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A MODIFICATION OF *S*-TYPE EIGENVALUE LOCALIZATION SET FOR TENSORS AND ITS APPLICATIONS

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Abstract

A new S-type eigenvalue localization set for tensors is given by partitioning the index set $N = \{1, 2, ..., n\}$ into a subset S and its complement. It is shown that the new set is tighter than those in Li et al. [35]. Based on this new set, we give a checkable sufficient condition for the positive (semi-)definiteness of tensors and an upper bound for the spectral radius of nonnegative tensors.

1. Introduction

Let $C(R)$ denote the set of all complex (real) numbers and $N = \{1, 2, ..., n\}$. We call $\mathcal{A} = (a_{i_1 \cdots i_m})$ to be a *complex* (*real*) *tensor of order m* and dimension *n*, denoted by $\mathcal{A} \in C^{[m, n]}(R^{[m, n]})$, if

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$$a_{i_1 \cdots i_m} \in C(R),$$

where $i_j = 1, \ldots, n$ for $j = 1, \ldots, m$. Obviously, a vector is a tensor of order 1 and a matrix is a tensor of order 2. A real tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$ is called *symmetric* [1, 2] if

$$a_{i_1 \cdots i_m} = a_{\pi(i_1 \cdots i_m)}, \quad \forall \pi \in \Pi_m,$$

where Π_m is the permutation group of *m* indices. Furthermore, a real tensor of order *m* and dimension *n* is called the *unit tensor*, if its entries are $\delta_{i_1 \cdots i_m}$ for $i_1, \ldots, i_m \in N$, where

$$\delta_{i_1 \cdots i_m} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

We now define a tensor-vector multiplication, i.e., for a tensor $\mathcal{A} = (a_{i_1 \cdots i_m}) \in C^{[m, n]}$, and a vector $x = (x_1, x_2, \ldots, x_n)^T \in C^n$, $\mathcal{A}x^{m-1}$ is an *n* dimensional vector whose *i*th component is

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, \ldots, i_m \in N} a_{ii_2 \cdots i_m} x_{i_2} \cdots x_{i_m}.$$

Moreover, if there are a complex number λ and a nonzero complex vector $x = (x_1, x_2, \ldots, x_n)^T$ such that

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then λ is called an *eigenvalue* of $\mathcal{A}$ and *x* an *eigenvector* of $\mathcal{A}$ associated with λ, where

$$x^{[m-1]} = (x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1})^T.$$

This definition was introduced by Qi in [1] where he assumed that $\mathcal{A} \in R^{[m, n]}$ is symmetric and *m* is even. Independently, in [3], Lim gave such a definition but restricted *x* to be a real vector and λ to be a real number. In this case, we call λ to be an *H-eigenvalue* of $\mathcal{A}$ and x to be an *H-eigenvector* of $\mathcal{A}$ associated with λ [1, 4, 5].

Eigenvalue problems of tensors have become an important topic of study in numerical multilinear algebra, and they have a wide range of practical applications; see [5-30]. For example, we can use the smallest H-eigenvalue of a tensor to determine its positive (semi-)definiteness. But it is not easy to compute the smallest H-eigenvalue of tensors when the order and dimension are very large, so we always try to obtain a set in the complex field which includes all eigenvalues of a given tensor.

In [1], Qi gave an eigenvalue localization set for real symmetric tensors, which is a generalization of the well known Geršgorin's eigenvalue localization set of matrices [31, 32]. This result can be easily generalized to general tensors [2, 33].

Theorem 1.1 [1, 2, 33]**.** *Let* $\mathcal{A} = (a_{i_1 \cdots i_m}) \in C^{[m, n]}$. *Then*

$$\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) := \bigcup_{i \in N} \Gamma_i(\mathcal{A}),$$

where $\sigma(\mathcal{A})$ *is the set of all the eigenvalues of* $\mathcal{A}$ *and*

$$\Gamma_i(\mathcal{A}) = \{z \in \mathbb{C} : |\, z - a_{i \cdots i} \,| \leq r_i(\mathcal{A})\}, \quad r_i(\mathcal{A}) = \sum_{\substack{i_2, \ldots, i_m \in N \\ \delta_{ii_2 \cdots i_m} = 0}} |\, a_{ii_2 \cdots i_m} \,|.$$

To obtain tighter sets than $\Gamma(\mathcal{A})$, Li et al. [33] extended the Brauer's eigenvalue localization set of matrices [32, 34] to tensors and gave the following Brauer-type eigenvalue localization set for tensors.

Theorem 1.2 [33]**.** *Let* $\mathcal{A} = (a_{i_1 \cdots i_m}) \in C^{[m, n]}$, $n \geq 2$. *Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) := \bigcup_{\substack{i, j \in N \\ j \neq i}} \mathcal{K}_{i, j}(\mathcal{A}),$$

where

$$\mathcal{K}_{i,j}(\mathcal{A}) = \{z \in \mathbb{C} : (|z - a_{i\cdots i}| - r_i^j(\mathcal{A}))|z - a_{j\cdots j}| \le |a_{ij\cdots j}| r_j(\mathcal{A})\},$$

and

$$r_i^j(\mathcal{A}) = \sum_{\substack{\delta_{ii_2\cdots i_m}=0 \\ \delta_{ji_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| = r_i(\mathcal{A}) - |a_{ij\cdots j}|.$$

Furthermore, $\mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$.

Theorem 1.3 [33]. *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in C^{[m,n]}$, $n \ge 2$, *and* S *be a nonempty proper subset of* N. *Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}) := \left(\bigcup_{i\in S,\, j\in \overline{S}} \mathcal{K}_{i,j}(\mathcal{A})\right) \cup \left(\bigcup_{i\in \overline{S},\, j\in S} \mathcal{K}_{i,j}(\mathcal{A})\right).$$

Furthermore, $\mathcal{K}^S(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$.

Theorem 1.4 [35]. *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in C^{[m,n]}$, $n \ge 2$, *and* S *be a nonempty proper subset of* N. *Then*

$$\sigma(\mathcal{A}) \subseteq \Omega^S(\mathcal{A}) := \left(\bigcup_{i\in S,\, j\in \overline{S}} \Omega^S_{i,j}(\mathcal{A})\right) \cup \left(\bigcup_{i\in \overline{S},\, j\in S} \Omega^{\overline{S}}_{i,j}(\mathcal{A})\right),$$

where

$$\Omega^S_{i,j}(\mathcal{A}) := \{z \in \mathbb{C} : (|z - a_{i\cdots i}|)(|z - a_{j\cdots j}| - r_j^{\overline{\Delta^S}}(\mathcal{A}))$$

$$\le r_i(\mathcal{A}) r_j^{\Delta^S}(\mathcal{A}),\ i \in S,\ j \in \overline{S}\}.$$

Li et al. [35] proved that

$$\Omega^S(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A}). \tag{1.1}$$

In this paper, we continue this work on the eigenvalue localization problem for tensors. In Section 2, a new eigenvalue inclusion set for tensors is obtained, and it has been proved to be tighter than those in Theorems 1.1-1.4 Based on the new set, a checkable sufficient condition for the positive (semi-)definiteness of tensors is given in Section 3, and an upper bound for the spectral radius of nonnegative tensor is given in Section 4.

2. A New Eigenvalue Localization Set

Given a nonempty proper subset S of N, $\overline{S} = N \backslash S$. We denote

$$\Delta^N := \{(i_2, i_3, \ldots, i_m) : \text{each } i_j \in N \text{ for } j = 2, \ldots, m\},$$

$$\Delta^S := \{(i_2, i_3, \ldots, i_m) : \text{each } i_j \in S \text{ for } j = 2, \ldots, m\},$$

and

$$\overline{\Delta^S} = \Delta^N \backslash \Delta^S.$$

This implies that for a tensor $\mathcal{A} = (a_{i_1 \cdots i_m}) \in C^{[m, n]}$, we have that for $i \in S$,

$$r_i(\mathcal{A}) = r_i^{\Delta^S}(\mathcal{A}) + r_i^{\overline{\Delta^S}}(\mathcal{A}), \quad r_i^j(\mathcal{A}) = r_i^{\Delta^S}(\mathcal{A}) + r_i^{\overline{\Delta^S}}(\mathcal{A}) - |a_{ij \cdots j}|,$$

where

$$r_i^{\Delta^S}(\mathcal{A}) = \sum_{\substack{(i_2, \ldots, i_m) \in \Delta^S \\ \delta_{ii_2 \cdots i_m} = 0}} |a_{ii_2 \cdots i_m}|, \quad r_i^{\overline{\Delta^S}}(\mathcal{A}) = \sum_{(i_2, \ldots, i_m) \in \overline{\Delta^S}} |a_{ii_2 \cdots i_m}|.$$

Theorem 2.1. *Let* $\mathcal{A} = (a_{i_1 \cdots i_m}) \in C^{[m, n]}$ *with* $n \geq 2$, S *be a nonempty proper subset of* N*. Then*

$$\sigma(\mathcal{A}) \subseteq \Psi^S(\mathcal{A}) := \left(\bigcup_{i\in S}\Psi_i^S(\mathcal{A})\right) \cup \left(\bigcup_{i\in\overline{S}}\Psi_i^{\overline{S}}(\mathcal{A})\right)$$

$$\cup \left(\bigcup_{\substack{i\in S\\ j\in\overline{S}}}(\Psi_{i,j}^S(\mathcal{A}) \cap \Gamma_i(\mathcal{A}))\right) \cup \left(\bigcup_{\substack{i\in\overline{S}\\ j\in S}}(\Psi_{i,j}^{\overline{S}}(\mathcal{A}) \cap \Gamma_i(\mathcal{A}))\right),$$

where

$$\Psi_i^S(\mathcal{A}) = \{z \in \mathbb{C} : |z - a_{i\cdots i}| \le r_i^{\Delta^S}(\mathcal{A})\},$$

$$\Psi_{i,j}^S(\mathcal{A}) = \{z \in \mathbb{C} : (|z - a_{i\cdots i}| - r_i^{\Delta^S}(\mathcal{A}))$$

$$\cdot (|z - a_{j\cdots j}| - r_j^{\overline{\Delta^S}}(\mathcal{A})) \le r_i^{\overline{\Delta^S}}(\mathcal{A}) r_j^{\Delta^S}(\mathcal{A})\}.$$

Proof. For any $\lambda \in \sigma(\mathcal{A})$, let $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n \backslash \{0\}$ be an associated eigenvector, i.e.,

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}. \tag{2.1}$$

Let $|x_p| = \max_{i\in S}|x_i|$ and $|x_q| = \max_{i\in\overline{S}}|x_i|$. Then at least one of x_p or x^q is nonzero. If

$$\lambda \in \left(\bigcup_{i\in S}\Psi_i^S(\mathcal{A})\right) \cup \left(\bigcup_{i\in\overline{S}}\Psi_i^{\overline{S}}(\mathcal{A})\right),$$

then it is obvious that $\lambda \in \Psi^S(\mathcal{A})$. Otherwise,

$$\lambda \notin \left(\bigcup_{i\in S}\Psi_i^S(\mathcal{A})\right) \cup \left(\bigcup_{i\in\overline{S}}\Psi_i^{\overline{S}}(\mathcal{A})\right),$$

and we have

$$|\lambda - a_{i\cdots i}| > r_i^{\Delta^S}(\mathcal{A}) \text{ each } i \in S,$$

and

$$|\lambda - a_{i\cdots i}| > r_i^{\Delta^{\overline{S}}}(\mathcal{A}) \text{ for each } i \in \overline{S}.$$

We next prove

$$\lambda \in \left(\bigcup_{\substack{i\in S \\ j\in \overline{S}}} (\Psi_{i,j}^{S}(\mathcal{A}) \cap \Gamma_i(\mathcal{A}))\right) \cup \left(\bigcup_{\substack{i\in \overline{S} \\ j\in S}} (\Psi_{i,j}^{\overline{S}}(\mathcal{A}) \cap \Gamma_i(\mathcal{A}))\right).$$

Case I. $x_p x_q \neq 0$ and $|x_p| \geq |x_q|$, that is, $|x_p| = \max_{i\in N} |x_i|$. By (2.1), we have

$$(\lambda - a_{p\cdots p}) x_p^{m-1} = \sum_{(i_2,\ldots,i_m)\in\Delta^{\overline{S}}} a_{pi_2\cdots i_m} x_{i_2} \cdots x_{i_m}$$

$$+ \sum_{\substack{(i_2,\ldots,i_m)\in\overline{\Delta^{\overline{S}}} \\ \delta_{pi_2\cdots i_m}=0}} a_{pi_2\cdots i_m} x_{i_2} \cdots x_{i_m}.$$

Taking modulus in the above equation and using the triangle inequality give

$$|\lambda - a_{p\cdots p}||x_p|^{m-1}$$

$$\leq \sum_{(i_2,\ldots,i_m)\in\Delta^{\overline{S}}} |a_{pi_2\cdots i_m}||x_{i_2}|\cdots|x_{i_m}| + \sum_{\substack{(i_2,\ldots,i_m)\in\overline{\Delta^{\overline{S}}} \\ \delta_{pi_2\cdots i_m}=0}} |a_{pi_2\cdots i_m}||x_{i_2}|\cdots|x_{i_m}|$$

$$\leq \sum_{(i_2,\ldots,i_m)\in\Delta^{\overline{S}}} |a_{pi_2\cdots i_m}||x_q|^{m-1} + \sum_{\substack{(i_2,\ldots,i_m)\in\overline{\Delta^{\overline{S}}} \\ \delta_{pi_2\cdots i_m}=0}} |a_{pi_2\cdots i_m}||x_p|^{m-1}$$

$$= r_p^{\Delta^{\overline{S}}}(\mathcal{A})|x_q|^{m-1} + r_p^{\overline{\Delta^{\overline{S}}}}(\mathcal{A})|x_p|^{m-1}.$$

Hence,

$$(|\lambda - a_{p\cdots p}| - r_p^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))|x_p|^{m-1} \le r_p^{\Delta^{\bar{S}}}(\mathcal{A})|x_q|^{m-1}. \tag{2.2}$$

Similarly, we have that

$$\begin{aligned}&(\lambda - a_{q\cdots q})x_q^{m-1}\\ &= \sum_{\substack{(i_2,\ldots,i_m)\in\Delta^{\bar{S}}\\ \delta_{qi_2\cdots i_m}=0}} a_{qi_2\cdots i_m}x_{i_2}\cdots x_{i_m} + \sum_{(i_2,\ldots,i_m)\in\overline{\Delta^{\bar{S}}}} a_{qi_2\cdots i_m}x_{i_2}\cdots x_{i_m}\end{aligned}$$

and that

$$|\lambda - a_{q\cdots q}||x_q|^{m-1} \le r_q^{\Delta^{\bar{S}}}(\mathcal{A})|x_q|^{m-1} + r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})|x_p|^{m-1},$$

i.e.,

$$(|\lambda - a_{q\cdots q}| - r_q^{\Delta^{\bar{S}}}(\mathcal{A}))|x_q|^{m-1} \le r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})|x_p|^{m-1}. \tag{2.3}$$

Note that for any $j \in \bar{S}$, $|\lambda - a_{j\cdots j}| - r_j^{\Delta^{\bar{S}}}(\mathcal{A}) > 0$, hence by (2.2) and (2.3), we get

$$(|\lambda - a_{p\cdots p}| - r_p^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))(|\lambda - a_{q\cdots q}| - r_q^{\Delta^{\bar{S}}}(\mathcal{A})) \le r_p^{\Delta^{\bar{S}}}(\mathcal{A})r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}),$$

which leads to $\lambda \in \Psi_{q,p}^{\bar{S}}(\mathcal{A}) \subseteq \Psi^S(\mathcal{A})$.

Case II. $x_p x_q \ne 0$ and $|x_q| \ge |x_p|$, that is, $|x_q| = \max_{i\in N}|x_i|$. Similar to the argument of Case I, we can obtain that

$$(|\lambda - a_{q\cdots q}| - r_q^{\overline{\Delta^S}}(\mathcal{A}))|x_q|^{m-1} \le r_q^{\Delta^S}(\mathcal{A})|x_p|^{m-1},$$

and

$$(|\lambda - a_{p\cdots p}| - r_p^{\Delta^S}(\mathcal{A}))|x_p|^{m-1} \le r_p^{\overline{\Delta^S}}(\mathcal{A})|x_q|^{m-1}.$$

Thus,

$$(|\lambda - a_{p\cdots p}| - r_p^{\Delta^S}(\mathcal{A}))(|\lambda - a_{q\cdots q}| - r_q^{\overline{\Delta^S}}(\mathcal{A})) \leq r_p^{\overline{\Delta^S}}(\mathcal{A}) r_q^{\Delta^S}(\mathcal{A}),$$

by the fact that for any $i \in S$, $|\lambda - a_{i\cdots i}| - r_i^{\Delta^S}(\mathcal{A}) > 0$. Hence, $\lambda \in \Psi_{p,q}^S(\mathcal{A}) \subseteq \Psi^S(\mathcal{A})$.

Case III. $|x_p||x_q| = 0$, without loss of generality, let $|x_q| = 0$ and $|x_p| \neq 0$. Then by (2.2),

$$|\lambda - a_{p\cdots p}| - r_p^{\overline{\Delta^{S}}}(\mathcal{A}) \leq 0.$$

Hence, for any $j \in \overline{S}$,

$$(|\lambda - a_{p\cdots p}| - r_p^{\overline{\Delta^{\overline{S}}}}(\mathcal{A}))(|\lambda - a_{j\cdots j}| - r_j^{\Delta^{\overline{S}}}(\mathcal{A})) \leq r_p^{\Delta^{\overline{S}}}(\mathcal{A}) r_j^{\overline{\Delta^{\overline{S}}}}(\mathcal{A}),$$

which leads to $\lambda \in \Psi_{j,p}^{\overline{S}}(\mathcal{A}) \subseteq \Psi^S(\mathcal{A})$. The conclusions follows by combining Cases I, II and III. □

To compare the sets $\Gamma(\mathcal{A})$ in Theorem 1.1, $\mathcal{K}(\mathcal{A})$ in Theorem 1.2, $\mathcal{K}^S(\mathcal{A})$ in Theorem 1.3, $\Omega^S(\mathcal{A})$ in Theorem 1.4 and $\Psi^S(\mathcal{A})$ in Theorem 2.1, we need the following conclusion.

Lemma 2.2 [36]**.** (I) *Let* $a, b, c \geq 0$ *and* $d > 0$. *If* $\frac{a}{b+c+d} \leq 1$, *then*

$$\frac{a-(b+c)}{d} \leq \frac{a-b}{c+d} \leq \frac{a}{b+c+d}.$$

(II) *Let* $a, b, c \geq 0$ *and* $d > 0$. *If* $\frac{a}{b+c+d} \geq 1$, *then*

$$\frac{a-(b+c)}{d} \geq \frac{a-b}{c+d} \geq \frac{a}{b+c+d}.$$

Theorem 2.3. *Let* $\mathcal{A} = (a_{i_1 \cdots i_m}) \in C^{[m,n]}$, $n \geq 2$. *Then*

$$\Psi^S(\mathcal{A}) \subseteq \Omega^S(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A}).$$

Proof. By (1.1), $\Omega^S(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$ holds. We only prove $\Psi^S(\mathcal{A}) \subseteq \Omega^S(\mathcal{A})$. Let $\lambda \in \Psi^S(\mathcal{A})$. If

$$\lambda \in \left(\bigcup_{i \in S} \Psi_i^S(\mathcal{A})\right) \cup \left(\bigcup_{i \in \overline{S}} \Psi_i^{\overline{S}}(\mathcal{A})\right),$$

without loss of generality, we suppose $\lambda \in \bigcup_{i \in S} \Psi_i^S(\mathcal{A})$ (we can also prove it similarly if $\lambda \in \bigcup_{i \in \overline{S}} \Psi_i^{\overline{S}}(\mathcal{A})$). Then there are index $p \in S$ and $q \in \overline{S}$ such that $\lambda \in \Psi_p^S(\mathcal{A})$, i.e.,

$$|\lambda - a_{p \cdots p}| \leq r_p^{\Delta^S}(\mathcal{A}).$$

For $r_p^{\overline{\Delta^S}}(\mathcal{A}) \geq 0$, so $0 < |\lambda - a_{p \cdots p}| \leq r_p(\mathcal{A})$. For $|\lambda - a_{q \cdots q}| \leq r_q(\mathcal{A})$, so $|\lambda - a_{q \cdots q}| - r_q^{\overline{\Delta^S}}(\mathcal{A}) \leq r_q^{\Delta^S}(\mathcal{A})$, then we have

$$(|\lambda - a_{q \cdots q}| - r_q^{\overline{\Delta^S}}(\mathcal{A}))|\lambda - a_{p \cdots p}| \leq r_p(\mathcal{A}) r_q^{\Delta^S}(\mathcal{A}),$$

which implies that $\lambda \in \Omega^S(\mathcal{A})$.

If $\lambda \notin \left(\bigcup_{i \in S} \Psi_i^S(\mathcal{A})\right) \cup \left(\bigcup_{i \in \overline{S}} \Psi_i^{\overline{S}}(\mathcal{A})\right)$, that is,

$$|\lambda - a_{i \cdots i}| > r_i^{\Delta^S}(\mathcal{A}) \text{ for each } i \in S, \tag{2.4}$$

and

$$|\lambda - a_{i \cdots i}| > r_i^{\Delta^{\overline{S}}}(\mathcal{A}) \text{ for each } i \in \overline{S},$$

then

$$\lambda \in \left(\bigcup_{\substack{i \in S \\ j \in \overline{S}}} (\Psi_{i,j}^{S}(\mathcal{A}) \cap \Gamma_i(\mathcal{A})) \right) \cup \left(\bigcup_{\substack{i \in \overline{S} \\ j \in S}} (\Psi_{i,j}^{\overline{S}}(\mathcal{A}) \cap \Gamma_i(\mathcal{A})) \right).$$

Without loss of generality, we suppose that

$$\lambda \in \bigcup_{\substack{i \in S \\ j \in \overline{S}}} (\Psi_{i,j}^{S}(\mathcal{A}) \cap \Gamma_i(\mathcal{A})).$$

Then there are $p \in S$ and $q \in \overline{S}$ such that $\lambda \in \Psi_{p,q}^{S}(\mathcal{A}) \cap \Gamma_p(\mathcal{A})$, i.e.,

$$| \lambda - a_{p \cdots p} | \leq r_p(\mathcal{A}), \tag{2.5}$$

and

$$(| \lambda - a_{p \cdots p} | - r_p^{\Delta^S}(\mathcal{A}))(| \lambda - a_{q \cdots q} | - r_q^{\overline{\Delta^S}}(\mathcal{A})) \leq r_p^{\overline{\Delta^S}}(\mathcal{A}) r_q^{\Delta^S}(\mathcal{A}). \tag{2.6}$$

If $r_p^{\overline{\Delta^S}}(\mathcal{A}) r_q^{\Delta^S}(\mathcal{A}) = 0$, then by inequalities (2.4), (2.5) and (2.6), we have $r_p^{\overline{\Delta^S}}(\mathcal{A}) > 0$, $r_q^{\Delta^S}(\mathcal{A}) = 0$, and $| \lambda - a_{q \cdots q} | - r_q^{\overline{\Delta^S}}(\mathcal{A}) \leq 0$, this leads to

$$(| \lambda - a_{q \cdots q} | - r_q^{\overline{\Delta^S}}(\mathcal{A}))| \lambda - a_{p \cdots p} | \leq 0 = r_p(\mathcal{A}) r_q^{\Delta^S}(\mathcal{A}),$$

which implies that $\lambda \in \Omega^S(\mathcal{A})$.

If $r_p^{\overline{\Delta^S}}(\mathcal{A}) r_q^{\Delta^S}(\mathcal{A}) > 0$, then by dividing inequality (2.6) by $r_p^{\overline{\Delta^S}}(\mathcal{A}) r_q^{\Delta^S}(\mathcal{A})$, we have

$$\frac{| \lambda - a_{p \cdots p} | - r_p^{\Delta^S}(\mathcal{A})}{r_p^{\overline{\Delta^S}}(\mathcal{A})} \frac{| \lambda - a_{q \cdots q} | - r_q^{\overline{\Delta^S}}(\mathcal{A})}{r_q^{\Delta^S}(\mathcal{A})} \leq 1. \tag{2.7}$$

If $\dfrac{|\lambda - a_{p\cdots p}| - r_p^{\Delta^S}(\mathcal{A})}{r_p^{\overline{\Delta^S}}(\mathcal{A})} \geq 1,$ by Lemma 2.2 and inequality (2.7), we get

$$\frac{|\lambda - a_{p\cdots p}|}{r_p(\mathcal{A})} \frac{|\lambda - a_{q\cdots q}| - r_q^{\overline{\Delta^S}}(\mathcal{A})}{r_q^{\Delta^S}(\mathcal{A})}$$

$$\leq \frac{|\lambda - a_{p\cdots p}| - r_p^{\Delta^S}(\mathcal{A})}{r_p^{\overline{\Delta^S}}(\mathcal{A})} \frac{|\lambda - a_{q\cdots q}| - r_q^{\overline{\Delta^S}}(\mathcal{A})}{r_q^{\Delta^S}(\mathcal{A})} \leq 1,$$

equivalently,

$$(|\lambda - a_{q\cdots q}| - r_q^{\overline{\Delta^S}}(\mathcal{A}))|\lambda - a_{p\cdots p}| \leq r_p(\mathcal{A}) r_q^{\Delta^S}(\mathcal{A}),$$

which implies that $\lambda \in \Omega^S(\mathcal{A})$.

If $\dfrac{|\lambda - a_{p\cdots p}| - r_p^{\Delta^S}(\mathcal{A})}{r_p^{\overline{\Delta^S}}(\mathcal{A})} \leq 1,$ equivalently, $|\lambda - a_{p\cdots p}| - r_p^{\Delta^S}(\mathcal{A}) \leq r_p^{\overline{\Delta^S}}(\mathcal{A}),$ on multiplication, inequality (2.5) leads to

$$|\lambda - a_{p\cdots p}|(|\lambda - a_{p\cdots p}| - r_p^{\Delta^S}(\mathcal{A})) \leq r_p(\mathcal{A}) r_p^{\overline{\Delta^S}}(\mathcal{A}),$$

which implies that $\lambda \in \Omega^S(\mathcal{A})$. Hence, $\Psi^S(\mathcal{A}) \subseteq \Omega^S(\mathcal{A})$. The proof is thus completed. □

Example 2.4. Let $\mathcal{A} = (a_{i_1 i_2 i_3}) \in \mathbb{C}^{[3,3]}$, where $a_{122} = a_{132} = 1$, $a_{233} = 2$, $a_{212} = a_{312} = 3$ and the other entries are zero. Let $S = \{1, 3\}$, $\overline{S} = \{2\}$. By Theorem 1.4, we have

$$\sigma(\mathcal{A}) \subseteq \Omega^S(\mathcal{A}) = \{z \in \mathbb{C} : (|z| - 3)|z| \leq 6\}.$$

By Theorem 2.1, we obtain

$$\sigma(\mathcal{A}) \subseteq \Psi^{S}(\mathcal{A}) = \{z \in \mathbb{C} : (|z| - 3)|z| \le 0\}.$$

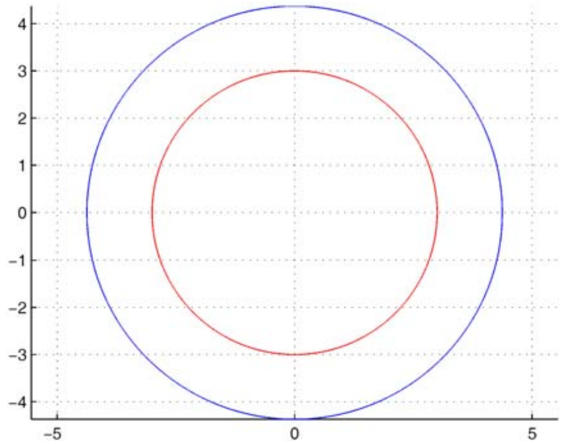

Figure 1. $\Omega^{S}(\mathcal{A})$ is represented by the blue boundary, while $\Psi^{S}(\mathcal{A})$ by the red. Obviously, $\Psi^{S}(\mathcal{A}) \subseteq \Omega^{S}(\mathcal{A})$.

3. Sufficient Condition for Positive (semi-)definiteness of Tensors

As the application of Theorem 2.1, a checkable sufficient condition for the positive (semi-)definiteness of tensors is given in this section.

Definition 3.1. Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in C^{[m, n]}$ with $n \ge 2$, and S be a nonempty proper subset of N. Then $\mathcal{A}$ is called an $S - DSDD_0$ $(S - DSDD)$ *tensor* if the following four statements hold:

(I) for each $i \in S$, $a_{i \ldots i} \ge (>) r_i^{\Delta^S}(\mathcal{A})$;

(II) for each $i \in \overline{S}$, $a_{i \ldots i} \ge (>) r_i^{\Delta^{\overline{S}}}(\mathcal{A})$;

(III) for each $i \in S$, $j \in \overline{S}$,

$$(a_{i \ldots i} - r_i^{\Delta^S}(\mathcal{A}))(a_{j \ldots j} - r_j^{\overline{\Delta^S}}(\mathcal{A})) \ge (>) r_i^{\overline{\Delta^S}}(\mathcal{A}) r_j^{\Delta^S}(\mathcal{A}), \text{ or } a_{i \ldots i} > r_i(\mathcal{A});$$

(IV) for each $i \in \overline{S}$, $j \in S$,

$$(a_{i\cdots i} - r_i^{\Delta^{\overline{S}}}(\mathcal{A}))(a_{j\cdots j} - r_j^{\overline{\Delta^{\overline{S}}}}(\mathcal{A})) \geq (>) r_i^{\overline{\Delta^{\overline{S}}}}(\mathcal{A}) r_j^{\Delta^{\overline{S}}}(\mathcal{A}), \text{ or } a_{i\cdots i} > r_i(\mathcal{A}).$$

Theorem 3.2. *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in R^{[m,n]}$ *with* $n \geq 2$, *and* S *be a nonempty proper subset of* N. *If* $\mathcal{A}$ *is an even-order symmetric* $S - DSDD$ $(S - DSDD_0)$ *tensor with* $a_{k\cdots k} > (\geq) 0$ *for all* $k \in N$, *then* $\mathcal{A}$ *is positive* (*semi*-)*definite*.

Proof. Without loss of generality, we only need to prove that $\mathcal{A}$ is positive definite (by another case, if $a_{k\cdots k} \geq 0$, we can prove that $\mathcal{A}$ is positive semi-definite). Let λ be an H-eigenvalue of $\mathcal{A}$. Suppose $\lambda \leq 0$. According to Theorem 2.1, we have $\lambda \in \Psi^S(\mathcal{A})$ which implies that there are $i_0, i_1 \in S$, $j_0, j_1 \in \overline{S}$ such that

$$\lambda \in \Psi_{i_0}^S(\mathcal{A}) \bigcup (\Psi_{i_0, j_0}^S(\mathcal{A}) \cap \Gamma_{i_0}(\mathcal{A})),$$

or

$$\lambda \in \Psi_{j_1}^{\overline{S}}(\mathcal{A}) \bigcup (\Psi_{j_1, i_1}^{\overline{S}}(\mathcal{A}) \cap \Gamma_{j_1}(\mathcal{A})).$$

Without loss of generality, we assume that

$$\lambda \in \Psi_{i_0}^S(\mathcal{A}) \bigcup (\Psi_{i_0, j_0}^S(\mathcal{A}) \cap \Gamma_{i_0}(\mathcal{A})),$$

that is,

$$|\lambda - a_{i_0\cdots i_0}| \leq r_{i_0}^{\Delta^S}(\mathcal{A}),$$

or

$$(|\lambda - a_{i_0\cdots i_0}| - r_{i_0}^{\Delta^S}(\mathcal{A}))(|\lambda - a_{j_0\cdots j_0}| - r_{j_0}^{\overline{\Delta^S}}(\mathcal{A})) \leq r_{i_0}^{\overline{\Delta^S}}(\mathcal{A}) r_{j_0}^{\Delta^S}(\mathcal{A}),$$

$$|\lambda - a_{i_0\cdots 0}| \leq r_{i_0}(\mathcal{A}).$$

On the other hand, since $\mathcal{A}$ is an $S-DSDD$ tensor with $a_{k\cdots k}>0$ for all $k\in N$, we have that for $i_0\in S$, $j_0\in\overline{S}$,

$$|\lambda-a_{i_0\cdots i_0}|\geq|a_{i_0\cdots i_0}|>r_{i_0}^{\Delta^S}(\mathcal{A}),$$

or

$$(|\lambda-a_{i_0\cdots i_0}|-r_{i_0}^{\Delta^S}(\mathcal{A}))(|\lambda-a_{j_0\cdots j_0}|-r_{j_0}^{\Delta^{\overline{S}}}(\mathcal{A}))$$

$$\geq(a_{i_0\cdots i_0}-r_{i_0}^{\Delta^S}(\mathcal{A}))(a_{j_0\cdots j_0}-r_{j_0}^{\Delta^{\overline{S}}}(\mathcal{A}))>r_{i_0}^{\Delta^{\overline{S}}}(\mathcal{A})r_{j_0}^{\Delta^S}(\mathcal{A}),$$

$$|\lambda-a_{i_0\cdots i_0}|\geq|a_{i_0\cdots i_0}|>r_{i_0}(\mathcal{A}).$$

This leads to a contradiction. Hence, $\lambda>0$, and $\mathcal{A}$ is positive definite. The conclusion follows. □

4. An Upper Bound for the Spectral Radius of Nonnegative Tensors

On the basis of the results in Section 2, we give an upper bound for nonnegative tensors in this section. Before that, we introduce some results of nonnegative tensors [2, 33], which are generalized from nonnegative matrices.

Theorem 4.1 [2, Theorem 2.3]**.** *If* $\mathcal{A}$ *is a nonnegative tensor of order m dimension n, then the spectral radius* $\rho(\mathcal{A})$ *is an eigenvalue with a nonnegative eigenvector* $x\neq 0$ *corresponding to it, where*

$$\rho(\mathcal{A})=\max\{|\lambda|:\lambda\in\sigma(A)\}.$$

Theorem 4.1 is called the *Perron-Frobenius theorem* for nonnegative tensors.

Lemma 4.2 [33, Lemma 3.2]**.** *If* $\mathcal{A}$ *is a nonnegative tensor of order m dimension n, then*

$$\rho(\mathcal{A})\geq\max_{i\in N}a_{i\cdots i}.$$

Meanwhile, an upper bound was given in [2].

Theorem 4.3 [2, Lemma 5.2]**.** *Let* $\mathcal{A} = (a_{i_1 \cdots i_m}) \in R^{[m,n]}$ *be nonnegative. Then*

$$\rho(\mathcal{A}) \leq \max_{i \in N} \sum_{i_2, \ldots, i_m \in N} a_{ii_2 \cdots i_m} = \max_{i \in N} R_i(\mathcal{A}).$$

Theorem 4.4 [33, Theorem 3.3]**.** *Let* $\mathcal{A} = (a_{i_1 \cdots i_m}) \in R^{[m,n]}$ *be nonnegative with* $n \geq 2$. *Then*

$$\rho(\mathcal{A}) \leq w = \max_{i, j \in N, i \neq j} \frac{1}{2} \{a_{i \cdots i} + a_{j \cdots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\},$$

where

$$\Delta_{i,j}(\mathcal{A}) = (a_{i \cdots i} - a_{j \cdots j} + r_i^j(\mathcal{A}))^2 + 4a_{ij \cdots j} r_j(\mathcal{A}).$$

Theorem 4.5 [33, Theorem 3.4]**.** *Let* $\mathcal{A} = (a_{i_1 \cdots i_m}) \in R^{[m,n]}$ *be nonnegative with* $n \geq 2$, *S be a nonempty proper subset of N. Then*

$$\rho(\mathcal{A}) \leq \phi = \max\{\phi^S, \phi^{\bar{S}}\},$$

where

$$\phi^S = \max_{i \in S, j \in \bar{S}} \frac{1}{2} \{a_{i \cdots i} + a_{j \cdots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\}.$$

Theorem 4.6 [33, Theorem 3.5]**.** *Let* $\mathcal{A} = (a_{i_1 \cdots i_m}) \in R^{[m,n]}$ *be nonnegative with* $n \geq 2$, *S be a nonempty proper subset of N. Then*

$$\phi \leq \omega \leq \max_{i \in N} R_i(\mathcal{A}).$$

Based on Theorem 2.1, a new sharp upper bound of $\rho(\mathcal{A})$ is established by the following theorem:

Theorem 4.7. *Let* $\mathcal{A} = (a_{i_1 \cdots i_m}) \in R^{[m,n]}$ *be nonnegative with* $n \geq 2$, *S be a nonempty proper subset of N. Then*

$$\rho(\mathcal{A}) \leq \psi = \max\{\psi^S, \psi^{\overline{S}}, W^S, W^{\overline{S}}\},$$

where $\psi^S = \max_{i \in S}\{a_{i \cdots i} + r_i^{\Delta^S}(\mathcal{A})\}$,

$$W^S = \max_{j \in \overline{S}} \min_{i \in S}\left\{\frac{1}{2}\left(a_{i \cdots i} + a_{j \cdots j} + r_i^{\Delta^S}(\mathcal{A}) + r_j^{\overline{\Delta^S}}(\mathcal{A}) + \Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A})\right), R_i(\mathcal{A})\right\},$$

and $\Lambda_{i,j}(\mathcal{A}) = (a_{i \cdots i} - a_{j \cdots j} + r_i^{\Delta^S}(\mathcal{A}) - r_j^{\overline{\Delta^S}}(\mathcal{A}))^2 + 4 r_i^{\overline{\Delta^S}}(\mathcal{A}) r_j^{\Delta^S}(\mathcal{A})$.

Proof. Since $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$, by Theorem 2.1, if $\rho(\mathcal{A}) \in \bigcup_{i \in S} \Psi_i^S(\mathcal{A})$, then there is $i_0 \in S$ such that $|\rho(\mathcal{A}) - a_{i_0 \cdots i_0}| \leq r_{i_0}^{\Delta^S}(\mathcal{A})$. Furthermore, by Lemma 4.2, we have

$$\rho(\mathcal{A}) - a_{i_0 \cdots i_0} \leq r_{i_0}^{\Delta^S}(\mathcal{A}),$$

equivalently,

$$\rho(\mathcal{A}) \leq a_{i_0 \cdots i_0} + r_{i_0}^{\Delta^S}(\mathcal{A}) \leq \max_{i \in S}\{a_{i \cdots i} + r_i^{\Delta^S}(\mathcal{A})\}. \tag{4.1}$$

We can prove it similarly that if $\rho(\mathcal{A}) \in \bigcup_{i \in \overline{S}} \Psi_i^{\overline{S}}(\mathcal{A})$, then there is $i_0 \in \overline{S}$ such that

$$\rho(\mathcal{A}) \leq a_{i_0 \cdots i_0} + r_{i_0}^{\Delta^{\overline{S}}}(\mathcal{A}) \leq \max_{i \in \overline{S}}\{a_{i \cdots i} + r_i^{\Delta^{\overline{S}}}(\mathcal{A})\}. \tag{4.2}$$

For the case that $\rho(\mathcal{A}) \in \bigcup_{\substack{i \in S \\ j \in \overline{S}}} (\Psi_{i,j}^S(\mathcal{A}) \cap \Gamma_i(\mathcal{A}))$, there are $p \in S$ and $q \in \overline{S}$ such that

$$|\,\rho(\mathcal{A}) - a_{p\cdots p}\,| \le r_p(\mathcal{A}), \tag{4.3}$$

and

$$(|\,\rho(\mathcal{A}) - a_{p\cdots p}\,| - r_p^{\Delta^S}(\mathcal{A}))(|\,\rho(\mathcal{A}) - a_{q\cdots q}\,| - r_q^{\overline{\Delta^S}}(\mathcal{A}))$$

$$\le r_p^{\overline{\Delta^S}}(\mathcal{A}) r_q^{\Delta^S}(\mathcal{A}). \tag{4.4}$$

By the inequality (4.3), we can get that

$$\rho(\mathcal{A}) \le a_{p\cdots p} + r_p(\mathcal{A}) = R_p(\mathcal{A}). \tag{4.5}$$

On the other hand, solving $\rho(\mathcal{A})$ in inequality (4.4), we can get

$$\rho(\mathcal{A}) \le \frac{1}{2}(a_{p\cdots p} + a_{q\cdots q} + r_p^{\Delta^S}(\mathcal{A}) + r_q^{\overline{\Delta^S}}(\mathcal{A}) + \Lambda_{p,q}^{\frac{1}{2}}(\mathcal{A})), \tag{4.6}$$

where

$$\Lambda_{p,q}(\mathcal{A}) = (a_{p\cdots p} - a_{q\cdots q} + r_p^{\Delta^S}(\mathcal{A}) - r_q^{\overline{\Delta^S}}(\mathcal{A}))^2 + 4 r_p^{\overline{\Delta^S}}(\mathcal{A}) r_q^{\Delta^S}(\mathcal{A}).$$

Combining inequality (4.5) with inequality (4.6), we have

$$\rho(\mathcal{A}) \le \min\left\{\frac{1}{2}(a_{p\cdots p} + a_{q\cdots q} + r_p^{\Delta^S}(\mathcal{A}) + r_q^{\overline{\Delta^S}}(\mathcal{A}) + \Lambda_{p,q}^{\frac{1}{2}}(\mathcal{A})),\ R_p(\mathcal{A})\right\},$$

that is,

$$\rho(\mathcal{A}) \le \max_{j\in\overline{S}} \min_{i\in S}\left\{\frac{1}{2}(a_{i\cdots i} + a_{j\cdots j} + r_i^{\Delta^S}(\mathcal{A}) + r_j^{\overline{\Delta^S}}(\mathcal{A}) + \Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A})),\ R_i(\mathcal{A})\right\}. \tag{4.7}$$

where

$$\Lambda_{i,j}(\mathcal{A}) = (a_{i\cdots i} - a_{j\cdots j} + r_i^{\Delta^S}(\mathcal{A}) - r_j^{\overline{\Delta^S}}(\mathcal{A}))^2 + 4 r_i^{\overline{\Delta^S}}(\mathcal{A}) r_j^{\Delta^S}(\mathcal{A}).$$

Similarly if

$$\rho(\mathcal{A}) \in \bigcup_{\substack{i\in\overline{S}\\ j\in S}} (\Psi_{i,j}^{\overline{S}}(\mathcal{A}) \cap \Gamma_i(\mathcal{A})),$$

that is,

$$\rho(\mathcal{A}) \leq \max_{j\in S} \min_{i\in\overline{S}} \left\{ \frac{1}{2}(a_{i\cdots i} + a_{j\cdots j} + r_i^{\Delta^{\overline{S}}}(\mathcal{A}) + r_j^{\overline{\Delta^{\overline{S}}}}(\mathcal{A}) + \overline{\Lambda}_{i,j}^{\frac{1}{2}}(\mathcal{A})),\ R_i(\mathcal{A}) \right\}, \tag{4.8}$$

where $\overline{\Lambda}_{i,j}(\mathcal{A}) = (a_{i\cdots i} - a_{j\cdots j} + r_i^{\Delta^{\overline{S}}}(\mathcal{A}) - r_j^{\overline{\Delta^{\overline{S}}}}(\mathcal{A}))^2 + 4r_i^{\overline{\Delta^{\overline{S}}}}(\mathcal{A}) r_j^{\Delta^{\overline{S}}}(\mathcal{A})$, the conclusion follows from inequalities (4.1), (4.2), (4.7) and inequality (4.8). □

Now we compare the upper bounds in Theorems 4.3, 4.4 and 4.5 with that in Theorem 4.7.

Theorem 4.8. *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in R^{[m,n]}$ *be nonnegative with* $n \geq 2$, *S be a nonempty proper subset of N. Then*

$$\psi \leq \phi \leq \omega \leq \max_{i\in N} R_i(\mathcal{A}). \tag{4.9}$$

Proof. It is obvious by Theorem 2.3 and Theorem 4.6. □

Example 4.9. Consider the nonnegative tensor

$$\mathcal{A} = [A(:,:,1),\ A(:,:,2),\ A(:,:,3)] \in R^{[3,3]},$$

where

$$A(:,:,1) = \begin{pmatrix} 0.0900 & 0.0606 & 0.5294 \\ 0.3209 & 0.7257 & 0.8300 \\ 0.5114 & 0.5566 & 0.8588 \end{pmatrix},$$

$$A(:,\ :,\ 2) = \begin{pmatrix} 0.7890 & 0.7522 & 0.2699 \\ 0.3178 & 0.1099 & 0.5246 \\ 0.4522 & 0.1097 & 0.9727 \end{pmatrix},$$

$$A(:,\ :,\ 3) = \begin{pmatrix} 0.7104 & 0.8504 & 0.2554 \\ 0.3119 & 0.9116 & 0.0887 \\ 0.2915 & 0.6393 & 0.8383 \end{pmatrix}.$$

We can compute the bounds with $S = \{1\},\ \overline{S} = \{2,\ 3\},$ then

$$\max_{i \in N} R_i(\mathcal{A}) = 5.2303, \quad \omega = 5.2074, \quad \phi = 5.1457, \quad \psi = 4.7611.$$

It is easy to see that the upper bound in Theorem 4.7 is sharper than those in Theorems 4.3, 4.4 and 4.5.

Remark 4.10. How to pick S to make the upper bound as sharper as possible is very interesting, but difficult when the dimension of the tensor $\mathcal{A}$ is large. In future, we will deal with this problem.

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References

[1] L. Qi, Eigenvalues of a real supersymmetric tensor, J. Symbolic Comput. 40 (2005), 1302-1324.

[2] Y. Yang and Q. Yang, Further results for Perron-Frobenius theorem for nonnegative tensors, SIAM. J. Matrix Anal. Appl. 31 (2010), 2517-2530.

[3] L. H. Lim, Singular values and eigenvalues of tensors: a variational approach, CAMSAP'05: Proceeding of the IEEE International Workshop on Computational Advances in MultiSensor Adaptive Processing 2005, pp. 129-132.

[4] L. Qi, H^+-eigenvalues of Laplacian and signless Laplacian tensors, Commun. Math. Sci. 12 (2014), 1045-1064.

[5] Q. Yang and Y. Yang, Further results for Perron-Frobenius theorem for nonnegative tensors II, SIAM. J. Matrix Anal. Appl. 32 (2011), 1236-1250.

[6] N. K. Bose and A. R. Modaress, General procedure for multivariable polynomial positivity with control applications, IEEE Trans. Automat. Control AC21 (1976), 596-601.

[7] D. Cartwright and B. Sturmfels, The number of eigenvalues of a tensor, Linear Algebra Appl. DOI:10.1016/j.laa.2011.05.040.

[8] Y. Chen, Y. Dai, D. Han and W. Sun, Positive semidefinite generalized diffusion tensor imaging via quadratic semidefinite programming, SIAM J. Imaging Sci. 6 (2013), 1531-1552.

[9] K. C. Chang, K. Pearson and T. Zhang, Perron-Frobenius theorem for nonnegative tensors, Commun. Math. Sci. 6 (2008), 507-520.

[10] L. De Lathauwer, B. D. Moor and J. Vandewalle, On the best rank-1 and rank-$(R_1, R_2, ..., R_N)$ approximation of higher-order tensors, SIAM J. Matrix Anal. Appl. 21 (2000), 1324-1342.

[11] W. Ding, L. Qi and Y. Wei, M-tensors and nonsingular M-tensors, Linear Algebra Appl. 439 (2013), 3264-3278.

[12] S. Friedland, S. Gaubert and L. Han, Perron-Frobenius theorem for nonnegative multilinear forms and extensions, (2010). http://arxiv.org/abs/0905.1626v3.

[13] S. Hu, Z. Huang, H. Ni and L. Qi, Positive definiteness of diffusion kurtosis imaging, Inverse Probl. Imaging 6 (2012), 57-75.

[14] S. Hu, Z. Huang, C. Ling and L. Qi, E-determinants of tensors, 2011. arXiv:1109.0348v3, math.NA.

[15] S. Hu and L. Qi, Algebraic connectivity of an even uniform hypergraph, J. Comb. Optim. 24 (2012), 564-579.

[16] S. Hu and L. Qi, The eigenvectors associated with the zero eigenvalues of the Laplacian and signless Laplacian tensors of a uniform hypergraph, Discrete Appl. Math. 169 (2014), 140-151.

[17] S. Hu, L. Qi and J. Shao, Cored hypergraphs, power hypergraphs and their Laplacian eigenvalues, Linear Algebra Appl. 439 (2013), 2980-2998.

[18] S. Hu, L. Qi and J. Xie, The largest Laplacian and signless Laplacian eigenvalues of a uniform hyper-graph, 2013. arXiv:1304.1315.

[19] E. I. Jury and M. Mansour, Positivity and nonnegativity conditions of a quartic equation and related problems, IEEE Trans. Automat. Control AC26 (1981), 444-451.

[20] W. H. Ku, Explicit criterion for the positive definiteness of a general quartic form, IEEE Trans. Automat. Control AC10 (1965), 372-373.

[21] C. Q. Li, F. Wang, J. X. Zhao, Y. Zhu and Y. T. Li, Criterions for the positive definiteness of real supersymmetric tensors, J. Comput. Appl. Math. 255 (2014), 1-14.

[22] Q. Ni, L. Qi and F. Wang, An eigenvalue method for the positive definiteness identification problem, IEEE Trans. Automat. Control 53 (2008), 1096-1107.

[23] L. Qi, Hankel tensors: associated Hankel matrices and Vandermonde decomposition, Commun. Math. Sci. 13 (2015), 113-125.

[24] L. Qi, J. Shao and Q. Wang, Regular uniform hypergraphs, *s*-cycles, *s*-paths and their largest Laplacian *H*-eigenvalues, Linear Algebra Appl. 443 (2014), 215-227.

[25] L. Qi, C. Xu and Y. Xu, Nonnegative tensor factorization, completely positive tensors and a hierarchically elimination algorithm, SIAM J. Matrix Anal. Appl. (to appear).

[26] L. Qi, G. Yu and E. X. Wu, Higher order positive semi-definite diffusion tensor imaging, SIAM J. Imaging Sci. 3 (2010), 416-433.

[27] L. Qi, G. Yu and Y. Xu, Nonnegative diffusion orientation distribution function, J. Math. Imaging Vision 45 (2013), 103-113.

[28] L. Qi and Y. S. Song, An even order symmetric *B* tensor is positive definite, Linear Algebra Appl. 457 (2014), 303-312.

[29] Y. Wang, L. Qi and X. Zhang, A practical method for computing the largest *M*-eigenvalue of a fourth-order partially symmetric tensor, Numer. Linear Algebra Appl. 16 (2009), 589-601.

[30] L. Zhang, L. Qi and G. Zhou, *M*-tensors and some applications, SIAM J. Matrix Anal. Appl. 35 (2014), 437-452.

[31] S. Geršgorin, Über die abgrenzung der eigenwerte einer matrix, Izv. Akad. Nauk SSSR Ser. Mat. 1 (1931), 749-754.

[32] R. S. Varga, Geršgorin and His Circles, Springer-Verlag, Berlin, 2004.

[33] C. Q. Li, Y. T. Li and X. Kong, New eigenvalue inclusion sets for tensors, Numer. Linear Algebra Appl. 21 (2014), 39-50.

[34] A. Brauer, Limits for the characteristic roots of a matrix II, Duke Math. J. 14 (1947), 21-26.

[35] C. Q. Li, A. Q. Jiao and Y. T. Li, An *S*-type eigenvalue localization set for tensors, Linear Algebra Appl. 493 (2016), 493-483.

[36] C. Q. Li and Y. T. Li, An eigenvalue localization set for tensor with applications to determine the positive (semi-)definiteness of tensors, Linear and Multilinear Algebra. DOI:10.1080/03081087.2105.1049582.