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WHEN THE RELATIVE INTEGRAL CLOSURE IS THE ONLY INTERMEDIATE RING

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Abstract

Let $A \subset B$ be (commutative) rings (with the same 1); let A^* denote the integral closure of *A* in *B*. Suppose that $A \subset A^*$ and $A^* \subset B$ are minimal ring extensions whose crucial maximal ideals are *M* and *N*, respectively. Then A^* is the only ring *C* such that $A \subset C \subset B$ if and only if $N \cap A = M$. This generalizes a recent for integral domains due to Ben Nasr and Zeidi [2]. We give examples with nontrivial zero-divisors to illustrate both possibilities (i.e., where $N \cap A$ may or may not be M).

1. Introduction

All rings considered in this note are commutative with identity; all subrings, inclusions of rings, and ring homomorphisms are unital. If $A \subseteq B$ is a ring extension, it is convenient to let $[A, B]$ denote the set of intermediate

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rings (that is, the set of rings C such that $A \subseteq C \subseteq B$). Recall from [1] that if $A \subseteq B$ is a ring extension, then $A \subseteq B$ is said to *satisfy FIP* if there are only finitely many rings contained between A and B (that is, if $|[A, B]| < \infty$). Whenever $A \subset B$ satisfies FIP, one has a finite (maximal) chain of rings $A = A_0 \subset \ldots \subset A_i \subset A_{i+1} \subset \ldots \subset A_n = B$ for some positive integer n, such that $A_i \subset A_{i+1}$ is a minimal ring extension for all $i = 0, ..., n-1$. (As usual, $\subset$ denotes proper inclusion. Some useful background on minimal ring extensions will be given in the next paragraph.) Not all such "compositions" of minimal ring extensions produce a ring extension $A \subset B$ that satisfies FIP. In [7], Dobbs and Shapiro focussed on the case $n = 2$. Indeed, if $A \subset C$ and $C \subset B$ are each minimal ring extensions [7, Theorem 4.1] gave 13 mutually exclusive conditions on these minimal ring extensions and their crucial maximal ideals to characterize when $A \subset B$ satisfies FIP. As $|[A, B]| \geq 3$ in general, much of the subsequent material in [7] began to examine the relationship between each of the 13 conditions from [7, Theorem 4.1] and the possible conclusion that $|[A, B]| = 3$ (that is, the possible conclusion that C is the only ring that is properly contained between A and B).

That study continued in [3, Theorem 2.9], where it was shown that there are two of the 13 conditions from [7, Theorem 4.1] which each implies that C is the only ring properly contained between A and B; there are seven (of the 13) conditions which each implies that C is not the only ring properly contained between A and B; and for each of the remaining four conditions, some examples satisfying the condition are such that C is the only ring properly contained between A and B while other examples satisfying the condition do not have this feature. The *main purpose of this note* is to characterize "$|[A, B]| = 3$" for one of those four "ambiguous" conditions, namely, the context where the minimal ring extension $A \subset C$ is integral and the minimal ring extension $C \subset B$ is integrally closed (in the sense that C is integrally closed in B). Equivalently, since integrality is transitive, one can describe this context as consisting of minimal ring extensions $A \subset C$ and $C \subset B$ where C is the integral closure of A in B. We will state the characterization of "$|[A, B]| = 3$" for this context after a brief paragraph of background information.

Recall (cf. [8]) that a ring extension $R \subset T$ is a *minimal ring extension* if there does not exist a ring properly contained between *R* and *T*. A minimal ring extension $R \subset T$ is either integral or integrally closed. If $R \subset T$ is a minimal ring extension, it follows from [8, Théorème 2.2(i) and Lemme 1.3] that there exists a unique maximal ideal *M* of *R* (called the *crucial maximal ideal* of $R \subset T$) such that the canonical injective ring homomorphism $R_M \to T_M$ $(:= T_{R \setminus M})$ can be viewed as a minimal ring extension while the canonical ring homomorphism $R_P \to T_P$ is an isomorphism for all prime ideals *P* of *R* except *M*.

For the context of interest, where *C* is the integral closure of *A* in *B* and $A \subset C$ and $C \subset B$ are minimal ring extensions having crucial maximal ideals *M* and *N*, respectively, Theorem 2.1 establishes that $|[A, B]| = 3$ if and only if $N \bigcap A = M$. In case *A* and *B* are (commutative integral) domains, this result was obtained recently by Ben Nasr and Zeidi [2, Corollary 2.11] as a consequence of the two main results in [2], namely, Theorems 2.1 and 2.7 of [2]. The proofs of those two theorems are extremely domain-theoretic in nature, as they involve several intersections of localizations which would be meaningless (for lack of a universe of discourse) in a more general ring-theoretic setting. On the other hand, our proof of the ring-theoretic generalization in Theorem 2.1 is comparatively short and involves no new technical results. To show that this is a meaningful generalization, Example 2.3 provides data showing that both of the possible conclusions (i.e., $|[A, B]| = 3$ and $|[A, B]| > 3$) can be realized by extensions involving rings that are not domains. For the sake of completeness, Remark 2.2 provides data that accomplish the same for ring extensions involving domains.

If *A* is a ring, then $\text{Spec}(A)$ (resp., $\text{Max}(A)$) denotes the set of prime (resp., maximal) ideals of *A*. For rings $A \subseteq B$, $\text{Supp}(B/A) := \text{Supp}_A(B/A)$ $:= \{P \in \text{Spec}(A) \mid A_P \subset B_P\}$. Following [11, p. 28], we let INC and GU, respectively, denote the incomparable and going-up properties of ring extensions. By the "dimension" of a ring, we mean its Krull dimension. As usual, if $\mathcal{S}$ is a set, then $|\mathcal{S}|$ denotes the cardinal number of $\mathcal{S}$. Any unexplained material is standard, as in [9, 11].

2. Results

Before giving our main result, we recall a definition and some related facts. A ring extension $R \subseteq T$ is said to satisfy FCP (also known as FC) if each chain of rings in $[R, T]$ is finite. While FIP $\Rightarrow$ FCP, the converse is false.

Theorem 2.1. *Let $A \subset B$ be rings, with A^* denoting the integral closure of A in B. Suppose that $A \subset A^*$ and $A^* \subset B$ are minimal ring extensions whose crucial maximal ideals are M and N, respectively. Then A^* is the only ring C such that $A \subset C \subset B$ if and only if $N \cap A = M$.*

Proof. By integrality, $A \subset A^*$ satisfies both GU and INC (cf. [11, Theorem 42]). It follows that $N \cap A \in \text{Max}(A)$, and so the condition that $N \cap A = M$ is equivalent to $N \cap A \subseteq M$. We will first prove the contrapositive of the "only if" assertion. Assume, then, that $N \cap A \neq M$; our task is to show that $[A, B] \setminus \{A, B, A^*\}$ is nonempty. By the above comment, $N \cap A \nsubseteq M$. Hence, by the Crosswise Exchange Lemma [5, Lemma 2.7], there exists $D \in [A, B]$ such that $A \subset D$ inherits from $A^* \subset B$ the property of being an integrally closed minimal ring extension. Thus, $D \in [A, B] \setminus \{A, B, A^*\}$ as desired.

Next, we will prove the contrapositive of the "if" assertion. Assume, then, that there exists a ring $E \in [A, B] \setminus \{A, B, A^*\}$; our task is to show that $N \cap A \neq M$. As mentioned in the introduction, the present context ensures that $A \subset B$ satisfies FIP. (This part of [7, Theorem 4.1] actually followed from the proof of [6, Proposition 2.1 (c)]. The main focus of [6] was on the FCP property.) In particular, $A \subset B$ satisfies FCP. Hence, so do $A \subset E$ and $E \subset B$. Since any decreasing chain in $[A, E]$ must terminate in finitely many steps, there exists $E_1 \in [A, E]$ such that $A \subset E_1$ is a minimal ring extension. It is straightforward to verify that $E_1 \in [A, B] \setminus \{A, B, A^*\}$; also, $E_1 \subset B$ satisfies FCP. Thus, it is harmless to change notation and take $E = E_1$; that is, to assume that $A \subset E$ is a minimal ring extension.

Let Q denote the crucial maximal ideal of $A \subset E$. Since $A^* \neq E$, we get $A^* \cap E = A$; that is, the extension $A \subset E$ is integrally closed. Hence, by [8, Théorème 2.2(ii)], no prime ideal of E can lie over Q, that is, Q is not in the image of the canonical map $\mathrm{Spec}(E) \to \mathrm{Spec}(A)$. Next, by considering increasing chains in $[E, B]$, we get (since $E \subset B$ satisfies FCP) a chain

$$E = A_0 \subset \ldots \subset A_i \subset A_{i+1} \subset \ldots \subset A_n = B$$

where n is a positive integer and $A_i \subset A_{i+1}$ is a minimal ring extension for all $i = 0, \ldots, n-1$. For each i, let Q_i denote the crucial maximal ideal of $A_i \subset A_{i+1}$. As no prime ideal of E can lie over Q, we have $Q_0 \cap A \neq Q$. Next, consider the finite maximal chain of minimal ring extensions

$$A \subset A_0 \subset \ldots \subset A_i \subset A_{i+1} \subset \ldots \subset A_n = B.$$

Applying [5, Corollary 3.2] to this chain, we get that

$$\mathcal{S} := \mathrm{Supp}_A(B/A) = \{Q\} \cup \{Q_i \cap A \mid i = 0, \ldots, n-1\}.$$

In particular, $\mathcal{S} \supseteq \{Q, Q_0 \cap A\}$, and so $|\mathcal{S}| \geq 2$. On the other hand, by applying [5, Corollary 3.2] to the chain $A \subset A^* \subset B$, we get that $\mathcal{S} = \{M, N \cap A\}$. Therefore, $|\{M, N \cap A\}| \geq 2$, whence $M \neq N \cap A$. □

We pause to collect some domain-theoretic data realizing the possible cases in Theorem 2.1, namely, where $N \cap A$ is or is not M.

Remark 2.2. (a) It was shown in [3, Theorem 2.4] (with nearly all the relevant work being done in [3, Lemma 2.3]) that if $A \subset C$ is an integral minimal ring extension and $C \subset B$ is an integrally closed minimal ring extension (so that C is necessarily the integral closure of A in B) and if A is quasi-local, then $|[A, B]| = 3;$ that is, C is the only ring H such that $A \subset H \subset B$. Therefore, by Theorem 2.1, $N \cap A = M$, where M and N denote the crucial maximal ideals of $A \subset C$ and $C \subset B$, respectively. (This equality is also clear directly since A is assumed quasi-local and integrality ensures that $N \cap A \in \text{Max}(A)$.) One way to build such data is to take A to be a (necessarily quasi-local one-dimensional) domain, with quotient field B, whose integral closure (in B) is a one-dimensional valuation domain C such that $A \subset C$ is a minimal ring extension. (One example of such data is found by using $A := \mathbb{R} + X\mathbb{C}[[X]]$, where X is an analytic indeterminate over $\mathbb{C}$; the integral closure of A is $C = \mathbb{C} + X\mathbb{C}[[X]] = \mathbb{C}[[X]]$. This ring will also play an auxiliary role in Example 2.3.) To complete the verification, it remains only to show that $C \subset B$ is an integrally closed minimal ring extension. This, in turn, is standard: cf. [11, Theorem 65], [9, Theorem 26.1 (2)].

(b) If $A \subset C$ is an integral minimal ring extension and $C \subset B$ is an integrally closed minimal ring extension (so that C is necessarily the integral closure of A in B), then it need not be the case that $|[A, B]| = 3$. An example illustrating this was essentially given in [7, Remark 4.2 (d)]. This involves taking $A := \mathbb{Z}[2i]$, $C := \mathbb{Z}[i]$ (the ring of Gaussian integers), and $B := \bigcap_{P \neq Q} C_P$, where the index set for this intersection consists of all the prime ideals P of C other than $Q := 3C$. It is well known that $A \subset C$ is an integral minimal ring extension. Hence, by [8, Théorème 2.2(ii)], its crucial maximal ideal is $M := (A : C) = (2, 2i)C = 2\mathbb{Z} + 2\mathbb{Z}i$. It was shown in [7, Remark 4.2(d)] that $C \subset B$ is an integrally closed minimal ring extension. Since every prime ideal of C except Q is lain over from B, it follows from [8, Théorème 2.2(ii)] that $N := Q$ is the crucial maximal ideal of $C \subset B$. Note that $N \cap A = 3\mathbb{Z} + 6\mathbb{Z}i \neq M$. So, by Theorem 2.1, $[A, B] \setminus \{A, B, C\}$ is nonempty. In fact, it was shown in [7, Remark 4.2(d)] that $A[1/3] \in [A, B] \setminus$

$\{A, B, C\}$. Thus, C is not the only ring H such that $A \subset H \subset B$. This completes the verification.

(c) Recall that the domain-theoretic case of Theorem 2.1 was given earlier by Ben Nasr and Zeidi in [2, Corollary 2.11]. To illustrate that result, they gave, in [2, Example 2.12], an example of a one-dimensional quasi-local domain (A, M) and a one-dimensional valuation overring B of A such that $A \subset A^*$ and $A^* \subset B$ are minimal ring extensions, each of which has crucial maximal ideal M. Two significant ways in which that example differs from our construction in (a) are the following: the ring A^* (resp., B) in [2, Example 2.12] is not quasi-local (resp., is not a field). In any event, one can fairly conclude that the main point of (a) was anticipated in [2, Example 2.12]. However, the same cannot be said of the point made in (b). Indeed, [2] did not address the possible existence of data that would fit the context of [2, Corollary 2.11] but fail to satisfy the equivalent conditions in that result. As explained in (a), results from [3] show that any such data (for instance, the data in (b)) must feature a base ring that is not quasi-local.

We close with examples showing that rings that have non-trivial zero-divisors can exhibit the same diversity of behavior as in parts (a) and (b) of Remark 2.2. Recall from [10] that a (necessarily quasi-local) domain D is said to be a *pseudo-valuation domain* if there is a (uniquely determined) valuation overring V of D (inside the quotient field of D) that has the same maximal ideal as D; V is referred to as the canonically associated valuation overring of D.

Example 2.3. Let (D, m) be a one-dimensional pseudo-valuation domain with quotient field K such that the integral closure of D (in K) is the canonically associated valuation overring V of D and also such that $D \subset V$ is a minimal ring extension. (For instance, take $D = \mathbb{R} + X\mathbb{C}[[X]]$, where X is an analytic indeterminate over $\mathbb{C}$.) Then:

(a) Let E be any nonzero ring. Put $A := D \times E$ and $B := K \times E$. Then the integral closure of A in B is $A^* = V \times E$, $A \subset A^*$ is an integral minimal ring extension whose crucial maximal ideal is $M := m \times E$, $A^* \subset B$ is an integrally closed minimal ring extension whose crucial maximal ideal is $N := m \times E$ $(= M)$, and A^* is the only ring C such that $A \subset C \subset B$.

(b) Put $A := D \times V$ and $B := V \times K$. Then the integral closure of A in B is $A^* = V \times V$, $A \subset A^*$ is an integral minimal ring extension whose crucial maximal ideal is $M := m \times V$, $A^* \subset B$ is an integrally closed minimal ring extension whose crucial maximal ideal is $N := V \times m$, and A^* is not the only ring C such that $A \subset C \subset B$. Indeed, the only such C other than A^* is $D \times K$.

Proof. It is well known that $\operatorname{Spec}(D) = \operatorname{Spec}(V)$ as sets. In particular, $\dim(V) = \dim(D) = 1$. (The latter conclusion also follows via integrality, as in [11, Theorem 48].) In addition, $V \subset K$ is an integrally closed minimal ring extension, necessarily with crucial maximal ideal m (cf. [11, Theorem 65], [9, Theorem 26.1 (2)]).

(a) The hypothesis that $E \neq 0$ has been made only to ensure that A, B and A^* are non-domains. It is straightforward to show that the integral closure of A in B is $A^* := V \times E$. Hence, A^* is integrally closed in B. Since the assignment $H \mapsto H \times E$ gives a bijection $[D, V] \to [A, A^*]$ and $D \subset V$ is a minimal ring extension, we now have that $A \subset A^*$ is an integral minimal ring extension. By [8, Théorème 2.2(ii)], $m = (D : V)$, the crucial maximal ideal of $D \subset V$. Thus, the crucial maximal ideal of $A \subset A^*$ is

$$(A : A^*) = (D \times E : V \times E) = (D : V) \times E = m \times E =: M.$$

Since the assignment $H \mapsto H \times E$ gives a bijection $[V, K] \to [A^*, B]$ and $V \subset K$ is a minimal ring extension, it now follows that $A^* \subset B$ is an integrally closed minimal ring extension. By [8, Théorème 2.2(ii)], the crucial maximal ideal of this extension is the only maximal ideal of A^* which is not lain over from B, namely, $m \times E =: N(= M)$. Of course, $N \cap A = M$, and so by Theorem 2.1, A^* is the only ring C such that $A \subset C \subset B$. A direct proof of the last assertion is also available, since $[A, B] = [D, K] \times \{E\}$.

(b) Insofar as possible, we will argue as in (a). It is straightforward to show that the integral closure of A in B is $A^* := V \times V$. Hence, A^* is integrally closed in B. To show that $A \subset A^*$ is a(n integral) minimal ring extension with crucial maximal ideal $M := m \times V$, one need only observe that $(A : A^*) = (D : V) \times V = M$. (Here is another way to show that the integral extension $A \subset A^*$ is minimal, with crucial maximal ideal M. Note that $A/M \cong D/m$ and $A^*/M \cong V/m$. Hence by [4, Proposition II.4] (cf. also [12, Theorem 3.3]), the minimality of $D \subset V$ implies that of $D/m \subset V/m$, hence that of $A/M \subset A^*/M$, hence that of $A \subset A^*$; the cited references can also be used to show that M is the crucial maximal ideal of $A \subset A^*$. Of course, this alternate reasoning could also have been used at the corresponding point in the proof of (a).) Next, to show that $A^* \subset B$ is a(n integrally closed) minimal ring extension with crucial maximal ideal $N := V \times m$, note that N is the only maximal ideal of A^* which is not lain over from B. Finally, since

$$N \cap A = (V \cap D) \times (m \cap V) = D \times m \neq m \times V = M,$$

Theorem 2.1 implies that A^* is not the only ring C such that $A \subset C \subset B$. In fact, the data have been arranged so that $[A, B] \setminus \{A, A^*, B\}$ contains only one element, namely, $D \times K$. □

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