > 0$, $c > 0$ and $d < 0$.

In this paper, the order of a system is considered as the list of each order of its equation. For example, for system (3), it is considered to be the (0.98, 0.98, 0.98)-order system when $q = 0.98$. By utilizing the predictor-corrector method, we obtain the attractors of system (3) when $q = 0.98$, $a = 35$, $b = 3$, $c = 28$ and $d = -7$, the phase portraits are depicted in Figure 1. From the phase portrait shown in Figure 1, we can see that there exist chaotic attractors in system (3) with $q = 0.98$, $a = 35$, $b = 3$, $c = 28$ and $d = -7$.

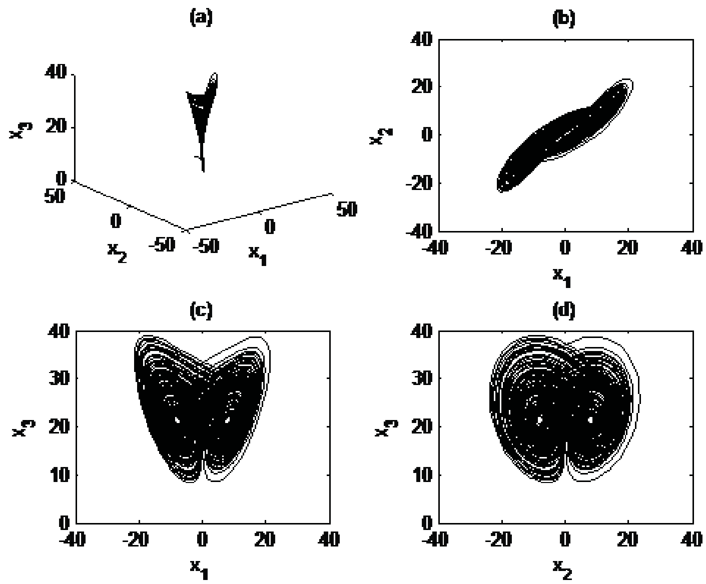

Figure 1. (a) Phase portraits of system (3); (b) $x_1 - x_2$ plan; (c) $x_1 - x_3$ plan; and (d) $x_2 - x_3$ plan.

4. Numerical Simulation

Here, taking the (0.98, 0.98, 0.98)-order Chen system as an example, we conduct the numerical simulation of the synchronization via Matlab. When the system parameters are chosen to be $q = 0.98$, $a = 35$, $b = 3$, $c = 28$ and $d = -7$, in time step of 0.01, with initial values of the state variables being $x_1(0) = -9$, $x_2(0) = -5$, $x_3(0) = 14$, $y_1(0) = 5$, $y_2(0) = 4$, $y_3(0) = 3$, respectively, we can observe that the drive system and the response system are in the asymptotic synchronization from Figure 2.

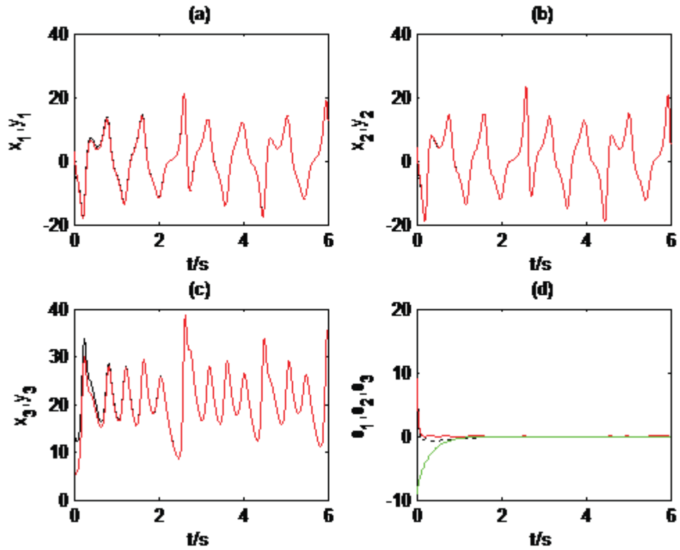

Figure 2. (a) Variation of (x_1, y_1) with time; (b) variation of (x_2, y_2) with time; (c) variation of (x_3, y_3) with time; and (d) the synchronization errors versus time t.

5. Circuit Design for the (0.98, 0.98, 0.98)-order Chen System

Because fractional calculus does not allow the direct calculation of the differential operators in time domain, we usually adopt the method of approximation conversion from time domain to frequency domain to design a circuit. Based on Bode diagrams, Liu [30] developed an effective algorithm to approximate the fractional-order transfer functions by utilizing frequency-domain techniques. Based on this, we deduced $1/s^q$ $(q = 0.1 \sim 0.99)$ with discrepancies 2dB and 3dB through linear approximation in frequency domain. Here, we utilize the approximation of $1/s^q$ with discrepancy 2dB to design the analog circuit.

According to [14], we can obtain

$$\frac{1}{s^{0.98}} = \frac{1.2974(s + 1125)}{(s + 1423)(s + 0.01125)}.$$

A circuit design where resistors and capacitors are connected in parallel was proposed in [15] to realize fractional calculus. When $q = 0.98$, the circuit unit is designed as shown in Figure 3, where $R_1 = 91.1873\,\mathrm{M\Omega}$, $R_2 = 190.933\,\Omega$, $C_1 = 975.32\,\mathrm{nF}$ and $C_2 = 3.6806\,\mu\mathrm{F}$.

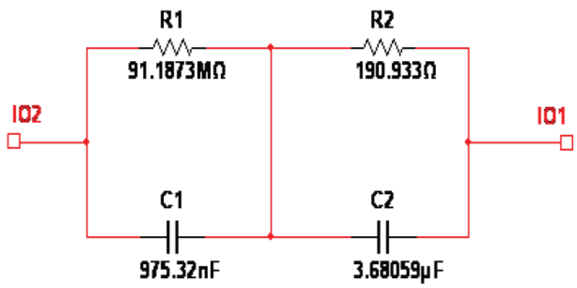

Figure 3. Circuit unit of $\dfrac{1}{s^{0.98}}$.

We select LF353P as the amplifier and AD633JN as the multiplier to design the synchronous circuit. In order to restrict the change of state variables to the operating voltage of the analog circuit, the state variables are reduced by 10 times, namely let $(x_1, x_1, x_3) \to (10x_1, 10x_2, 10x_3)$ and $(y_1, y_2, y_3) \to (10y_1, 10y_2, 10y_3)$. When $q = 0.98$, $a = 35$, $b = 3$, $c = 28$ and $d = -7$, the circuit schematics of the drive system (3) and the response system (4) are given, respectively, in Figures 4 and 5, where the values of resistors are indicated.

For the drive system, the circuit equation is as follows:

$$\begin{cases} \dfrac{d^{0.98}x_1}{dt^q} = -\dfrac{R_2R_5}{R_1R_3R_6}x_1 + \dfrac{R_2}{R_1R_4}x_2, \\[2ex] \dfrac{d^{0.98}x_2}{dt^q} = -\dfrac{R_8R_{11}}{R_7R_9R_{14}}x_1 + \dfrac{R_8}{R_7R_{10}}x_2 - \dfrac{R_8R_{11}}{R_7R_9R_{12}}(x_1x_3), \\[2ex] \dfrac{d^{0.98}x_3}{dt^q} = -\dfrac{R_{15}R_{18}}{R_{14}R_{16}R_{19}}x_3 + \dfrac{R_{15}}{R_{14}R_{17}}(x_1x_2). \end{cases} \tag{9}$$

The circuit equation of the response system is as follows:

$$\begin{cases} \dfrac{d^{0.98}y_1}{dt^q} = -\dfrac{R_{21}R_{24}}{R_{20}R_{22}R_{26}}\,y_1 - \dfrac{R_{21}R_{24}}{R_{20}R_{22}R_{25}}\,x_1 + \dfrac{R_{21}}{R_{20}R_{23}}\,x_2, \\ \dfrac{d^{0.98}y_2}{dt^q} = -\dfrac{R_{29}R_{32}}{R_{28}R_{30}R_{33}}\,y_2 - \dfrac{R_{29}R_{32}}{R_{28}R_{30}R_{34}}\,x_1 \\ \qquad\qquad + \dfrac{R_{29}}{R_{28}R_{31}}\,x_2 - \dfrac{R_{29}R_{32}}{R_{28}R_{30}R_{35}}\,(x_1x_3), \\ \dfrac{d^{0.98}y_3}{dt^q} = -\dfrac{R_{37}R_{40}}{R_{36}R_{38}R_{41}}\,y_3 + \dfrac{R_{37}}{R_{36}R_{39}}\,(x_1x_2). \end{cases} \tag{10}$$

The circuit simulation result of the (0.98, 0.98, 0.98)-order Chen system in Figure 4 is shown in Figure 6, we observed that the system is in chaotic state. We also observed in Figure 7, the synchronization of states variables of drive and response system which indicated that complete synchronization was achieved.

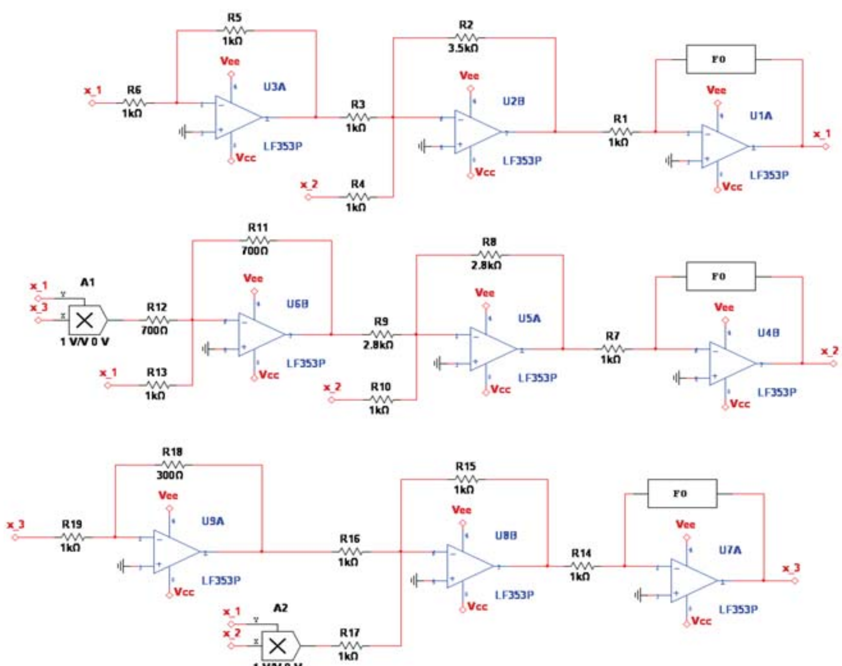

Figure 4. Circuit diagram of system (3).

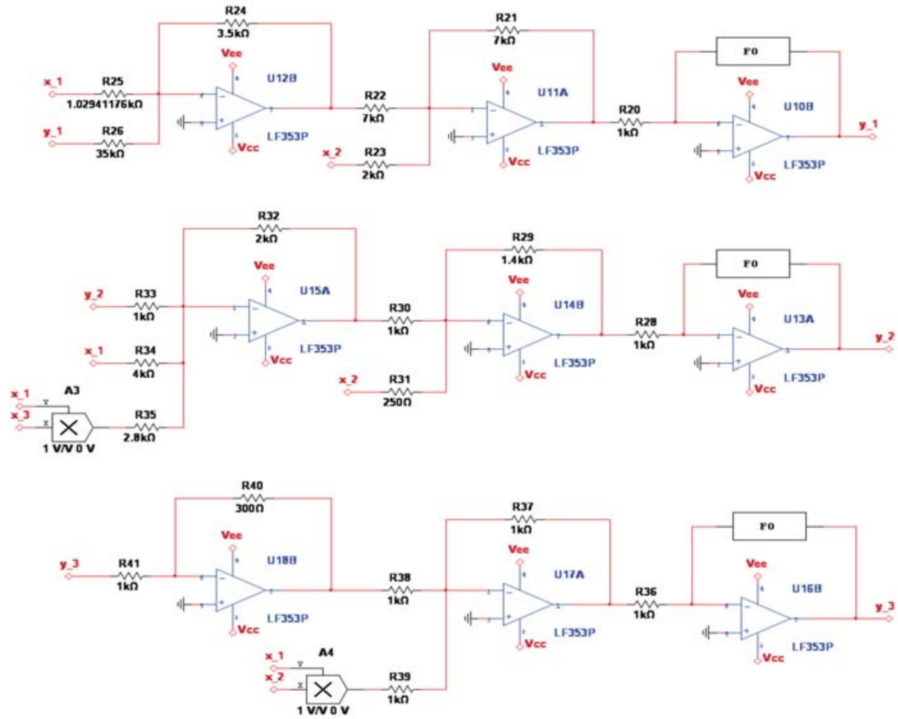

Figure 5. Circuit diagram of system (4).

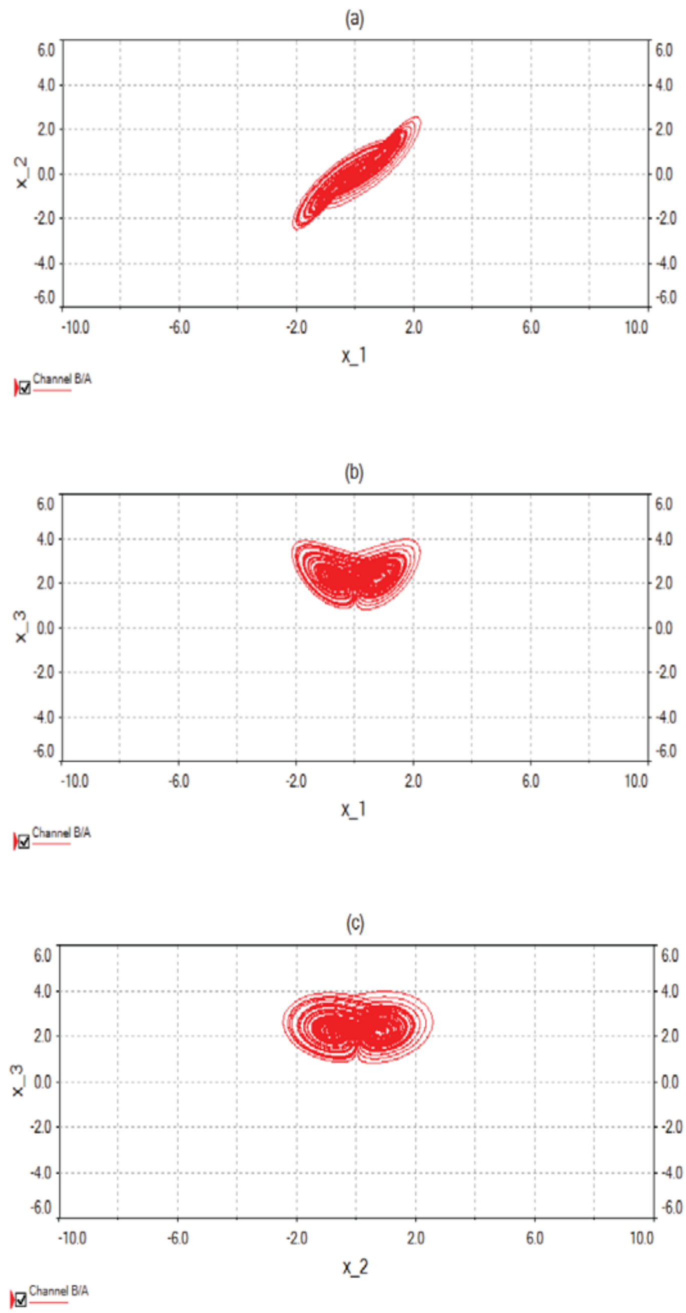

Figure 6. Circuit simulation of system (3): phase portraits.

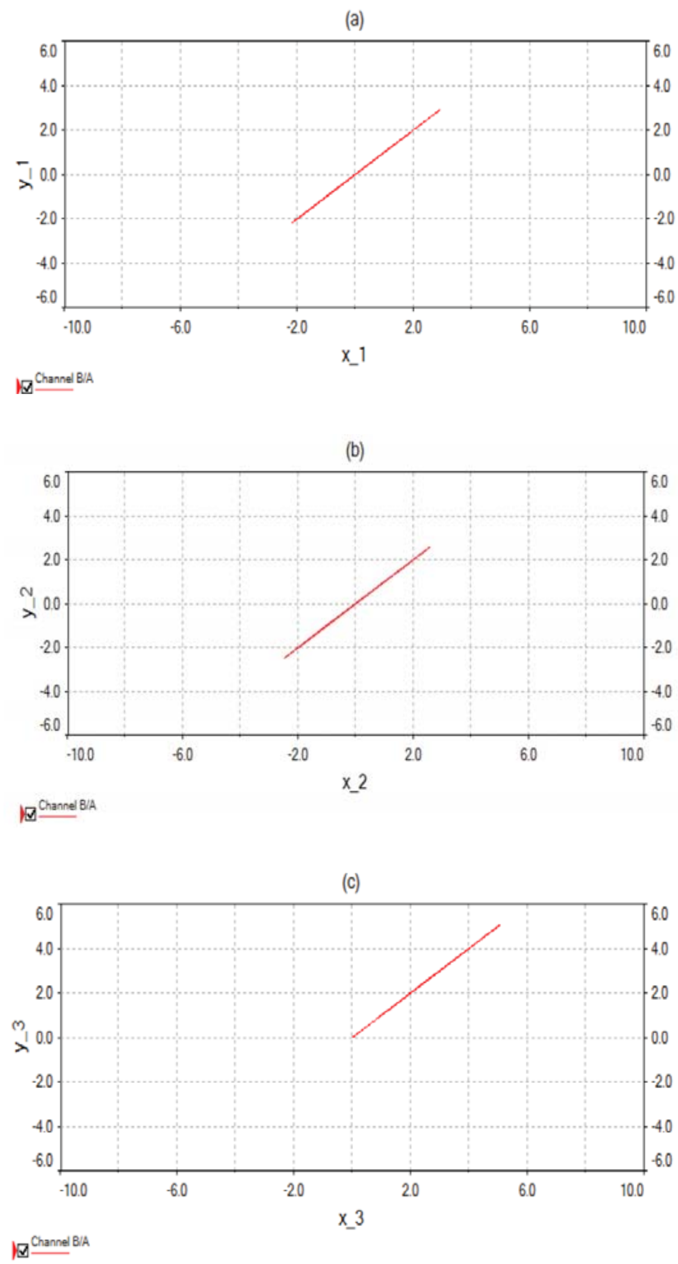

Figure 7. Circuit simulation of synchronization between system (3) and system (4).

6. Conclusion

In this paper, we investigated the synchronization of the fractional-order chaotic Chen system via predictor-corrector method. The results obtained show how the synchronization scheme is fast and simple. To verify numerical simulation, the electronic circuit of the systems has been designed and the results are in accordance with numerical investigation.

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