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PROPERTIES OF WEIBULL MODELS

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Abstract

In this paper, we analyze the hazard rate and reversed hazard rate of some well-known Weibull models, which are widely used in reliability analysis. The comparison of reversed hazard rate with hazard rate, and aging intensity function is done with the help of numerical examples.

1. Introduction

In the context of reliability theory, some well-known functions are available, viz., survival function, hazard rate function, reversed hazard rate function, mean residual life function to study lifetime distributions or statistical data. The notations used throughout the paper are mentioned in the

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sequel. We denote a continuous lifetime random variable by X with probability density function $f_X(\cdot)$, cumulative distribution function $F_X(\cdot)$, survival function $\overline{F}_X(\cdot)$, hazard rate function $r_X(\cdot) = f_X(\cdot)/\overline{F}_X(\cdot)$, and reversed hazard rate function $\mu_X(\cdot) = f_X(\cdot)/F_X(\cdot)$. The hazard rate $r_X(t)$ is widely used in aging analysis of a device, whereas the importance of $\mu_X(t)$ is found in the Forensic Science, where exact time of failure (i.e., death in case of living beings) of a system is of importance (cf. Gupta and Arnold [6], Navarro et al. [15] and Marshall and Olkin [11]). $r_X(t)$ and $\overline{F}_X(t)$ are related by the well known relationship

$$\overline{F}_X(t) = e^{-\int_0^t r_X(u)\,du}. \tag{1.1}$$

Recently, the role of aging intensity function (AI) in analyzing aging phenomenon quantitatively is significantly discussed by Jiang et al. [9], Nanda et al. [13] and Bhattacharjee et al. [1]. AI function $L_X(\cdot)$ (cf. Jiang et al. [9]) is defined as $L_X(t) = r_X(t)/H_X(t)$ with

$$H_X(t) = \frac{1}{t}\left(\int_0^t r_X(u)\,du\right).$$

It can be seen that

$$L_X(t) = \frac{-t f_X(t)}{\overline{F}_X(t)\ln \overline{F}_X(t)} \text{ for } t > 0.$$

Pham and Lai [16] established that (1.1) facilitates to generate the Weibull-type lifetime distributions. Keeping in view on traditional Weibull distribution function, a good number of authors have proposed new statistical distributions. In view of the aforementioned fact, it is quite important to mention that Nadarajah and Kotz [12] recently pointed out that the proposed distributions that are published in reliability engineering journals, are either not new or arise from a representation suggested by Gurvich et al. [7]. According to them, Gurvich et al.'s [7] work needs to be recognized by the readers of reliability journals as they were the beginners to present a class of distributions generalizing the traditional Weibull distribution.

In Section 2, we derive the reversed hazard rate of some well-known Weibull models, which are widely used in reliability analysis. Here, we analyze some system properties having component lives from Weibull family. We explore the properties of series and parallel systems consisting of independent, and non-identically distributed random variables, viz., exponential, Weibull family. The comparison of reversed hazard rate with hazard rate, and aging intensity function is done with the help of numerical examples in Section 3. The paper ends with concluding remarks in Section 4.

2. Recent Weibull Models

Some of the families of recent Weibull models have been highlighted in Nadarajah and Kotz [12] and Pham and Lai [16]. The following notations will be used throughout the paper:

Survival function	Notation used
$\overline{F}_X(t) = \exp(-at^b),\ a, b > 0,\ t \geq 0$	: $X \sim W_2(a, b)$ (Weibull [18])
$\overline{F}_X(t) = \exp(-at^b e^{\lambda t}),\ a > 0,\ b > 0,\ \lambda \geq 0,\ t \geq 0$	: $X \sim W_3(a, b, \lambda)$ (cf. Lai et al. [10])
$\overline{F}_X(t) = 1 - \exp\left\{-\left(\frac{\beta}{t}\right)^{\alpha}\right\},\ \alpha, \beta > 0,\ t \geq 0$	: $X \sim W_I(a, b)$ (cf. Jiang and Murthy [8])

*As reported in Pham and Lai [16]

Below we state quite a few interesting results with proofs being not so important, thereby omitting the proofs of some results. These results have wide applications in reliability and survival analysis. The notations used in this section are same as what is defined earlier.

Theorem 2.1. *If* $X \sim W_2(\alpha, \beta)$, *then its reversed hazard rate is a decreasing function of t.*

Theorem 2.2. *If* $X \sim W_3(a, \beta, b)$, *then the reversed hazard rate is a decreasing function of t.*

Proof. Here

$$f_X(t) = at^{\beta-1}(bt+\beta)e^{bt - a\exp(bt)t^{\beta}},$$

$$\mu_X(t) = \frac{ae^{bt}t^{\beta-1}(bt+\beta)}{e^{ae^{bt}t^{\beta}} - 1}, \tag{2.1}$$

so that

$$\frac{d}{dt}(\mu_X(t)) = \frac{W(t)}{(e^{ae^{bt}t^{\beta}} - 1)^2}, \tag{2.2}$$

where

$$\begin{aligned} W(t) &= ae^{bt}t^{\beta-2}\{-ae^{bt+ae^{bt}t^{\beta}}t^{\beta}(bt+\beta)^2 + (-1+e^{ae^{bt}t^{\beta}})\{-\beta+(bt+\beta)^2\}\} \\ &= ae^{bt}t^{\beta-2}[(bt+\beta)^2\{-ae^{bt+ae^{bt}t^{\beta}}t^{\beta} - 1 + e^{ae^{bt}t^{\beta}}\} - \beta\{-1+e^{ae^{bt}t^{\beta}}\}] \\ &= ae^{bt}t^{\beta-2}[(bt+\beta)^2 W_1(t) - \beta\{-1+e^{ae^{bt}t^{\beta}}\}] \end{aligned} \tag{2.3}$$

with $W_1(t) = \{-ae^{bt+ae^{bt}t^{\beta}}t^{\beta} - 1 + e^{ae^{bt}t^{\beta}}\}$. Note that

$$\frac{d}{dt}(W_1(t)) = -a^2e^{2bt+ae^{bt}t^{\beta}}t^{2\beta-1}(bt+\beta),$$

which is negative for $t \geq 0$. Thus, $W_1(t)$ is decreasing in *t*. Also, $W_1(0) = 0$, which gives $W_1(t) \leq W_1(0) = 0$. Hence, in (2.3), we find that $W(t) \leq 0$, which leads to the fact that $\mu_X(t)$ in (2.1) is a decreasing function of *t*. □

Theorem 2.3. *If* $X \sim W_I(\beta, \alpha)$, *then* $\mu_X(t)$ *is a decreasing function of t.*

The next remark explores about a parallel system consisted by independent and non-identically distributed exponential random variables.

Remark 2.1. The failure rate of a parallel system consisting of two components where the lifetimes of each component are independent and non-identically distributed exponential random variables is non-monotonic.

Considering a parallel system consisting of two components having lifetimes (exponential random variables) with survival functions $\overline{F}_{X_1}(t) = e^{-2t}$ and $\overline{F}_{X_2}(t) = e^{-3t}$, $t \geq 0$, respectively. Then $\overline{F}_X(t) = 1 - (1 - e^{-2t}) \cdot (1 - e^{-3t})$, $t \geq 0$, where $X = \max\{X_1, X_2\}$. The failure rate $r_X(t)$ versus t is plotted in Figure 1, which shows that it is non-monotonic.

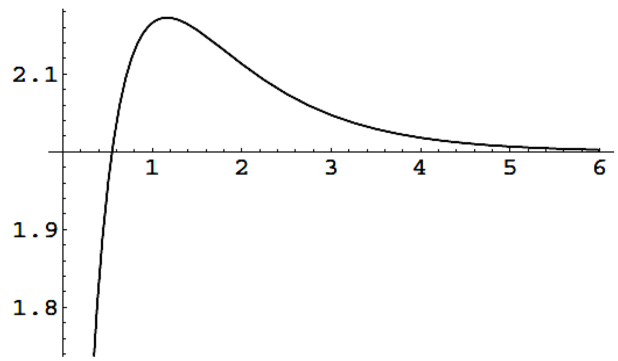

Figure 1. Plot of failure function for X in Remark 2.1.

The next remark describes the nature of failure rate of a series system formed with two independent components having 2-parameter Weibull (with different shape and scale parameters) distribution.

Remark 2.2. If $X_1 \sim W_2(a_1, b_1)$ and $X_2 \sim W_2(a_2, b_2)$ are independent, then the hazard rate of $Z = \min\{X_1, X_2\}$ is:

$$r_Z(x) = a_1 b_1 x^{b_1 - 1} + a_2 b_2 x^{b_2 - 1}.$$

Moreover, we have

$$z'_Z(x) = a_1 b_1 (b_1 - 1) x^{b_1 - 2} + a_2 b_2 (b_2 - 1) x^{b_2 - 2}.$$

It is evident that $r_Z(x)$ is

(i) DFR if $b_1 < 1$ and $b_2 < 1$,

(ii) IFR if $b_1 > 1$ and $b_2 > 1$,

(iii) non-monotonic with stationary point

$$x^* = \left\{\frac{a_1 b_1(b_1 - 1)}{a_2 b_2(b_2 - 1)}\right\}^{1/(b_2 - b_1)}$$

if $b_1 > 1$ and $b_2 < 1$ (i.e., W_1 and W_2 are IFR and DFR, respectively) or if $b_1 < 1$ and $b_2 > 1$.

The next remark says that *a parallel system formed with two independent components having IFR two-parameter Weibull distributions possesses a non-monotonic failure rate*.

Remark 2.3. The failure rate of a parallel system consisting of two components where the lifetimes of each component are independent and non-identically distributed Weibull random variables (each being IFR) is non-monotonic.

Considering a parallel system consisting of two components having lifetimes (Weibull random variables) with survival functions $\overline{F}_{X_1}(t) = e^{-t^{1.5}}$ and $\overline{F}_{X_2}(t) = e^{-t^{1.2}}$ for $t \geq 0$, respectively. Then $\overline{F}_X(t) = 1 - (1 - e^{-t^{1.2}}) \cdot (1 - e^{-t^{1.5}})$, for $t \geq 0$, where $X = \max\{X_1, X_2\}$. The failure rate $r_X(t)$ versus t is plotted in Figure 2, which shows that it is non-monotonic.

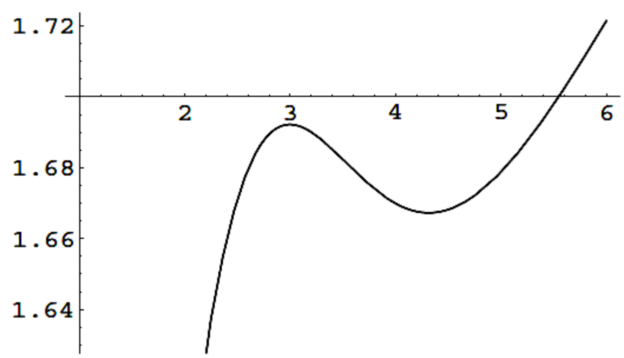

Figure 2. Plot of failure rate function of X with time t in Remark 2.3.

The next remark highlights the behavior of a 2-component parallel system with components having independent Weibull distributions, both of which have DFR property.

Remark 2.4. The failure rate of a parallel system consisting of two components where the lifetimes of each component are independent and identically distributed Weibull random variables (each being DFR) is non-monotonic.

Considering a parallel system consisting of two components having lifetimes (Weibull random variables) with survival functions $\overline{F}_{X_i}(t) = e^{-t^{0.8}}$ for $i = 1, 2,$ and $t \geq 0.$ Then $\overline{F}_X(t) = 1 - (1 - e^{-t^{0.8}})^2,$ $t \geq 0,$ where $X = \max\{X_1, X_2\}.$ The failure rate $r_X(t)$ versus t is plotted in Figure 3, which shows that it is non-monotonic.

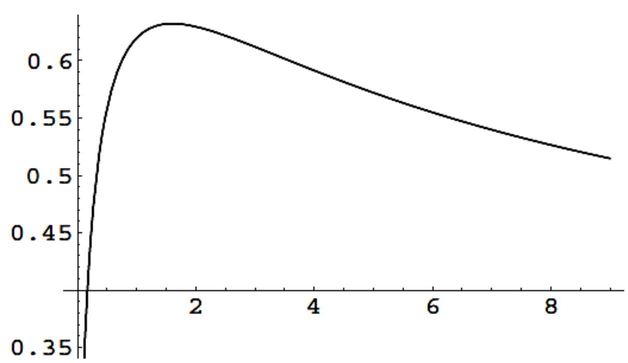

Figure 3. Plot of failure rate function of X with time t in Remark 2.4.

The next remark highlights the behavior of a 2-component parallel system with components having independent Weibull distributions, one of which is IFR and other being DFR.

Remark 2.5. The failure rate of a parallel system consisting of two components where the lifetimes of each component are independent and non-identically distributed Weibull random variables is non-monotonic.

Considering a parallel system consisting of two components having lifetimes (Weibull random variables) with survival functions $\overline{F}_{X_1}(t) = e^{-t^{0.8}}$ and $\overline{F}_{X_2}(t) = e^{-t^2}$, $t \geq 0$, respectively. Then $\overline{F}_X(t) = 1 - (1 - e^{-t^2}) \cdot (1 - e^{-t^{0.8}})$, $t \geq 0$, where $X = \max\{X_1, X_2\}$. The failure rate $r_X(t)$ versus t is plotted in Figure 4, which shows that it is non-monotonic.

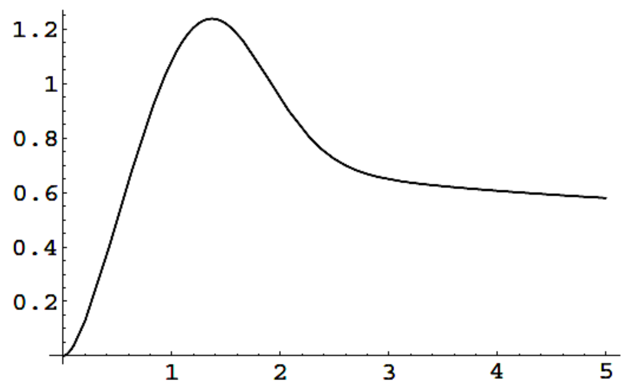

Figure 4. Plot of failure rate function for X in Remark 2.5.

The next theorem says that *a series system formed with n components having a BT* 3*-parameter Weibull distribution is BT*.

To justify the importance of the next theorem, we give an example to stress upon the fact that the sum of two non-negative and non-monotonic functions need not be non-monotonic.

Example 2.1. Let us take

$$f(t) = \begin{cases} 0.8 - 3t & \text{if } t \leq 0.06, \\ \frac{62}{6} t & \text{if } t \geq 0.06 \end{cases} \tag{2.4}$$

and

$$g(t) = \begin{cases} 3t & \text{if } t \leq 0.06, \\ 0.48 - 5t & \text{if } 0.06 \leq t \leq 0.08, \\ 0.08 & \text{if } t \geq 0.08. \end{cases} \tag{2.5}$$

Note that $f(t)$ and $g(t)$ are non-negative and non-monotonic functions, respectively. But

$$f(t) + g(t) = \begin{cases} 0.8 & \text{if } t \leq 0.06, \\ 0.48 + \dfrac{32t}{6} & \text{if } 0.06 \leq t \leq 0.08, \\ 0.08 + \dfrac{62t}{6} & \text{if } t \geq 0.08 \end{cases} \tag{2.6}$$

is monotonic in $t \geq 0$.

As a result of the discussion, we find the next theorem to be an interesting one.

Theorem 2.4. *Let* $X_i \sim W_3(a_i, b_i, \lambda_i)$ *for* $i = 1, 2, ..., n$ *be BT. Then* $X = \min\{X_i : 1 \leq i \leq n\}$ *is BT.*

Proof. Note that

$$r_X(t) = \frac{1}{t}\left[\sum_{i=1}^{n} a_i e^{\lambda_i t} t^{b_i} (b_i + \lambda_i t)\right],$$

so that

$$\frac{d}{dt}(r_X(t)) = \frac{1}{t^2}\sum_{i=1}^{n}[a_i e^{\lambda_i t} t^{b_i} \{b_i(b_i - 1) + 2tb_i\lambda_i + t^2\lambda_i^2\}]$$

$$= \frac{1}{t^2}\sum_{i=1}^{n}[a_i e^{\lambda_i t} t^{b_i} \{b_i^2 - b_i + 2tb_i\lambda_i + t^2\lambda_i^2\}]$$

$$= \frac{1}{t^2}\sum_{i=1}^{n}[a_i e^{\lambda_i t} t^{b_i} \{(b_i + t\lambda_i)^2 - b_i\}]. \tag{2.7}$$

If X_i for $1 \leq i \leq n$ is BT $(0 < b_i < 1)$, so is X. □

The next remark says that *a series system formed with two independent components having an IFR and a DFR* 3*-parameter Weibull distribution possesses a bathtub failure rate*.

Remark 2.6. Let $X_1 \sim W_3(a, b_1, \lambda)$ and $X_2 \sim W_3(a, b_2, \lambda)$ having IFR and DFR, respectively. Then the failure rate of $X = \min\{X_i : 1 \le i \le 2\}$ is non-monotonic. Let $X_1 \sim W_3(2, 0.2, 2)$ and $X_2 \sim W_3(2, 5, 2)$. Then $r_X(t) = e^{2t}(0.4 + 4t + 10t^{4.8} + 4t^{5.8})/t^{0.8}$ for $t \ge 0$, where $X = \min\{X_i : 1 \le i \le 2\}$, which is non-monotone as is shown in Figure 5.

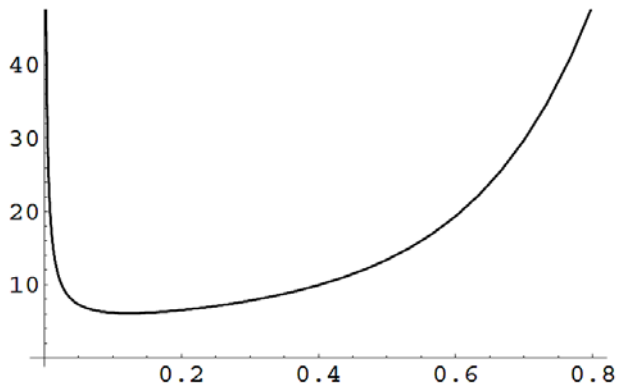

Figure 5. Plot of failure function for *X* with time *t* in Remark 2.6.

The next remark says that *a series system formed with two independent components having an IFR and a DFR* 3-*parameter Weibull distribution* (*with different parameters*) *possesses a bathtub failure rate*.

Remark 2.7. Let $X_1 \sim W_3(a_1, b_1, \lambda_1)$ and $X_2 \sim W_3(a_2, b_2, \lambda_2)$ having IFR and DFR, respectively. Then the failure rate of $X = \min\{X_i : 1 \le i \le 2\}$ is non-monotonic. Let $X_1 \sim W_3(2, 0.2, 2.5)$ and $X_2 \sim W_3(4, 5, 3)$. Then $r_X(t) = 4e^{3t}t^4(5 + 3t) + \dfrac{2e^{2.5t}(0.2 + 2.5t)}{t^{0.8}}$ for $t \ge 0$, where $X = \min\{X_i : 1 \le i \le 2\}$, which is non-monotone as is shown in Figure 6.

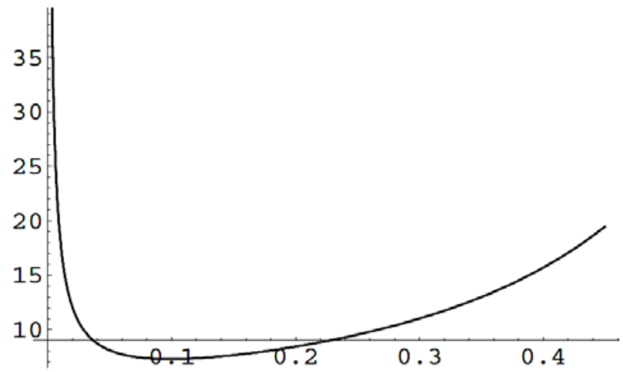

Figure 6. Plot of failure function for X in Remark 2.7.

3. Numerical Examples

One can estimate $\overline{F}_X(t)$, $r_X(t)$, $\mu_X(t)$ and $L_X(t)$ with the help of logical estimates, as highlighted in the present section. Let N units be put to test at $t = 0$. Further, let the number of units having survived at ordered times t_j be $N_s(t_j)$. The estimates for $\overline{F}_X(t)$, $r_X(t)$ and $\mu_X(t)$ are, respectively, given as follows:

$$\hat{\overline{F}}_X(t) = \frac{N_s(t_j)}{N} \text{ for } t_j < t < t_j + \Delta t_j,$$

$$\hat{r}_X(t) = \frac{\{N_s(t_j) - N_s(t_j + \Delta t_j)\}}{N_s(t_j)\Delta t_j} \text{ for } t_j < t < t_j + \Delta t_j,$$

$$\hat{\mu}_X(t) = \frac{\{N_s(t_j) - N_s(t_j + \Delta t_j)\}}{\{N - N_s(t_j)\}\Delta t_j} \text{ for } t_j < t < t_j + \Delta t_j.$$

Thus, an estimate for $L_X(t)$, for $t > 0$, is

$$\hat{L}_X(t) = \frac{-t\{N_s(t_j) - N_s(t_j + \Delta t_j)\}}{N_s(t_j)\Delta t_j \ln \frac{N_s(t_j)}{N}} \text{ for } t_j < t < t_j + \Delta t_j.$$

A failure data of seventy compressors collected from Ebeling [4] are observed at 5-month intervals with failures as shown in Table 1. Estimates of $r(t)$, $\mu(t)$, $L(t)$ computed for the failure data in Table 1 are plotted in Figure 7. The hypothetical data given in Table 2 depict failures in one thousand B-52 bombers (i.e., $N = 1000$) performing various 24-hr missions (cf. Shooman [17]) as shown in Table 2. $r(t)$, $\mu(t)$, $L(t)$ computed for the failure data in Table 2 are plotted in Figure 8.

Table 1. Failure data of compressors

Time till failure	$N_s(t_j)$	$N_s(t_j) - N_s(t_j + \Delta t_j)$	$\hat{f}_X(t)$	$\hat{r}_X(t)$	$\hat{\mu}_X(t)$	$\hat{L}_X(t)$
0-5	70	3	0.0085714	0.0086	-	-
5-10	67	7	0.02	0.0209	0.4666667	0.47714t
10-15	60	8	0.022857	0.0267	0.16	0.17321t
15-20	52	9	0.025714	0.03460	0.1	0.11640t
20-25	43	13	0.0371429	0.0605	0.0962963	0.12415t
25-30	30	18	0.051429	0.1200	0.09	0.1416t
30-35	12	12	0.034286	0.2	0.0413793	0.11341t

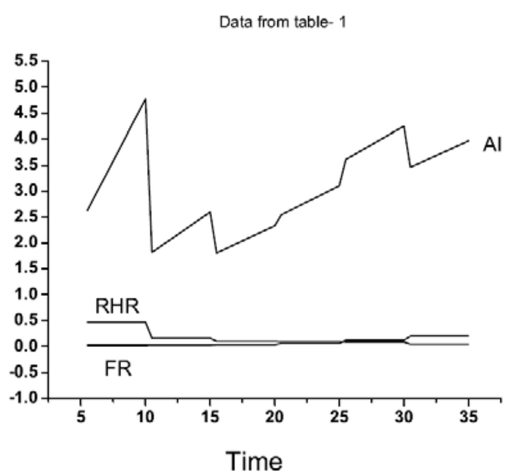

Figure 7. Estimates of $r(t)$, $\mu(t)$, $L(t)$ versus t plotted for the data in Table 1.

Table 2. Failure data of B-52 bombers

Time till Failure	$N_s(t_j)$	$N_s(t_j)$ $-N_s(t_j+\Delta t_j)$	$\hat{f}_X(t_j)$	$\hat{r}_X(t_j)$	$\hat{\mu}_X(t_j)$	$\hat{L}_X(t_j)$
0-2	1000	222	0.111	0.111	-	-
2-4	778	45	0.0225	0.028920308	0.101351	0.1152t
4-6	733	32	0.016	0.021828104	0.0599251	0.0703t
6-8	701	27	0.0135	0.019258203	0.045150	0.0542t
8-10	674	21	0.0105	0.015578635	0.0322086	0.0395t
10-12	653	15	0.0075	0.011485452	0.0216138	0.0270t
12-14	638	17	0.0085	0.013322884	0.023481	0.0297t
14-16	621	7	0.0035	0.005636071	0.009235	0.0118t
16-18	614	14	0.007	0.011400651	0.018135	0.0234t
18-20	600	9	0.0045	0.0075	0.01125	0.0147t
20-22	591	8	0.004	0.00676819	0.00978	0.0129t
22-24	583	3	0.0015	0.002572899	0.0035971	0.0048t

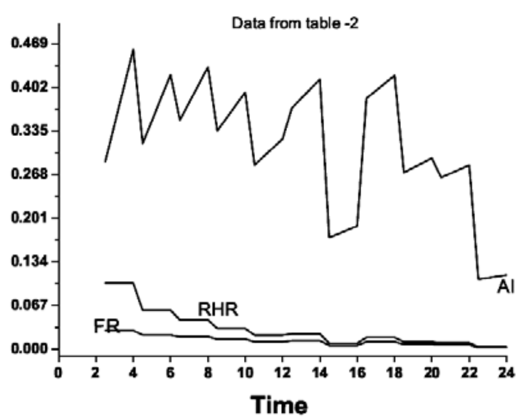

Figure 8. Estimates of $r(t)$, $\mu(t)$, $L(t)$ versus t plotted for the data in Table 2.

4. Concluding Remarks

We look at the system properties of some well known Weibull models, each one of these has wide applications in appropriate scenario. Similar results can be studied for other Weibull models. The comparison of $L(t)$, $h(t)$, $\mu(t)$ is studied for two numerical examples. Figure 7 and Figure 8 show that the AI function considerably differs from failure rate and reversed hazard rate functions for the given data sets. The present work can be extended for other Weibull distributions to help researchers to conclude about the nature of reversed hazard rate and other aging properties.

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