**Far East Journal of Mathematical Sciences (FJMS)**

Published: December 2016
http://dx.doi.org/10.17654/MS100122029
Volume 100, Number 12, 2016, Pages 2029-2040 **ISSN: 0972-0871**

# WATER INFILTRATION FROM PERIODIC TRAPEZOIDAL CHANNELS WITH DIFFERENT TYPES OF ROOT-WATER UPTAKE

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## Abstract

In this paper, a problem involving steady water infiltration flow in a homogeneous soil with water absorption by plant root is considered. The problem involves infiltration from periodic trapezoidal channels. Four different root distributions are incorporated into the problem. The governing equation of the problem is the Richards equation, which can be studied more conveniently by transforming the equation into a modified Helmholtz equation. A dual reciprocity boundary element method (DRBEM) and a predictor-corrector scheme are employed simultaneously to solve the modified Helmholtz equation, subject to a set of boundary conditions, numerically. Using the solutions obtained, numerical values of the suction potential are then computed.

Received: August 12, 2016; Revised: October 13, 2016; Accepted: October 19, 2016

2010 Mathematics Subject Classification: 76M15.

Keywords and phrases: dual reciprocity boundary element method, infiltration, root-water uptake, predictor-corrector, modified Helmholtz equation.

Communicated by Inna K. Shingareva; Editor: Advances and Applications in Fluid Mechanics, Published by Pushpa Publishing House, Allahabad, India

## 1. Introduction

Dual reciprocity boundary element methods (DRBEMs) have been applied to solve various problems numerically. Ang and Ang employed a DRBEM to solve a generalized nonlinear Schrödinger equation [2]. A Laplace transform DRBEM was applied to obtain numerical solutions of two-dimensional microscale thermal problems [1]. Brebbia and Nardini solved problems involving solid mechanics using a DRBEM [5]. In this paper, a DRBEM approach with a predictor-corrector scheme is applied simultaneously to solve a problem involving steady infiltration from periodic trapezoidal channels with different root distributions.

Steady water infiltration problems from periodic irrigation channels in a homogeneous soil have been examined by Solekhudin and Ang [10, 11]. Four different geometries of irrigation channels were considered [11]. In that study, water absorption by plant roots was not considered. Solekhudin and Ang also studied a steady infiltration problem from periodic trapezoidal channels with root-water uptake [10]. Effect of water absorption by plant roots on water content in the soil was examined. The results in these studies gave useful insight into the steady state distribution of water content in the soil. However, water content or moisture content from the start of infiltration process was not provided in the results.

Time-dependent water infiltration problems from irrigation channels in a homogeneous soil have been studied by a number of researchers, such as Clements and Lobo [7] and Solekhudin and Ang [13]. While Clements and Lobo studied infiltration from a single channel, Solekhudin and Ang investigated infiltration from periodic channels with water absorption by plant roots. However, in their study, Solekhudin and Ang considered only a distribution of plant roots.

In this paper, we consider a problem involving steady infiltration from periodic trapezoidal channels with four different types of root-water uptake, which is a continuation of the study conducted by Solekhudin and Ang [10]. In their study, Solekhudin and Ang considered only a type of root-water uptake. In order to solve the problem, the governing equation, which is a Richards equation, is transformed into a modified Helmholtz equation. The modified Helmholtz equation is then solved numerically using a DRBEM with a predictor-corrector scheme simultaneously. Using the numerical solutions obtained, numerical values of suction potential are computed.

## 2. Problem Formulation

Study in this paper is a continuation of what was discussed in Solekhudin and Ang [10]. Hence, the problem formulation is very similar. Referred to a Cartesian coordinate *OXYZ* with *OZ* positively downward, consider a homogeneous soil, Pima clay loam (PCL). On the surface of soil, periodic trapezoidal irrigation channels are constructed. The cross-sectional perimeter of the channels is $2L$. The channels are completely filled with water. It is assumed that the channels are sufficiently long, and there are a large number of such channels. Between two channels, a row of crops, with roots of depth $Z_m$ and width $2X_m$, are planted. The distance between two adjacent rows of plants is $2(L + D)$. It is assumed that the geometry of the channels and root zone do not vary in the *OY* direction and are symmetric about the planes $X = \pm k(L + D)$, for $k = 0, 1, 2, \ldots$. These descriptions are shown in Figure 1.

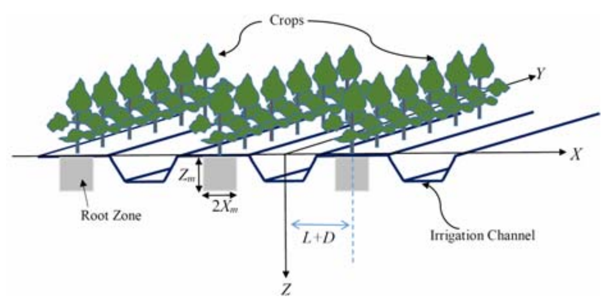



**Figure 1.** Periodic trapezoidal channels with crops.

Due to the symmetry of the problem, it is sufficient to consider a semi infinite region defined by $0 \le X \le L + D$ and $Z \ge 0$. This region is denoted by *R* bounded by $C = C_1 \cup C_2 \cup C_3 \cup C_4$ as shown in Figure 2. It is assumed that fluxes over the surface of channels are constant, that is, $v_0$. There are no fluxes across the surface of soil outside the channels. The fluxes over $X = 0$ and $X = L + D$ are also 0, as the problem symmetric about them. Following Batu in [4], the derivatives $\partial\Theta/\partial X \to 0$ and $\partial\Theta/\partial Z \to 0$ as $X^2 + Z^2 \to \infty$, where $\Theta$ is the matric flux potential (MFP).

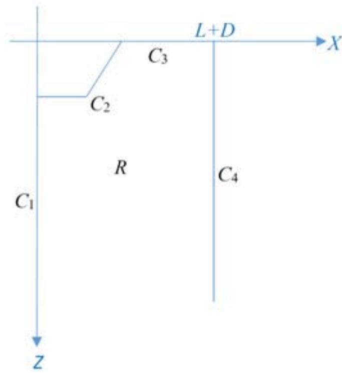



**Figure 2.** Region $R$ with boundary $C$.

In this paper, we consider four different types of root-water uptake, denoted by Root A, Root B, Root C and Root D. These types of roots are as reported by Vrugt et al. [15]. The problem in this study is to determine suction potentials in the soil for infiltration from periodic irrigation channels with different types of root-water uptake. In addition, a comparison between the suction potentials from different root-uptake models is presented.

## 3. Basic Equations

In this section, governing equations of the problem are presented, as well as a method of solution, which are as discussed in [10]. However, for completeness and for the convenience of readers, some details of the equations and the method are reproduced and presented in this section. Steady infiltration in a homogeneous soil is governed by Richards equation of the form [10]

$$\frac{\partial}{\partial X}\left(K\frac{\partial\psi}{\partial X}\right) + \frac{\partial}{\partial Z}\left(K\frac{\partial\psi}{\partial Z}\right) - \frac{\partial K}{\partial Z} = S(X, Z, \psi(X, Z)), \tag{1}$$

where $K$ is the hydraulic conductivity, $\psi(X, Z)$ is the suction potential, and $S$ is the root-water uptake function. Here, $S$ is adopted from the model proposed by Vrugt et al. [15], that is,

$$S(X, Z, \psi) = \gamma(\psi)\frac{L_t\beta(X, Z)T_{pot}}{\int_0^{Z_m}\int_{L+D-X_m}^{L+D}\beta(X, Z)dXdZ}, \tag{2}$$

where $L_t$ is the width of the soil surface associated with transpiration process, $\beta$ is the spatial root-water uptake distribution, $T_{pot}$ is the transpiration potential, and $\gamma$ is the root-water stress response function reported by Utset et al. [14]. In this study, $\beta$ is formulated as

$$\beta(X, Z) = \left(1 - \frac{L + D - X}{X_m}\right)\left(1 - \frac{Z}{Z_m}\right)e^{-\left(\frac{p_Z}{Z_m}| Z^* - Z | + \frac{p_X}{X_m}| X^* - (L+D-X) |\right)},$$

$$\text{for } L + D - X_m \le X \le L + D,\ 0 \le Z \le Z_m,$$

where $p_X$, $p_Z$, $X^*$ and $Z^*$ are fitting parameters.

Using the Kirchhoff transformation

$$\Theta = \int_{-\infty}^{\psi} K(t)dt, \tag{3}$$

where $\Theta$ is the matric flux potential (MFP), an exponential relationship between $K$ and $\psi$,

$$K = K_s e^{\alpha\psi}, \tag{4}$$

where $K_s$ is the saturated hydraulic conductivity, the suction potential can be formulated as

$$\psi = \frac{1}{\alpha}\ln\left(\frac{\alpha\Theta}{K_s}\right). \tag{5}$$

Using equations (3), (4) and substituting dimensionless variables

$$x = \frac{\alpha}{2}X;\ z = \frac{\alpha}{2}Z;\ \Phi = \frac{\pi\Theta}{v_0 L};\ u = \frac{2\pi}{v_0\alpha L}U;\ v = \frac{2\pi}{v_0\alpha L}V;\ f = \frac{2\pi}{v_0\alpha L}F, \quad (6)$$

into equation (1) yield

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial z^2} - 2\frac{\partial\Phi}{\partial z} = \gamma^*(\Phi)s^*(x, z), \quad (7)$$

where

$$s^*(x, z) = \frac{2\pi}{\alpha L}\frac{l_t\beta^*(x, z)}{\int_0^{z_m}\int_{b-x_m}^{b}\beta^*(x, z)dxdz}\frac{T_{pot}}{v_0}, \quad (8)$$

$$\gamma^*(\Phi) = \gamma\left(\frac{1}{\alpha}\ln\left(\frac{\alpha v_0 L\Phi}{\pi K_s}\right)\right),$$

and

$$\beta^*(x, z) = \left(1 - \frac{b-x}{x_m}\right)\left(1 - \frac{z}{z_m}\right)e^{-\left(\frac{p_z}{z_m}\left|\frac{2z^*}{\alpha} - \frac{2z}{\alpha}\right| + \frac{p_x}{x_m}\left|\frac{2x^*}{\alpha} - \frac{2(b-x)}{\alpha}\right|\right)}. \quad (9)$$

Here $l_t = \frac{\alpha}{2}L_t$, $x^* = \frac{\alpha}{2}X^*$, $z^* = \frac{\alpha}{2}Z^*$, $x_m = \frac{\alpha}{2}X_m$, $z_m = \frac{\alpha}{2}Z_m$, $p_x = \frac{\alpha}{2}p_X$, $p_z = \frac{\alpha}{2}p_Z$ and $b = \frac{\alpha}{2}(L + D)$.

Applying transformation

$$\Phi = \phi e^z \quad (10)$$

into equation (7) yields

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} - \phi = \gamma^*(\phi)s^*(x, z)e^{-z}. \quad (11)$$

Equation (11) is subject to boundary conditions

$$\frac{\partial \phi}{\partial n} = \frac{2\pi}{\alpha L} e^{-z} - n_2 \phi \text{ on the surface of the channel,} \tag{12}$$

$$\frac{\partial \phi}{\partial n} = -\phi \text{ on the surface of soil outside the channel,} \tag{13}$$

$$\frac{\partial \phi}{\partial n} = 0, \; x = 0 \text{ and } z \geq 0, \tag{14}$$

$$\frac{\partial \phi}{\partial n} = 0, \; x = b \text{ and } z \geq 0, \tag{15}$$

$$\frac{\partial \phi}{\partial n} = -\phi, \; 0 \leq x \leq b \text{ and } z = \infty. \tag{16}$$

The term $n_2$ in boundary condition (12) is the vertical component of normal vector pointing out region $R$.

Following Solekhudin [12], an integral equation for solving equation (11) is

$$\begin{aligned} &\lambda(\xi, \eta)\phi(\xi, \eta) \\ &= \iint_R \varphi(x, z; \xi, \eta)[\phi(x, z) + \gamma^*(\phi)s^*(x, z)e^{-z}]dxdz \\ &+ \int_C \left[\phi(x, z)\frac{\partial}{\partial n}(\varphi(x, z; \xi, \eta)) - \varphi(x, z; \xi, \eta)\frac{\partial}{\partial n}(\phi(x, z))\right] ds(x, z), \end{aligned} \tag{17}$$

where $\varphi(x, z; \xi, \eta) = \frac{1}{4\pi}\ln[(x-\xi)^2 + (y-\eta)^2]$ is the fundamental solution of Laplace equation, and

$$\lambda(\xi, \eta) = \begin{cases} 1/2, & \text{for } (\xi, \eta) \text{ on smooth part of } C, \\ 1, & \text{for } (\xi, \eta) \in R. \end{cases}$$

Equation (17) may be solved using a DRBEM and a predictor-corrector simultaneously. Readers may refer to [10] for the detail of the method.

## 4. Results and Discussion

In this section, some numerical results for steady suction potential associated with infiltration from periodic trapezoidal channels with root-water uptake in a homogeneous soil are presented. The homogeneous soil considered in this study is Pima clay loam (PCL). For PCL, the values of $\alpha$ and $K_s$ are $0.014\text{cm}^{-1}$ and 9.9cm/day, respectively. These values are as those reported in [3] and [6]. We set $L = D = 50\text{cm}$, and the width and the depth of the channels are $4L/\pi$ and $3L/2\pi$, respectively. The potential transpiration rate, $T_{pot}$, is 4mm/day, which was also used by Li et al. [8] and Šimunek and Hopmans [9] in their studies.

Four different types of root-water uptake models are considered, namely Root A, Root B, Root C and Root D. Parameters of the root-water uptake models are adopted from [15], which are summarized in Table 1.

**Table 1.** Parameter values for four different root-water uptakes

| | Fitting parameters | | | | | |
|---|---|---|---|---|---|---|
| Root type | $Z_m$ | $X_m$ | $z^*$ | $x^*$ | $p_Z$ | $p_X$ |
| Root A | 100cm | 50cm | 0cm | 0cm | 1.0 | 1.0 |
| Root B | 100cm | 50cm | 20cm | 0cm | 1.0 | 1.0 |
| Root C | 100cm | 50cm | 0cm | 25cm | 1.0 | 4.0 |
| Root D | 100cm | 50cm | 20cm | 25cm | 5.0 | 2.0 |

The method, the DRBEM with a predictor-corrector scheme, is employed to obtain numerical results. To employ the method, boundary is discretized into 203 elements, and 619 interior collocation points are chosen. Some of the results are presented in Table 2 and Figure 3.

Table 2 shows numerical values of the dimensionless MFP, $\Phi$, in the soil with four different root-water uptake models at selected points. It can be seen, that the values of $\Phi$ at $x = 0.3$ are higher than those at $x = 0.5$, for the same level of $z$. This means that water content at $x = 0.3$ is higher than those at $x = 0.5$. This implies that water content at area near the channels is higher than those further. It can also be seen that values of $\Phi$ in the soil with Root A and Root B are higher than those in the soil with Root C and Root D. This means that water content in the soil with Root A and Root B is higher than those in the soil with Root C and Root D. This implies that Root C and Root D absorbed more water than Root A and Root B.

**Table 2.** Values of dimensionless MFP, Φ, at selected points

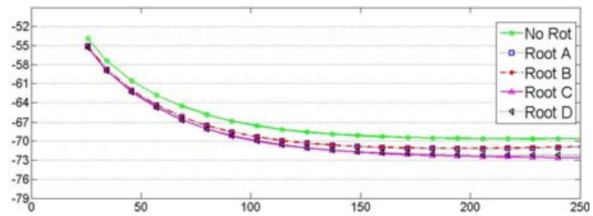
| | Dimensionless MFP | | | |
|---|---|---|---|---|
| Point | Root A | Root B | Root C | Root D |
| (0.3, 0.1) | 2.17060577 | 2.17297878 | 2.16334646 | 2.16032566 |
| (0.3, 0.5) | 2.23193525 | 2.23343304 | 2.21722247 | 2.21691233 |
| (0.3, 1.0) | 2.20149406 | 2.20277101 | 2.17621675 | 2.17972661 |
| (0.3, 1.5) | 2.20632116 | 2.20853770 | 2.16623013 | 2.17517351 |
| (0.3, 2.0) | 2.22473855 | 2.22913051 | 2.16061783 | 2.17887787 |
| (0.5, 0.1) | 1.55175600 | 1.55338900 | 1.53578563 | 1.53094377 |
| (0.5, 0.5) | 2.03307073 | 2.03380112 | 2.01690967 | 2.01726916 |
| (0.5, 1.0) | 2.16760522 | 2.16883399 | 2.14232086 | 2.14599560 |
| (0.5, 1.5) | 2.20081228 | 2.20302964 | 2.16050158 | 2.16956313 |
| (0.5, 2.0) | 2.22432137 | 2.22863232 | 2.15945537 | 2.17789928 |

Figure 3 shows values of the suction potential, ψ, along the *Z* axis at selected values of *X*. The values of *X* are 25cm, 50cm, 75cm and 90cm. It can be seen that there are decreases in the values of ψ due to water absorption by the plant roots. The magnitude of decrease reaches a threshold value at a certain value of *Z*, after which the decrease is constant. These results indicate that water content in the soil decline due to water absorption by the plant roots. The decreases in ψ depend on the type of root-water uptake model, ranged from 1 to 3. The higher decrease in ψ occurs when Root C or Root D is incorporated into the infiltration problem. Root A and Root B result in smaller decrease in ψ.

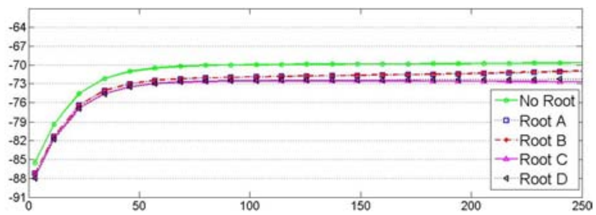
Values of suction potential in the soil without root-water uptake converge to about –70, while the values of ψ in the soil with root-water uptakes converge to values between –71 to –73. For $X = 50\text{cm}$, $X = 75\text{cm}$, and $X = 90\text{cm}$, it is observed that at smaller values of *X*, the values of ψ converge more rapidly than those at higher values of *X*. This result indicates that for the location nearer the channels, maximum water content is achieved at shallower level than those further.

Values of ψ decrease as *Z* increases for $X = 25\text{cm}$, while for $X = 50\text{cm}$, $X = 75\text{cm}$ and $X = 95\text{cm}$, there are increases in values of ψ as

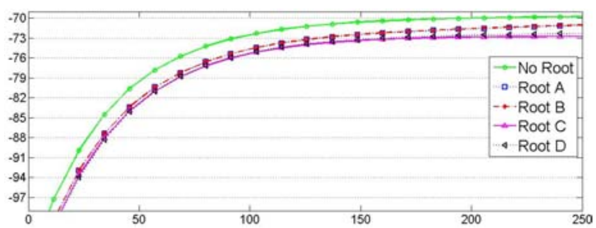
$Z$ increases. This result shows that in the area under the irrigation channels, for instance at $X = 25\text{cm}$, water content at shallower level of soil is higher than those at deeper level. However, at shallower level of soil further from the channels, water contents are smaller than those at deeper level. These occur due to the assumption that the source of infiltration is only the irrigation channels.



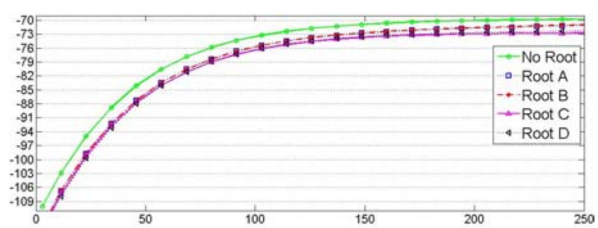

(a) $X = 25\text{cm}$




(b) $X = 50\text{cm}$




(c) $X = 75\text{cm}$




(d) $X = 90\text{cm}$

**Figure 3.** Suction potential at selected $X$ along $Z$ axis.

## 5. Concluding Remark

A problem involving steady infiltration from periodic trapezoidal channels with four different types of root uptake has been solved by applying a set of transformations and a DRBEM with a predictor-corrector scheme. The method is applied to obtain numerical values of dimensionless MFP and suction potential.

The results obtained indicate the difference in the amount of water absorbed by the plant roots. Root C and Root D absorb more water than Root A and Root B. The results also indicate that maximum water content at location near the channel attained at shallower level of soil than those further. Studies on the water contents and water absorbed by plant roots at top level of soil should be explored in the future research by including the effect of time of infiltration.

## Acknowledgment

The author thanks the anonymous referees for their valuable suggestions which led to the improvement of the manuscript.

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