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ON THE ASSOCIATED PRIME IDEALS OF GENERALIZED d-COHOMOLOGY MODULES

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Abstract

Let M and N be R-modules, where R is a commutative Noetherian ring with identity element. We provide conditions on modules so that associated prime ideals of generalized d-cohomology module $H_d^i(M, N)$, where d is a nonnegative integer, are finite.

1. Introduction

Throughout this note, R denotes a Noetherian (commutative with nonzero identity) ring and d a nonnegative integer. Let $\mathcal{C}(R)$ denote the category of R-modules, and M be a finitely generated R-module. The singular set $S_k^*(M)$ $(k \geq 0)$ contains all prime ideals of R satisfying $\operatorname{depth}(M_{\mathfrak{p}}) + \dim(R/\mathfrak{p}) \leq k$. Let

$$\sum = \{\mathfrak{a} : \mathfrak{a} \text{ is an ideal of } R \text{ with } \dim(R/\mathfrak{a}) \leq d\}.$$

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Then with the reverse inclusion, the set $\sum$ is a system of ideals of R in the sense of [2, p. 21]. Following [1], for an R-module M, let $L_d(M) = \{m \in M \mid \exists \mathfrak{a} \in \sum, \mathfrak{a}m = 0\}$, and for $i \geq 0$, $H_d^i(-)$ be the ith right derived functor of $L_d(-)$. In [9], the d-transform $T_d(M) = \varinjlim_{\mathfrak{a} \in \sum} Hom_R(\mathfrak{a}, M)$ on the category of R-modules was defined.

Now, we define $L_d(-, -), T_d(-, -) : \mathcal{C}(R) \times \mathcal{C}(R) \to \mathcal{C}(R)$ by

$$L_d(M, N) := \varinjlim_{\mathfrak{a} \in \sum} Hom_R(M/\mathfrak{a}M, N),$$

$$T_d(M, N) := \varinjlim_{\mathfrak{a} \in \sum} Hom_R(\mathfrak{a}M, N).$$

Also, for R-module M, let $H_d^i(M, -) := \mathcal{R}^i L_d(M, -)$ be defined for all nonnegative integers i. It is clear that

$$H_d^i(M, N) = \varinjlim_{\mathfrak{a} \in \sum} Ext_R^i(M/\mathfrak{a}M, N),$$

and call it to be the *ith generalized d-cohomology* module of M, N with support of dimension $\leq d$.

Banica and Stoia [1] have studied d-cohomology module $H_d^i(M)$. Zamani et al. [7-9] have explored $T_d(M, N)$ and $H_d^i(M, N)$ intimately. The aim of this paper is to study the associated prime ideals of $H_d^i(M, N)$ and $T_d(M, N)$ whenever M, N are finitely generated R-modules. Also, this provides some results on sets $Supp(T_d(M, N))$ and $Supp(H_d^1(M, N))$.

2. Preliminaries

In this paper, the associated prime ideals $T_d(M, N)$ and $H_d^i(M, N)$ are studied. It is obtained that $Ass_R(T_d(M, N))$ and $Ass_R(H^i(M, N))$ are finite under certain conditions for all $i \geq 0$. It is noted that a finite dimensional Noetherian ring R is said to be *biequidimensional* if $\dim(R/\mathfrak{p}) + \dim(R_{\mathfrak{p}}) = \dim(R)$ for all $\mathfrak{p} \in Spec(R)$, and $\dim(R/\mathfrak{p}) = \dim(R)$, for all $\mathfrak{p} \in Ass(R)$, where $Ass(R)$ denotes the set of all associated prime ideals of R.

The results, collected in Proposition 1, include some connections of the generalized d-transform functors and modules.

Proposition 1. *Let M, N be two R-modules. If M is finitely generated, then*

(i) $T_d(T_d(M, N)) \cong T_d(M, N)$.

(ii) $T_d(Hom_R(M, N)) \cong Hom_R(M, T_d(N))$.

(iii) $T_d(Hom_R(M, N)) \cong T_d(M, N)$.

Proof. (i) Using the definition, [6, Theorem 2.75] and [4, Satz 3], tit follows that:

$$\begin{aligned} T_d(T_d(M, N)) &= \varinjlim_{\mathfrak{a}\in\Sigma} Hom_R(\mathfrak{a}, T_d(M, N)) \\ &\cong \varinjlim_{\mathfrak{a}\in\Sigma}(Hom_R(\mathfrak{a}, \varinjlim_{\mathfrak{b}\in\Sigma} Hom_R(\mathfrak{b}M, N))) \\ &\cong \varinjlim_{\mathfrak{a}\in\Sigma}(\varinjlim_{\mathfrak{b}\in\Sigma} Hom_R(\mathfrak{a}, Hom_R(\mathfrak{b}M, N))) \\ &\cong \varinjlim_{\mathfrak{a}\in\Sigma}(\varinjlim_{\mathfrak{b}\in\Sigma} Hom_R(\mathfrak{a} \otimes \mathfrak{b}M, N)) \\ &\cong \varinjlim_{\mathfrak{b}\in\Sigma}(\varinjlim_{\mathfrak{a}\in\Sigma} Hom_R(\mathfrak{b}M, Hom_R(\mathfrak{a}, N))) \\ &\cong \varinjlim_{\mathfrak{b}\in\Sigma}(Hom_R(\mathfrak{b}M, \varinjlim_{\mathfrak{a}\in\Sigma} Hom_R(\mathfrak{a}, N))) \\ &\cong \varinjlim_{\mathfrak{b}\in\Sigma} Hom_R(\mathfrak{b}M, T_d(N)) \cong T_d(M, T_d(N)). \end{aligned}$$

Now, using [9, Theorem 2.15], we have

$$T_d(T_d(M, N)) \cong T_d(M, N).$$

(ii) By using the definition and part (i), we obtain

$$\begin{aligned} T_d(Hom_R(M, N)) &= \varinjlim_{\mathfrak{a}\in\Sigma} Hom_R(\mathfrak{a}, Hom_R(M, N)) \\ &\cong \varinjlim_{\mathfrak{a}\in\Sigma} Hom_R(\mathfrak{a}\otimes M, N) \\ &\cong \varinjlim_{\mathfrak{a}\in\Sigma} Hom_R(M, Hom_R(\mathfrak{a}, N)) \\ &\cong Hom_R(M, \varinjlim_{\mathfrak{a}\in\Sigma} Hom_R(\mathfrak{a}, N)) \\ &\cong Hom_R(M, T_d(N)). \end{aligned}$$

(iii) The exact sequence

$$0 \to L_d(M, N) \to Hom_R(M, N) \to T_d(M, N) \xrightarrow{\alpha} H_d^1(M, N)$$

provides the following exact sequence

$$0 \to L_d(M, N) \to Hom_R(M, N) \to T_d(M, N) \to Im\,\alpha \to 0. \qquad (\sharp)$$

Since $L_d(L_d(M, N)) = L_d(M, N)$ and $L_d(Im\,\alpha) = Im\,\alpha$, by [9, Corollary 2.6], $T_d(L_d(M, N)) = 0$ and $T_d(Im\,\alpha) = 0$. Hence, by part (i) on applying the functor $T_d(-)$ to the exact sequence $(\sharp)$, we have

$$T_d(Hom_R(M, N)) \cong T_d(T_d(M, N)) \cong T_d(M, N).$$

Now, by Proposition 1, the result follows. □

Theorem 2. *Let M, N be two finitely generated R-modules. Then* $Ass_R(T_d(M, N)) = Supp(M) \cap Ass_R(N/L_d(N))$ *and so* $Ass_R(T_d(M, N))$ *is finite.*

Proof. By [8, Theorem 1], $Ass_R(T_d(N)) = Ass_R(N/L_d(N))$. Now, by Proposition 1,

$$\begin{aligned} Ass_R(T_d(M, N)) &= Ass_R(T_d(Hom_R(M, N))) \\ &= Ass_R(Hom_R(M, T_d(N))) \end{aligned}$$

$$= Supp(M) \cap Ass_R(T_d(N))$$

$$= Supp(M) \cap Ass_R(N/L_d(N)).$$

Clearly, if N is a finitely generated R-module, then $Ass_R(N/L_d(N))$ is finite and so $Ass_R(T_d(M, N))$ is finite. □

Corollary 3. *Let* M, N *be two finitely generated* R*-modules. Then* $Supp(T_d(M, N)) \subseteq Supp(N/L_d(N))$.

Proof. Let $\mathfrak{p} \in Supp(T_d(M, N))$. Then there exists $\mathfrak{q} \in Ass_R(T_d(M, N))$ so that $\mathfrak{q} \subseteq \mathfrak{p}$. Using Theorem 2, $\mathfrak{q} \in Ass_R(N/L_d(N))$ and so there exists $0 \neq n + L_d(N) \in N/L_d(N)$ such that $\mathfrak{q} = Ann_R(n + L_d(N))$. It is clear that $\frac{n + L_d(N)}{1} \neq 0$ in the $R_{\mathfrak{p}}$-module $(N/L_d(N))_{\mathfrak{p}}$. Hence, $\mathfrak{p} \in Supp(N/L_d(N))$ and so $Supp(T_d(M, N)) \subseteq Supp(N/L_d(N))$. □

Corollary 4. *Let* M, N *be two finitely generated* R*-modules. If* $Ext^1_R(M, N) = 0$, *then* $Supp(H^1_d(M, N)) \subseteq Supp(N/L_d(N))$.

Proof. Using Proposition 1 and [9, Corollary 2], we can easily show that

$$Supp(T_d(M, N)) = Supp(T_d(\mathrm{Hom}_R(M, N)))$$

$$= Supp(Hom_R(M, N)/L_d(Hom_R(M, N))).$$

Now, the short exact sequence

$$0 \to Hom_R(M, N)/L_d(Hom_R(M, N)) \to T_d(M, N) \to H^1_d(M, N) \to 0$$

yields

$$Supp(T_d(M, N)) = Supp(Hom_R(M, N)/L_d(Hom_R(M, N)))$$

$$\cup\, Supp(H^1_d(M, N)).$$

Hence, by Corollary 3,

$$Supp(H^1_d(M, N)) \subseteq Supp(T_d(M, N)) \subseteq Supp(N/L_d(N)).$$ □

By definition, we can easily show that

$$Ass_R(L_d(M,\ N)) = Ass_R(Hom_R(M,\ L_d(N)))$$

$$= Supp(M) \bigcap Ass_R(L_d(N)).$$

Below is the following general theorem.

Theorem 5. *Let* $M,\ N$ *be two R-modules and t be a positive integer. If M is finitely generated, then*

$$Ass_R(H_d^t(M,\ N)) \subseteq \bigcup_{i=0}^{t} Ass_R(Ext_R^i(M,\ H_d^{t-i}(N))).$$

Proof. By [6, Theorem 10.47], there is a convergent spectral sequence

$$E_2^{r,s} \coloneqq Ext_R^r(M,\ H_d^s(N)) \underset{r}{\Rightarrow} H_d^{r+s}(M,\ N).$$

For all $i \geq 2$, we consider the exact sequence

$$0 \to Ker\, d_i^{0,t} \to E_i^{0,t} \xrightarrow{d_i^{0,t}} E_i^{i,t-i+1}. \qquad (*)$$

Since $E_i^{0,t} = Ker\, d_{i-1}^{0,t} / Im\, d_{i-1}^{1-i,t+i-2}$ and $E_t^{i,j} = 0$ for all $j < 0$, we may use $(*)$ to obtain

$$Ker\, d_{t+2}^{i,t-i} \cong E_{t+2}^{i,t-i} \cong \cdots \cong E_\infty^{i,t-i}$$

for all $0 \leq i \leq t$. There exists a finite filtration

$$0 = \varphi^{t+1}H^t \subseteq \varphi^t H^t \subseteq \cdots \subseteq \varphi^1 H^t \subseteq \varphi^0 H^t = H_d^t(M,\ N)$$

such that $E_\infty^{i,t-i} \cong \varphi^i H^t / \varphi^{i+1} H^t$ for all $0 \leq i \leq t$. Now, the exact sequence

$$0 \to \varphi^{i+1}H^t \to \varphi^i H^t \to E_\infty^{i,t-i} \to 0$$

$(0 \leq i \leq t)$ in conjunction with

$$E_\infty^{i,t-i} \cong Ker\, d_{t+2}^{i,t-i} \subseteq Ker\, d_2^{i,t-i} \subseteq E_2^{i,t-i}$$

yields

$$\begin{aligned} Ass_R(\varphi^i H^t) &\subseteq Ass_R(\varphi^{i+1}H^t) \cup Ass_R(E_\infty^{i,t-i}) \\ &\subseteq Ass_R(\varphi^{i+1}H^t) \cup Ass_R(E_2^{i,t-i}) \\ &\Rightarrow Ass_R(H_d^t(M, N)) \subseteq Ass_R(\varphi^1 H^t) \cup Ass_R(E_2^{0,t}) \\ &\subseteq Ass_R(\varphi^2 H^t) \cup Ass_R(E_2^{1,t-1}) \cup Ass_R(E_2^{0,t}) \subseteq \cdots \\ &\subseteq Ass_R(0) \cup Ass_R(E_2^{0,t}) \cup Ass_R(E_2^{1,t-1}) \cup \cdots \cup Ass_R(E_2^{t,0}). \end{aligned}$$

Then

$$Ass_R(H_d^t(M, N)) \subseteq \bigcup_{i=0}^t Ass_R(E_2^{i,t-i}) = \bigcup_{i=0}^t Ass_R(Ext_R^i(M, H_d^{t-i}(N)))$$

and the proof is complete. □

Corollary 6. *Let R be a ring, quotient of a regular biequidimensional ring, M, N be two finitely generated R-modules and t be a positive integer. If* $\dim(S_{t+d}^*(N)) \leq d$, *then* $Ass_R(H_d^t(M, N))$ *is finite.*

Proof. By [1, Theorem of finiteness], $H_d^i(N)$ is finitely generated for all $i \leq t$ and so $Ext_R^i(M, H_d^{t-i}(N))$ is finitely generated. Hence, by Theorem 5, $Ass_R(H_d^t(M, N))$ is finite for all $t > 0$. □

Corollary 7. *Let* M, N *be two finitely generated R-modules and t be a positive integer. If* $L_d(N) = N$, *then* $Ass_R(H_d^t(M, N))$ *is finite.*

Proof. By [9, Corollary 2.5], we have $H_d^i(N) = 0$ for all $i \geq 1$. Hence, by Theorem 5, we have

$$\begin{aligned} Ass_R(H_d^t(M, N)) &\subseteq \bigcup_{i=0}^{t} Ass_R(Ext_R^i(M, H_d^{t-i}(N))) \\ &= Ass_R(Ext_R^t(M, L_d(N))). \end{aligned}$$

Since *M* and $L_d(N)$ are finitely generated, the proof is complete. □

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