



## **PASTING AND REVERSING APPROACH TO MATRIX THEORY**

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### **Abstract**

The aim of this paper is to study some aspects of matrix theory through pasting and reversing using linear mappings. We obtain new properties and new sets in matrix theory. In particular, we introduce new linear mappings: palindromicing and antipalindromicing mappings, which allow us to obtain palindromic and antipalindromic vectors and matrices.

### **1. Introduction**

Pasting and reversing as mathematical operations were introduced by the first author in [2] and after extended in [1, 3-7]. Recently, in [10] were studied some properties of palindromic and antipalindromic polynomials. In this paper, we obtain new results in the framework of elementary matrix

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theory which arise from links with pasting and reversing. In particular, we study pasting and reversing from linear mappings, proving some interesting results in matrix theory and introducing new linear mappings such as palindromicing and antipalindromicing mappings for vectors and matrices, among others<sup>1</sup>. For a complete theoretical background in matrix theory, see [8, 9].

We start considering the vector space  $V = \mathbb{K}^n$ , where  $\mathbb{K}$  is a field of characteristic zero, and we write  $W \leq V$  to say that  $W$  is a *subspace* of  $V$ . In this way,  $\mathcal{M}_{n \times m}(\mathbb{K})$  denotes the set of  $n \times m$  matrices with elements belonging to  $\mathbb{K}$ . We should keep in mind that when we write  $\mathcal{M}_{n \times m}$ , we mean  $\mathcal{M}_{n \times m}(\mathbb{K})$ . In the same way, for  $(i, j) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ , we know that  $\delta_{ii} = 1$  while  $\delta_{ij} = 0$  for all  $i \neq j$ . Finally, *floor* and *ceiling* functions, denoted by  $\lfloor \cdot \rfloor$  and  $\lceil \cdot \rceil$ , respectively, are defined as

$$\lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \leq x\}, \quad \lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geq x\}.$$

## 2. Main Results

We introduce the reversing mapping, denoted by  $\mathcal{R}$ , as follows:

$$\begin{aligned} \mathcal{R} : \quad V &\rightarrow V \\ (v_1, v_2, \dots, v_n) &\mapsto (v_n, \dots, v_2, v_1). \end{aligned}$$

**Proposition 2.1.** *The following statements hold:*

- (1)  $\mathcal{R} \in \text{Aut}(V)$ .
- (2) *The transformation matrix of  $\mathcal{R}$  is given by*

$$M_{\mathcal{R}} = (\delta_{i, n-j+1})_{n \times n}.$$

- (3) *Minimal and characteristic polynomials of  $\mathcal{R}$  are given, respectively, by*

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$$Q(\lambda) = \lambda^2 - 1, \quad P(\lambda) = (\lambda + 1)^{\left\lfloor \frac{n}{2} \right\rfloor} (\lambda - 1)^{\left\lceil \frac{n}{2} \right\rceil}.$$

$$(4) \ker(\mathcal{R} - id)^\perp = \ker(\mathcal{R} + id), \quad id(v) = v, \quad \forall v \in V.$$

$$(5) V = \ker(\mathcal{R} - id) \oplus \ker(\mathcal{R} + id).$$

$$(6) \dim \ker(\mathcal{R} - id) = \left\lceil \frac{n}{2} \right\rceil, \quad \dim \ker(\mathcal{R} + id) = \left\lfloor \frac{n}{2} \right\rfloor.$$

**Proof.** Assume  $v, w \in V$ ,  $\alpha \in K$  such that  $v = (v_1, \dots, v_n)$ ,  $w = (w_1, \dots, w_n)$ . By definition of reversing, we have  $\mathcal{R} : V \rightarrow V$  it follows that  $\mathcal{R}(v + w) = \mathcal{R}v + \mathcal{R}w$  and  $\mathcal{R}(\alpha v) = \alpha \mathcal{R}v$ . Owing to  $\mathcal{R} : V \rightarrow V$  and  $\mathcal{R}^2 v = v$ , it implies that  $\mathcal{R}$  is left-right invertible, then  $\mathcal{R}$  is an automorphism of  $V$  and

$$\mathcal{R}(v_1, v_2, \dots, v_{n-1}, v_n) = (v_1, v_2, \dots, v_{n-1}, v_n)(\delta_{i, n-j+1})_{n \times n}.$$

We observe that  $\mathcal{R}^2 = id$  and  $\mathcal{R} \neq id$ , therefore  $Q(\lambda) = \lambda^2 - 1$  is the minimal polynomial of  $\mathcal{R}$ , that is,  $Q(\delta_{i, n-j+1}) = \mathbf{0}_n \in \mathcal{M}_{n \times m}$  ( $\mathbf{0}_n$  is the zero matrix of size  $n \times n$ ). Now, assuming  $n = 2$ , we have that the characteristic polynomial of  $\mathcal{R}$  is given by  $P(\lambda) = (\lambda + 1)(\lambda - 1)$ , assuming  $n = 3$ , we obtain  $P(\lambda) = (\lambda + 1)(\lambda - 1)^2$ . Therefore, the characteristic polynomial of  $\mathcal{R}$  is obtained inductively and it is given by

$$P(\lambda) = \det(\delta_{i, n-j+1} - \lambda I_n) = \begin{cases} (\lambda + 1)^m (\lambda - 1)^m, & \text{for } n = 2m, \\ (\lambda + 1)^m (\lambda - 1)^{m+1}, & \text{for } n = 2m + 1. \end{cases}$$

That is, by definition of floor and ceiling functions, we conclude

$$P(\lambda) = (\lambda + 1)^{\left\lfloor \frac{n}{2} \right\rfloor} (\lambda - 1)^{\left\lceil \frac{n}{2} \right\rceil}.$$

Assume  $v \in \ker(\mathcal{R} - id)$  and  $w \in \ker(\mathcal{R} + id)$ , for instance  $\mathcal{R}v = v$ ,  $\mathcal{R}(w) = -w$  and  $v \cdot w = -v \cdot w = 0$ . In this way,  $\ker(\mathcal{R} - id)^\perp = \ker(\mathcal{R} + id)$ .

Due to  $\mathcal{R}^2 - id = (\mathcal{R} - id)(\mathcal{R} + id)$ , then  $V = \ker(\mathcal{R} - id) \oplus \ker(\mathcal{R} + id)$ , which implies that  $\dim \ker(\mathcal{R} - id) = \left\lceil \frac{n}{2} \right\rceil$ ,  $\dim \ker(\mathcal{R} + id) = \left\lfloor \frac{n}{2} \right\rfloor$  and  $\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor = n$ .  $\square$

**Remark 2.2.** Proposition 2.1 summarizes some results given in [5, Section 1], without the formalism of endomorphism. In particular,  $W_p := \ker(\mathcal{R} - id)$ ,  $W_a := \ker(\mathcal{R} + id)$  and  $\mathcal{R}$  is an endomorphism associated to a *permutation matrix*. Recall that  $A_\sigma$  is a permutation matrix, defined over a given  $\sigma \in S_n$ , whether its associated linear mapping  $\mathcal{R}_\sigma$  is given by

$$\mathcal{R}_\sigma : \begin{array}{ccc} V & \rightarrow & V \\ (v_1, \dots, v_n) & \mapsto & (v_{\sigma(1)}, \dots, v_{\sigma(n)}). \end{array}$$

Reversing corresponds to  $\mathcal{R}_\sigma = \mathcal{R}$ , where the permutation matrix is  $A_\sigma = M_\mathcal{R}$  and the permutation  $\sigma$  is given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ n & n-1 & n-2 & \cdots & 2 & 1 \end{pmatrix}.$$

To illustrate this formalism, we rewrite the following properties obtained in [5]:

- (1)  $\mathcal{R}^2(v) = v$ .
- (2)  $\mathcal{R}(av + bw) = a\mathcal{R}(v) + b\mathcal{R}(w)$ ,  $a, b \in K$ ,  $v, w \in V$ .
- (3)  $v \cdot w = \mathcal{R}(v) \cdot \mathcal{R}(w)$ .
- (4)  $\mathcal{R}(v \times w) = \mathcal{R}(w) \times \mathcal{R}(v)$ ,  $\forall v, w \in K^3$ .

**Definition 2.3.** A mapping is *palindromicing* (resp. *antipalindromicing*) whether it transforms any vector of a given vector space into a palindromic (resp. antipalindromic) vector. These mappings are called *canonical*, denoted

by  $\mathcal{F}_p$  and  $\mathcal{F}_a$ , respectively, whether they are linear mappings and for all  $v \in V$ , they satisfy  $v = (\mathcal{F}_p + \mathcal{F}_a)(v)$  and  $\mathcal{R}(v) = (\mathcal{F}_p - \mathcal{F}_a)(v)$ .

We can see that palindromicing and antipalindromicing mappings are epimorphism from  $V$  to  $\ker(\mathcal{R} + id)$  but they are not isomorphisms due to they are not monomorphisms. From now on we only consider canonical palindromicing and canonical antipalindromicing mappings, which will be called *palindromicing* and *antipalindromicing mappings*.

**Proposition 2.4.** *The following statements hold:*

(1)  $\mathcal{F}_p$  and  $\mathcal{F}_a$  are given by

$$\begin{aligned} \mathcal{F}_p : V &\rightarrow \ker(\mathcal{R} - id) & \mathcal{F}_a : V &\rightarrow \ker(\mathcal{R} + id) \\ v &\mapsto \frac{1}{2}(v + \mathcal{R}(v)), & v &\mapsto \frac{1}{2}(v - \mathcal{R}(v)). \end{aligned}$$

(2)  $\ker(\mathcal{F}_p) = im(\mathcal{F}_a)$ ,  $\ker(\mathcal{F}_a) = im(\mathcal{F}_p)$ .

(3) The companion matrices of  $\mathcal{F}_p$  and  $\mathcal{F}_a$  are  $M_{\mathcal{R}} + I_n$  and  $M_{\mathcal{R}} - I_n$ .

**Proof.** Consider  $v = (v_1, \dots, v_n)$ . We proceed according to each item:

(1) We can see that  $\mathcal{F}_p$  and  $\mathcal{F}_a$  are linear mappings due to  $\mathcal{R}$  and  $id$  are linear mappings. Now,

$$\mathcal{F}_p(v) = \frac{1}{2}(v_1 + v_n, v_2 + v_{n-1}, \dots, v_{n-1} + v_2, v_n + v_1)$$

is a palindromic vector,  $\mathcal{F}_a(v) = \frac{1}{2}(v_1 - v_n, v_2 - v_{n-1}, \dots, v_{n-1} - v_2, v_n - v_1)$  is an antipalindromic vector, for instance  $\mathcal{F}_p$  and  $\mathcal{F}_a$  are epimorphisms from  $V$  to  $\ker(\mathcal{R} - id)$  and from  $V$  to  $\ker(\mathcal{R} + id)$ . Furthermore, we see that  $(\mathcal{F}_p + \mathcal{F}_a)(v) = \mathcal{F}_p(v) + \mathcal{F}_a(v) = v$  and  $(\mathcal{F}_p - \mathcal{F}_a)(v) = \mathcal{F}_p(v) - \mathcal{F}_a(v) = \mathcal{R}(v)$ . Thus,  $\mathcal{F}_p$  is the palindromicing mapping and  $\mathcal{F}_a$  is the antipalindromicing mapping.

(2) Assume  $v \in \ker \mathcal{F}_p$  and  $w \in \ker \mathcal{F}_a$ . We see that  $\mathcal{F}_p(v) = 0$  implies that  $\mathcal{R}(v) = -v$ , thus  $v \in W_a = \ker(\mathcal{R} + id) = im(\mathcal{F}_a)$ . Similarly, we see that  $\mathcal{F}_a(w) = 0$  implies that  $\mathcal{R}(w) = w$ , thus  $w \in W_p = \ker(\mathcal{R} - id) = im(\mathcal{F}_p)$ .

(3) It follows directly from the companion matrices of the linear mappings  $\mathcal{R}$  and  $id$ .  $\square$

**Definition 2.5** (Pasting of vectors). Let  $V$ ,  $W$  and  $Z$  be  $\mathbb{K}$ -vector spaces. Consider  $v = (v_1, \dots, v_n) \in V$  and  $w = (w_1, \dots, w_n) \in W$ . Pasting is the mapping  $\mathcal{P}$  such that

$$\mathcal{P} : \begin{array}{ccc} V \times W & \rightarrow & Z \\ (v, w) & \mapsto & \mathcal{P}(v, w) = z, \end{array}$$

where  $z = (v_1, \dots, v_n, w_1, \dots, w_m)$ . Furthermore,

$$\mathcal{P}(V, W) := \{\mathcal{P}(v, w) : v \in V, w \in W\}.$$

**Theorem 2.6.** Consider  $V = \mathbb{K}^n$ ,  $W = \mathbb{K}^m$ ,  $Z = \mathbb{K}^{n+m}$ ,  $V' \leq Z$  and  $W' \leq Z$  generated by

$$\mathcal{B}_l = \{e_1^{n+m}, \dots, e_n^{n+m}\} \quad \text{and} \quad \mathcal{B}_r = \{e_{n+1}^{n+m}, \dots, e_{n+m}^{n+m}\},$$

respectively, where  $e_i^r$  is the  $i$ th vector of the canonical basis of  $\mathbb{K}^r$ . The mappings  $\varphi_1, \varphi_2, \varphi$  and  $\mathcal{S}$  are given by

$$\varphi_1 : V \rightarrow V', \quad \varphi_2 : W \rightarrow W',$$

$$v \rightarrow \varphi_1(v) = v', \quad v_i = v'_i, \quad \forall i \leq n, \quad w \rightarrow \varphi_2(w) = w', \quad w_i = w'_{n+i}, \quad \forall i \leq m,$$

$$\varphi : V \times W \rightarrow V' \times W', \quad \mathcal{S} : V' \times W' \rightarrow Z,$$

$$(v, w) \rightarrow \varphi(v, w) = (\varphi_1(v), \varphi_2(w)), \quad (v', w') \rightarrow \mathcal{S}(v', w') = v' + w'.$$

The following statements hold:

- (1) Mappings  $\varphi_1$  and  $\varphi_2$  are linear isomorphisms.
- (2) The transformation matrices of  $\varphi_1$  and  $\varphi_2$  are given by  $M_{\varphi_1} = (\delta_{i,j})_{n \times (n+m)}$  and  $M_{\varphi_2} = (\delta_{i,j-n})_{m \times (n+m)}$ , respectively.
- (3)  $\text{im}(\varphi_1) \oplus \text{im}(\varphi_2) = V$ .
- (4)  $(\text{im}(\varphi_1))^\perp = \text{im}(\varphi_2)$ .
- (5) The diagram

$$\begin{array}{ccc}
 V \times W & \xrightarrow{\mathcal{P}} & Z \\
 \downarrow \varphi = (\varphi_1, \varphi_2) & \nearrow S & \\
 V' \times W' & & 
 \end{array}$$

is commutative, i.e.,  $\mathcal{P} = S \circ \varphi$ .

**Proof.** We proceed according to each item:

- (1) Assume  $\alpha, \beta \in \mathbb{K}$ ,  $v_1, v_2 \in V$  and  $w_1, w_2 \in W$  such that  $v_1 = (v_{11}, \dots, v_{1n})$ ,  $v_2 = (v_{21}, \dots, v_{2n})$ ,  $w_1 = (w_{11}, \dots, w_{1n})$ ,  $w_2 = (w_{21}, \dots, w_{2n})$ . By definitions of  $\varphi_1$  and  $\varphi_2$ , we have that

$$\varphi_1(v_1) = (v_{11}, \dots, v_{1n}, \underbrace{0, \dots, 0}_{m \text{ times}}), \quad \varphi_1(v_2) = (v_{21}, \dots, v_{2n}, \underbrace{0, \dots, 0}_{m \text{ times}}),$$

$$\varphi_2(w_1) = (\underbrace{0, \dots, 0}_{n \text{ times}}, w_{11}, \dots, w_{1n}), \quad \varphi_2(w_2) = (\underbrace{0, \dots, 0}_{n \text{ times}}, w_{21}, \dots, w_{2n}).$$

Therefore,  $\varphi_1(\alpha v_1 + \beta v_2) = \alpha \varphi_1(v_1) + \beta \varphi_1(v_2)$  and  $\varphi_2(\alpha w_1 + \beta w_2) = \alpha \varphi_2(w_1) + \beta \varphi_2(w_2)$ . Now, due to  $\varphi_1(v) = \mathbf{0}_{V'}$  if and only if  $v = \mathbf{0}_V$  and  $\varphi_2(w) = \mathbf{0}_{W'}$  if and only if  $w = \mathbf{0}_W$ , we get  $\ker \varphi_1 = \{\mathbf{0}_V\}$  and  $\ker \varphi_2 = \{\mathbf{0}_W\}$ . Finally, for all  $v' \in V'$  and for all  $w' \in W'$ , we get that there exist  $v \in V$  and  $w \in W$  such that  $\varphi_1(v) \in V'$  and  $\varphi_2(w) \in W'$ , i.e.,  $\text{im}(\varphi_1) = V'$

and  $\text{im}(\varphi_2) = W'$ . Thus, we conclude that  $\varphi_1$  and  $\varphi_2$  are linear mappings, monomorphisms and epimorphisms.

(2) We see that for all  $v \in V$  and for all  $w \in W$ , we obtain

$$\varphi_1(v) = v(\delta_{i,j})_{n \times (n+m)} = vM_{\varphi_1}, \quad \varphi_2(w) = w(\delta_{i,j-n})_{m \times (n+m)} = wM_{\varphi_2}.$$

(3) By item (1), we have that  $\text{im}(\varphi_1) = V'$  and  $\text{im}(\varphi_2) = W'$ . We see that  $V' \cap W' = \{\mathbf{0}_Z\}$  and  $Z = \{\alpha v + \beta w : \alpha, \beta \in \mathbb{K}, v \in V, w \in W\}$ .

(4) Owing to  $v' \cdot w' = 0$  for all  $v' \in V'$  and  $w' \in W'$  and by previous item, we have that  $W'$  is the orthogonal subspace of  $V'$ .

(5) Let  $v \in V$  and  $w \in W$ . Then

$$\begin{aligned} (S \circ \varphi)(v, w) &= S(\varphi_1(v), \varphi_2(w)) \\ &= \varphi_1(v) + \varphi_2(w) \\ &= \mathcal{P}(v, w). \end{aligned}$$

Therefore,  $\mathcal{P} = S \circ \varphi$ . □

**Corollary 2.7.** *If  $v, w \in \mathbb{K}^n$ , then  $\varphi$  is a linear mapping and its transformation matrix is given by  $M_\varphi = [m_{ij}]_{n \times 4n}$ , where*

$$m_{ij} = \begin{cases} 1, & i = j, \\ 1, & i = j - 3n, \\ 0, & \text{otherwise.} \end{cases}$$

**Proof.** Due to  $\varphi_1$  and  $\varphi_2$  are linear mappings, it follows that  $\varphi(\alpha v + \beta w) = (\varphi_1(\alpha v + \beta w), \varphi_2(\alpha v + \beta w)) = \alpha \varphi(v) + \beta \varphi(w)$ . Moreover,

$$\varphi(v) = ((v_1, \dots, v_n, \underbrace{0, \dots, 0}_{n \text{ times}}, \underbrace{0, \dots, 0}_{n \text{ times}}), v_1, \dots, v_n) = vM_\varphi,$$

where  $M_\varphi = [m_{ij}]_{n \times 4n}$  defined above. □



To illustrate this formalism, we recover the following results obtained in [5]:

- (1)  $\mathcal{P}(V, W) \cong K^{n+m}$ ,
- (2)  $\dim(\mathcal{P}(V, W)) = \dim V + \dim W$ ,
- (3)  $\mathcal{R}(\mathcal{P}(v, w)) = \mathcal{P}(\mathcal{R}(v), \mathcal{R}(w))$ ,
- (4)  $\mathcal{P}(\mathcal{P}(v, w), z) = \mathcal{P}(v, \mathcal{P}(w, z))$ .

**Definition 2.8** (Reversing of matrices). Assume  $A = (a_{ij})_{n \times m}$ ,  $a_{ij} \in \mathbb{K}$ . Reversing for rows of  $A$ , denoted by  $\mathcal{R}_r(A)$ , reversing for columns of  $A$ , denoted by  $\mathcal{R}_c(A)$ , and reversing of  $A$ , denoted by  $\mathcal{R}(A)$ , are given by

$$\begin{aligned}\mathcal{R}_r(A) &= (a_{i(m+1-j)})_{n \times m}, & \mathcal{R}_c(A) &= (a_{(n+1-i)j})_{n \times m}, \\ \mathcal{R}(A) &= (a_{(n+1-i)(m+1-j)})_{n \times m},\end{aligned}$$

respectively.

To avoid confusion, assume reversing of vectors and reversing of matrices transformations denoted by  $\mathcal{R}_v$  and  $\mathcal{R}_m$ , respectively. The “vectorization” mapping, denoted by  $\rho_{nm}$ , transforms a matrix belonging to  $\mathcal{M}_{n \times m}(\mathbb{K})$  in a vector belonging to  $\mathbb{K}^{nm}$ . From the following lemma, we get the analogue version for matrices of Theorem 2.6.

**Lemma 2.9** (Vectorization mapping). *Mapping  $\rho_{nm}$  is a linear isomorphism between  $\mathcal{M}_{n \times m}$  and  $\mathbb{K}^{nm}$ .*

**Proof.** It follows due to  $\mathcal{M}_{n \times m}$  and  $\mathbb{K}^{nm}$  are isomorphic vector spaces, for instance  $\rho_{nm}$  is a linear transformation, is a monomorphism and is an epimorphism.  $\square$

**Definition 2.10** (Pasting mappings by rows, columns and blocks). Assume  $A \in \mathcal{M}_{n \times m}$ ,  $B \in \mathcal{M}_{n \times p}$  and  $C \in \mathcal{M}_{q \times m}$ , pasting by rows of  $A$

with  $B$ , denoted by  $\mathcal{P}_r(A, B)$  is given by

$$\mathcal{P}_r(A, B) = \begin{pmatrix} (S \circ \varphi)(a_1, b_1) \\ \vdots \\ (S \circ \varphi)(a_n, b_n) \end{pmatrix}, \text{ where } a_i \in K^m, b_i \in K^p.$$

Pasting by columns of  $A$  with  $C$ , denoted by  $\mathcal{P}_c(A, C)$ , is given by

$$\mathcal{P}_c(A, C) = ((S \circ \varphi)(a_1, c_1) \cdots (S \circ \varphi)(a_m, c_m)),$$

where  $a_j \in K^n, c_j \in K^q$ .

Pasting by blocks of  $B$  with  $C$ , denoted by  $\mathcal{P}_b(B, C)$ , is given by

$$\mathcal{P}_b(B, C) = \mathcal{P}_c(\mathcal{P}_r(B, \mathbf{0}_{n \times m}), \mathcal{P}_r(\mathbf{0}_{q \times p}, C)).$$

**Remark 2.11.** We can see that previous definition allows to recover the formalism related with pasting and reversing of vectors, that is,

$$\mathcal{R}_r(A) = \begin{pmatrix} \mathcal{R}(a_1) \\ \vdots \\ \mathcal{R}(a_n) \end{pmatrix} \text{ with } a_i \in \mathbb{K}^m,$$

$$\mathcal{R}_c(A) = (\mathcal{R}(a_1) \cdots \mathcal{R}(a_m)) \text{ with } a_j \in \mathbb{K}^n,$$

$$\mathcal{R}(A) = (\mathcal{R}(a_m) \cdots \mathcal{R}(a_1)) \text{ with } a_j \in \mathbb{K}^n,$$

$$\mathcal{P}_r(A, B) = \begin{pmatrix} \mathcal{P}(a_1, b_1) \\ \vdots \\ \mathcal{P}(a_n, b_n) \end{pmatrix}, \text{ where } a_i \in K^m, b_i \in K^p$$

and

$$\mathcal{P}_c(A, C) = (\mathcal{P}(a_1, c_1) \cdots \mathcal{P}(a_m, c_m)), \text{ where } a_j \in K^n, c_j \in K^q.$$

We rewrite the following properties (results) obtained in [5] and [3].

$$(1) \mathcal{R}_r^2(A) = A,$$

$$(2) \mathcal{R}_c^2(A) = A,$$

- (3)  $\mathcal{R}_r(\mathcal{P}_r(A, B)) = \mathcal{P}_r(\mathcal{R}_r(B), \mathcal{R}_r(A)),$
- (4)  $\mathcal{R}_c(\mathcal{P}_c(A, B)) = \mathcal{P}_c(\mathcal{R}_c(B), \mathcal{R}_c(A)),$
- (5)  $\mathcal{P}_r(\mathcal{P}_r(A, B), C) = \mathcal{P}_r(A, \mathcal{P}_r(B, C)),$
- (6)  $\mathcal{P}_c(\mathcal{P}_c(A, B), C) = \mathcal{P}_c(A, \mathcal{P}_c(B, C)),$
- (7)  $\mathcal{R}_r(\alpha A + \beta B) = \alpha \mathcal{R}_r(A) + \beta \mathcal{R}_r(B),$
- (8)  $\mathcal{R}_c(\alpha A + \beta B) = \alpha \mathcal{R}_c(A) + \beta \mathcal{R}_c(B),$
- (9)  $\mathcal{P}_r(\mathcal{M}_{n \times m}, \mathcal{M}_{n \times p}) = \mathcal{M}_{n \times (m+p)},$
- (10)  $\mathcal{P}_c(\mathcal{M}_{n \times m}, \mathcal{M}_{l \times m}) = \mathcal{M}_{(n+l) \times m},$
- (11)  $\mathcal{P}_r(A, B) = A(\mathcal{P}_r(\mathcal{P}_c(I_n, \mathbf{0}_{(n-m) \times m}), \mathbf{0}_{n \times p}) + \mathcal{P}_r(\mathbf{0}_{n \times m}, B),$
- (12)  $\mathcal{P}_c(A, B) = A(\mathcal{P}_c(\mathcal{P}_r(I_n, \mathbf{0}_{n \times (m-q)}), \mathbf{0}_{n \times p}) + \mathcal{P}_c(\mathbf{0}_{n \times m}, B),$
- (13)  $(\mathcal{R}_r(A))^T = \mathcal{R}_c(A^T),$
- (14)  $(\mathcal{R}_c(A))^T = \mathcal{R}_r(A^T),$
- (15)  $(\mathcal{P}_c(A, B)) = \mathcal{P}_r(A^T, B^T),$
- (16)  $(\mathcal{P}_r(A, B)) = \mathcal{P}_c(A^T, B^T),$
- (17)  $\mathcal{R}_r(AB) = A\mathcal{R}_r(B),$
- (18)  $\mathcal{R}_c(AB) = \mathcal{R}_c(A)B,$
- (19)  $\det(\mathcal{R}_c(A)) = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \det A,$
- (20)  $\det(\mathcal{R}_r(A)) = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \det A,$
- (21)  $(\mathcal{R}_c(A))^{-1} = \mathcal{R}_r(A^{-1}),$

$$(22) (\mathcal{R}_r(A))^{-1} = \mathcal{R}_c(A^{-1}),$$

$$(23) \mathcal{R}_r(M^{(k)}) = M^{(n-k+1)}\mathcal{R}(I_{n-1}), 1 \leq k \leq n,$$

$$(24) \bigwedge_{i=1}^{n-1} \mathcal{R}_r(v_i) = (-1)^{\left\lceil \frac{3n}{2} \right\rceil} \mathcal{R}_r \left( \bigwedge_{i=1}^{n-1} (v_i) \right),$$

$$(25) \mathcal{R}(A) = M_{\mathcal{R}_c} A M_{\mathcal{R}_r},$$

$$(26) \mathcal{R}(\mathcal{R}(A)) = A,$$

$$(27) \mathcal{R}(\mathcal{P}(A, B)) = \mathcal{P}(\mathcal{R}(B), \mathcal{R}(A)),$$

$$(28) \mathcal{P}(\mathcal{P}(A, B), C) = \mathcal{P}(A, \mathcal{P}(B, C)),$$

$$(29) b, c \in K, p = n, q = m, \mathcal{R}(bA + cB) = b\mathcal{R}(A) + c\mathcal{R}(B),$$

$$(30) \mathcal{P}(A, B) = \mathcal{M}_{r \times s}(K),$$

$$(31) \mathcal{R}(I_n) = I_n,$$

$$(32) \mathcal{R}(A) = \mathcal{R}_c(\mathcal{R}_r(A)),$$

$$(33) \mathcal{R}(A) = \mathcal{R}_r(\mathcal{R}_c(A)),$$

$$(34) \mathcal{R}(AB) = \mathcal{R}(A)\mathcal{R}(B),$$

$$(35) (\mathcal{R}(A))^{-1} \mathcal{R}(A^{-1}),$$

$$(36) \det(\mathcal{R}(A)) = \det A,$$

$$(37) \text{Tr}(\mathcal{R}(A)) = \text{Tr}A,$$

$$(38) \mathcal{R}(A^T) = (\mathcal{R}(A))^T,$$

$$(39) \mathcal{R}(\mathcal{P}_b(A, B)) = \mathcal{P}_B(\mathcal{R}(B), \mathcal{R}(A)),$$

$$(40) \mathcal{P}_b(\mathcal{P}_b(A, B), C) = \mathcal{P}_b(A, \mathcal{P}_b(B, C)),$$

$$(41) \mathcal{P}_b(A, B) \in \mathcal{M}_{(n+p) \times (m+q)},$$

$$(42) \mathcal{P}_b(A, B)^T = \mathcal{P}_b(A^T, B^T),$$

$$(43) \det(\mathcal{P}_b(A, B)) = \det A \det B,$$

$$(44) \text{Tr}(\mathcal{P}_b(A, B)) = \text{Tr}A + \text{Tr}B,$$

$$(45) \mathcal{P}_b(A, B)^{-1} = \mathcal{P}_b(A^{-1}, B^{-1}).$$

**Proposition 2.12.** *Consider  $V = \mathcal{M}_{n \times m}$ . The following statements hold:*

$$(1) \{\mathcal{R}_c, \mathcal{R}_r, \mathcal{R}\} \subset \text{Aut}(V).$$

(2) *The transformation matrices of  $\mathcal{R}$ ,  $\mathcal{R}_r$  and  $\mathcal{R}_c$ , are given, respectively, by*

$$M_{\mathcal{R}} = (\delta_{i, nm-j+1})_{nm \times nm}, \quad M_{\mathcal{R}_r} = (\delta_{i, m-j+1})_{m \times m},$$

$$M_{\mathcal{R}_c} = (\delta_{n-i+1, j})_{n \times n},$$

where  $M_{\mathcal{R}}$  acts over  $A$  written as vector in  $\mathbb{K}^{nm}$ ,  $M_{\mathcal{R}_r}$  acts over  $A$  multiplying it by right and  $M_{\mathcal{R}_c}$  acts over  $A$  multiplying it by left.

$$(3) \mathcal{R}(A) = M_{\mathcal{R}_c} A M_{\mathcal{R}_r}.$$

$$(4) M_{\mathcal{R}_c} M_{\mathcal{R}_r} = I_n \Leftrightarrow n = m.$$

$$(5) \ker(\mathcal{R} - id)^\perp = \ker(\mathcal{R} + id).$$

$$(6) \ker(\mathcal{R} - id) = \ker(\mathcal{R}_r - \mathcal{R}_c).$$

$$(7) \ker(\mathcal{R} + id) = \ker(\mathcal{R}_r + \mathcal{R}_c).$$

$$(8)$$

$$V = \ker(\mathcal{R} - id) \oplus \ker(\mathcal{R} + id) = (\ker(\mathcal{R}_c - id) \cap \ker(\mathcal{R}_r - id))$$

$$\oplus (\ker(\mathcal{R}_c + id) \cap \ker(\mathcal{R}_r + id)) \oplus (\ker(\mathcal{R}_c + id) \cap \ker(\mathcal{R}_r - id))$$

$$\oplus (\ker(\mathcal{R}_c - id) \cap \ker(\mathcal{R}_r + id)).$$

(9)

$$\dim \ker(\mathcal{R} - id) = \left\lceil \frac{nm}{2} \right\rceil, \quad \dim \ker(\mathcal{R} + id) = \left\lfloor \frac{nm}{2} \right\rfloor,$$

$$\dim(\ker(\mathcal{R}_c - id) \cap \ker(\mathcal{R}_r - id)) = \left\lceil \frac{n}{2} \right\rceil \left\lceil \frac{m}{2} \right\rceil,$$

$$\dim(\ker(\mathcal{R}_c + id) \cap \ker(\mathcal{R}_r + id)) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor,$$

$$\dim(\ker(\mathcal{R}_c + id) \cap \ker(\mathcal{R}_r - id)) = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{m}{2} \right\rceil,$$

$$\dim(\ker(\mathcal{R}_c - id) \cap \ker(\mathcal{R}_r + id)) = \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{m}{2} \right\rfloor.$$

**Proof.** It follows from Proposition 2.1 and the vectorization mapping  $\rho_{nm}$  given in Lemma 2.9.  $\square$

**Remark 2.13.** In [5] were defined and studied the subspaces associated with items (3)-(5) of Proposition 2.12. We see that for square matrices  $M_{\mathcal{R}_c} = M_{\mathcal{R}_r}$ , although they are acting in different ways (left for columns and right for rows). From now on for square matrices we write  $M_{\mathcal{R}}$ .

**Proposition 2.14.** *The following statements hold:*

(1) Consider  $A \in M_{n \times m}$  and  $B \in M_{n \times r}$ , then

$$(\mathcal{P}_r(A, B))^T = \mathcal{P}_c(A^T, B^T).$$

(2) Consider  $A \in M_{n \times m}$ , then  $\mathcal{R}(\text{Adj}A) = \text{Adj}(\mathcal{R}(A))$ .

(3) Consider  $A'$  as the augmented matrix of  $A$  with  $b$ , then  $A' = \mathcal{P}_r(A, b)$ .

**Proof.** We proceed according to each item.

(1) Assume  $A \in M_{n \times m}$  and  $B \in M_{n \times p}$ . Then

$$(\mathcal{P}_r(A, B))^T = (\mathcal{P}(a_i, b_i))^T, \text{ where } a_i \in K^m, b_i \in K^p.$$

Now, by Theorem 2.6, we have

$$\begin{aligned} (\mathcal{P}(a_i, b_i))^T &= ((S \circ \varphi)(a_i, b_i))^T \\ &= (S(\varphi_1(a_i), \varphi_2(b_i)))^T \\ &= (\varphi_1(a_i) + \varphi_2(b_i))^T. \end{aligned}$$

Therefore, by properties of transpose matrices, we arrive to

$$(\varphi_1(a_i) + \varphi_2(b_i))^T = \varphi_1(a_i)^T + \varphi_2(b_i)^T$$

and for instance

$$\begin{aligned} \varphi_1(a_i)^T + \varphi_2(b_i)^T &= S(\varphi_1(a_i)^T, \varphi_2(b_i)^T), \\ (S \circ \varphi)(a_i^T, b_i^T) &= \mathcal{P}(a_i^T, b_i^T). \end{aligned}$$

Therefore,

$$(\mathcal{P}_r(A, B))^T = \mathcal{P}_c(A^T, B^T).$$

(2) Due to  $\text{Adj}A = |A|A^{-1}$ , we have that

$$\text{Adj}(\mathcal{R}(A)) = |\mathcal{R}(A)|(\mathcal{R}(A))^{-1}.$$

Due to property (36), we get  $|\mathcal{R}(A)|(\mathcal{R}(A))^{-1} = |A|(\mathcal{R}(A))^{-1}$ . Now, by property (35), we obtain  $|A|(\mathcal{R}(A))^{-1} = |A|\mathcal{R}(A^{-1}) = \mathcal{R}(|A|A^{-1})$ , which implies that  $\mathcal{R}(\text{Adj}A) = \text{Adj}(\mathcal{R}(A))$ .

(3) We see that

$$A' = \left[ \begin{array}{ccc|c} a_{11} & \cdots & a_{1m} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nm} & b_n \end{array} \right].$$

Moreover,  $\varphi_1(A) \in M_{n \times (m+1)}$  and  $\varphi_2(A) \in M_{n \times (m+1)}$ . In this way,  $A' = \varphi_1(A) + \varphi_2(A) = S(\varphi_1(A), \varphi_2(A))$ , for instance  $(S \circ \varphi)(A, b) = \mathcal{P}_r(A, b)$ , as  $A' = \mathcal{P}_r(A, b)$ .  $\square$

**Remark 2.15.** We note that item (2) is not true for reversing by rows and columns due to  $\mathcal{R}(A^{-1}) \neq (\mathcal{R}_r(A))^{-1}$  and  $\mathcal{R}_c(A^{-1}) \neq (\mathcal{R}_c(A))^{-1}$ .

Considering the vector space of  $n \times m$  matrices, we say that a mapping is *palindromicing by rows* (resp. by *columns*) whether it transforms any  $n \times m$  matrix into a palindromic matrix by rows (resp. by columns), i.e., it is epimorphism from  $\mathcal{M}_{n \times m}$  to  $W_p^r(n \times m)$  (resp.  $W_p^c(n \times m)$ ). In a similar way, we say that a mapping is *antipalindromicing by rows* (resp. by *columns*) whether it transforms any  $n \times m$  matrix into an antipalindromic matrix by rows (resp. by columns), i.e., it is epimorphism from  $\mathcal{M}_{n \times m}$  to  $W_a^r(n \times m)$  (resp.  $W_a^c(n \times m)$ ). Moreover, we say that palindromicing and antipalindromicing mappings by rows (resp. by columns) are *canonical*, denoted by  $\mathcal{F}_p^r$  and  $\mathcal{F}_a^r$  (resp.  $\mathcal{F}_p^c$  and  $\mathcal{F}_a^c$ ), respectively, whether they are linear mappings and for all  $A \in \mathcal{M}_{n \times m}$  they satisfy  $A = (\mathcal{F}_p^r + \mathcal{F}_a^r)(A) = (\mathcal{F}_p^c + \mathcal{F}_a^c)(A)$ ,  $\mathcal{R}_r(A) = (\mathcal{F}_p^r - \mathcal{F}_a^r)(A)$  and  $\mathcal{R}_c(A) = (\mathcal{F}_p^c - \mathcal{F}_a^c)(A)$ . From now on we only consider canonical palindromicing and antipalindromicing mappings by rows (resp. by columns), which will be called *palindromicing* and *antipalindromicing mappings* by rows (resp. by columns). Nevertheless, we can consider palindromicing and antipalindromicing mappings for matrices with respect  $\mathcal{R} = \mathcal{R}_r \circ \mathcal{R}_c$ , denoted as  $\mathcal{F}_p$  and  $\mathcal{F}_a$ , respectively.

**Proposition 2.16** (Palindromicing and antipalindromicing mappings in  $\mathcal{M}_{n \times m}$ ). *The following statements hold:*

(1)  $\mathcal{F}_p^r, \mathcal{F}_a^r, \mathcal{F}_p^c, \mathcal{F}_a^c, \mathcal{F}_p$  and  $\mathcal{F}_a$  are given by

$$\begin{aligned} \mathcal{F}_p^r: \mathcal{M}_{n \times m} &\rightarrow W_p^r(n \times m) & \mathcal{F}_a^r: \mathcal{M}_{n \times m} &\rightarrow W_a^r(n \times m) \\ A &\mapsto \frac{1}{2}(A + \mathcal{R}_r(A)), & A &\mapsto \frac{1}{2}(A - \mathcal{R}_r(A)), \end{aligned}$$



$$\begin{aligned}
\mathcal{F}_p^c: \mathcal{M}_{n \times m} &\rightarrow W_p^c(n \times m) & \mathcal{F}_a^c: \mathcal{M}_{n \times m} &\rightarrow W_a^c(n \times m) \\
A &\mapsto \frac{1}{2}(A + \mathcal{R}_c(A)), & A &\mapsto \frac{1}{2}(A - \mathcal{R}_c(A)), \\
\mathcal{F}_p: \mathcal{M}_{n \times m} &\rightarrow \mathbf{PA}(n \times m) & \mathcal{F}_a: \mathcal{M}_{n \times m} &\rightarrow \mathbf{aPA}(n \times m) \\
A &\mapsto \frac{1}{2}(A + \mathcal{R}(A)), & A &\mapsto \frac{1}{2}(A - \mathcal{R}(A)).
\end{aligned}$$

(2)

$$\ker(\mathcal{F}_p^r) = \text{im}(\mathcal{F}_a^r), \ker(\mathcal{F}_a^r) = \text{im}(\mathcal{F}_p^r), \ker(\mathcal{F}_p^c) = \text{im}(\mathcal{F}_a^c),$$

$$\ker(\mathcal{F}_a^c) = \text{im}(\mathcal{F}_p^c), \ker(\mathcal{F}_p) = \text{im}(\mathcal{F}_a), \ker(\mathcal{F}_a) = \text{im}(\mathcal{F}_p).$$

(3)

$$\mathcal{F}_p^r(A) = A(M_{\mathcal{R}} + I_n), \mathcal{F}_a^r(A) = A(M_{\mathcal{R}} - I_n), \mathcal{F}_p^c(A) = (M_{\mathcal{R}} + I_m)A,$$

$$\mathcal{F}_a^c(A) = (M_{\mathcal{R}} - I_m)A, \mathcal{F}_p(A) = M_{\mathcal{R}}AM_{\mathcal{R}} + A \text{ and}$$

$$\mathcal{F}_a(A) = M_{\mathcal{R}}AM_{\mathcal{R}} - A.$$

**Proof.** We can see any matrix  $A \in \mathcal{M}_{n \times m}$  as an array of  $n$  row vectors belonging to  $\mathbb{K}^m$  or an array of  $m$  column vectors belonging to  $\mathbb{K}^n$  or a vector belonging to  $\mathbb{K}^{nm}$ . Thus, the results are obtained in virtue of Proposition 2.4 and Lemma 2.9.  $\square$

It is well known that main diagonal ( $\text{diag}$ ) and trace ( $\text{Tr}$ ) of matrices are linear mappings, see [8, 9], which lead us to the following lemma:

**Lemma 2.17.** *Let  $D$  be a diagonal matrix and  $A = (a_{ij})_{n \times n}$ . Then the following statements hold:*

$$(1) \mathcal{R}(\text{diag}(A)) = \text{diag}(\mathcal{R}(A)),$$

$$(2) \text{Tr}(A) = \text{Tr}(\mathcal{R}(A)),$$

$$(3) \mathcal{R}_c(D) = \mathcal{R}_r(D) \Leftrightarrow \text{diag}(D) = \text{diag}(\mathcal{R}(D)),$$

$$(4) \mathcal{R}(\text{diag}(D)) = \text{diag}(D) \Leftrightarrow \mathcal{R}D = D.$$

**Proof.** We proceed according to each item:

(1) By Proposition 2.12, we see that  $\mathcal{R}(\text{diag}(A)) = M_{\mathcal{R}}\text{diag}(A)M_{\mathcal{R}}$ , for instance  $\mathcal{R}(\text{diag}(A)) = \text{diag}(M_{\mathcal{R}}AM_{\mathcal{R}}) = \text{diag}(\mathcal{R}(A))$ .

(2) It follows from previous item.

(3) Consider

$$D = \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix}.$$

By definition of reversing by columns and rows, we have

$$\mathcal{R}_c(D) = \begin{pmatrix} 0 & 0 & \cdots & 0 & d_{nn} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{11} & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and

$$\mathcal{R}_r(D) = \begin{pmatrix} 0 & 0 & \cdots & 0 & d_{11} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{nn} & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Owing to  $\mathcal{R}_c(D) = \mathcal{R}_r(D)$ , we have that  $d_{kk} = d_{(n-k+1)(n-k+1)}$ , then  $\text{diag}(D) = (d_{11}, d_{22}, \dots, d_{22}, d_{11}) = \text{diag}(\mathcal{R}(D))$ . Conversely, consider

$$D = \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix},$$

due to  $\text{diag}(D) = \text{diag}(\mathcal{R}(D))$ , then  $\text{diag}(D) = (d_{11}, d_{22}, \dots, d_{22}, d_{11})$ . Now, applying  $\mathcal{R}_c$  and  $\mathcal{R}_r$  over  $D$ , we have

$$\mathcal{R}_c(D) = \begin{pmatrix} 0 & 0 & \cdots & 0 & d_{11} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{11} & 0 & \cdots & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & d_{11} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{11} & 0 & \cdots & 0 & 0 \end{pmatrix} = \mathcal{R}_r D.$$

(4) By item (1), we have that  $\mathcal{R}(\text{diag}(D)) = \text{diag}(\mathcal{R}(D))$  and by hypothesis, we have that  $\text{diag}(\mathcal{R}(D)) = \text{diag}(D)$ . Now, by item (3), we obtain that  $\mathcal{R}_c(D) = \mathcal{R}_r(D)$  and applying  $\mathcal{R}_c$ , we get  $\mathcal{R}_c^2(D) = \mathcal{R}(D)$ . Therefore,  $D = \mathcal{R}(D)$ . Conversely, due to  $\mathcal{R} = \mathcal{R}_c \circ \mathcal{R}_r$ ,  $\mathcal{R}_c(\mathcal{R}_r(D)) = D$ . Applying  $\mathcal{R}_c$ , we have  $\mathcal{R}_r(D) = \mathcal{R}_c(D)$  and items (3) and (1) lead us to  $\text{diag}(D) = \text{diag}(\mathcal{R}(D)) = \mathcal{R}(\text{diag}(D))$ .  $\square$

**Remark 2.18.** In item (3) of Lemma 2.17, we can see that the main diagonal of  $D$  is palindromic, while in item (4), we observe an equivalence between the palindromic of the diagonal matrix with the palindromic of the main diagonal of the matrix.

**Theorem 2.19.** Let  $p$ ,  $q$ ,  $r$  and  $s$  be the characteristic polynomials of  $n \times n$  matrices  $A$ ,  $\mathcal{R}(A)$ ,  $\mathcal{R}_c(A)$  and  $\mathcal{R}_r(A)$ , respectively. Then the following statements hold:

- (1)  $\text{diag}(\mathcal{R}(A) - \lambda I) = \mathcal{R}(\text{diag}(A - \lambda I))$ ,
- (2)  $p = q$ ,
- (3)  $r = s$ ,
- (4)  $p(A) = p(\mathcal{R}(A))$ ,
- (5)  $p(\mathcal{R}_r(A)) = p(\mathcal{R}_c(A))$ .

**Proof.** We proceed according to each item:

(1) Due to  $\text{diag}$  is a linear mapping and  $\lambda I$  is a palindromic matrix, we obtain  $\text{diag}(\mathcal{R}(A) - \lambda I) = \text{diag}(\mathcal{R}(A)) - \text{diag}(\mathcal{R}(\lambda I))$ . By item (1) in Lemma 2.17, we conclude  $\text{diag}(\mathcal{R}(A) - \lambda I) = \mathcal{R}(\text{diag}(A - \lambda I))$ .

(2) Consider  $A = (a_{ij})_{n \times n}$ , thus  $A - \lambda I = (b_{ij})$ , where

$$b_{ij} = \begin{cases} a_{ij} - \lambda, & i = j, \\ a_{ij}, & i \neq j. \end{cases}$$

Now  $\mathcal{R}(A) - \lambda I = (c_{ij})$ , where

$$c_{ij} = \begin{cases} a_{(n-i+1)(n-j+1)} - \lambda, & n - i + 1 = n - j + 1, \\ a_{(n-i+1)(n-j+1)}, & n - i + 1 \neq n - j + 1. \end{cases}$$

This leads us to

$$c_{ij} = \begin{cases} a_{(n-i+1)(n-j+1)} - \lambda, & i = j, \\ a_{(n-i+1)(n-j+1)}, & i \neq j. \end{cases}$$

Due to  $\mathcal{R}(A - \lambda I) = \mathcal{R}(A) - \lambda I$ , in virtue of property (36), we have  $\det(\mathcal{R}(A - \lambda I)) = \det(\mathcal{R}(A) - \lambda I)$ , which implies that  $p = q$ .

(3) Owing to  $\mathcal{R}_c^2 = I$ , we can write  $\det(\mathcal{R}_c A - \lambda I)$  as

$$\det(\mathcal{R}_c A - \lambda \mathcal{R}_c^2 I) = \det(\mathcal{R}_c (A - \lambda \mathcal{R}_c I)) = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \det(A - \lambda \mathcal{R}_c I).$$

Due to  $\mathcal{R}_c I = \mathcal{R}_r I$  and  $[\delta_{i, n-j+1}]_{n \times n} \cdot I = I \cdot [\delta_{i, n-j+1}]_{n \times n}$ , we obtain

$$(-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \det(A - \lambda \mathcal{R}_c I) = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \det(A - \lambda \mathcal{R}_r I) = \det(\mathcal{R}_r A - \lambda I),$$

which implies  $r = s$ .

Items (4) and (5) correspond to the application of Cayley-Hamilton theorem for reversing, followed by items (1) and (2).  $\square$

**Remark 2.20.** Analysis of reversing to characteristic polynomials can be done exactly as in [4] and [10], in where were also studied palindromic and antipalindromic polynomials.

**Theorem 2.21** (Reversing Jordan form). *Assume  $V = \mathbb{K}^n$ ,  $\mathcal{B}_p$  and  $\mathcal{B}_a$  canonical basis of palindromic and antipalindromic vectors of  $V$ . Let  $\mathfrak{M}_p$  and  $\mathfrak{M}_a$  be matrices formed by the pasting by rows (resp. by columns) of palindromic and antipalindromic vectors by column (resp. by rows), respectively. Then, up to isomorphisms, Jordan form and similarity matrix of  $M_{\mathcal{R}}$  are given by*

$$J_{\mathcal{R}} = \mathcal{P}_b(I_{\lceil \frac{n}{2} \rceil}, -I_{\lfloor \frac{n}{2} \rfloor}) \quad \text{and} \quad P_{\mathcal{R}} = \mathcal{P}_r(\mathfrak{M}_p, \mathfrak{M}_a), \text{ respectively.}$$

Furthermore,  $P_{\mathcal{R}}$  is a symmetric matrix.

**Proof.** Assume  $v \in V$  as column vectors. In virtue of Proposition 2.1, we obtain  $J_{\mathcal{R}}$ . Now, due to the eigenvalues 1 and  $-1$  correspond to palindromic and antipalindromic eigenvectors, respectively, we can choose those belonging to  $\mathcal{B}_p$  and  $\mathcal{B}_a$ , respectively, that is,  $v_{p_i} \in \mathcal{B}_p$  and  $v_{a_i} \in \mathcal{B}_a$ . Thus,  $\mathfrak{M}_p = \mathcal{P}_r(v_{p_1}, \dots, v_{p_{\lceil \frac{n}{2} \rceil}})$  and  $\mathfrak{M}_a = \mathcal{P}_r(v_{a_1}, \dots, v_{a_{\lfloor \frac{n}{2} \rfloor}})$ .

Finally,  $P_{\mathcal{R}}$  is obtained as  $P_{\mathcal{R}} = \mathcal{P}(\mathfrak{M}_p, \mathfrak{M}_a)$ , which is symmetric. Under assumption of  $v \in V$  as row vectors, the proof is similar.  $\square$

**Theorem 2.22.** *Suppose that the  $n \times n$  matrix  $A$  admits a Jordan form. The following statements hold:*

(1) *Jordan form is preserved under reversing, where similarity matrix associated to reversing of  $A$  is reversing by columns of similarity matrix associated to  $A$ .*

(2) *If the Jordan form is palindromic (resp. antipalindromic) and the similarity matrix associated to  $A$  is palindromic (resp. antipalindromic), then  $A$  is palindromic (resp. antipalindromic).*

**Proof.** Due to  $A$  admits a Jordan form, there exists a similarity matrix  $P$ , i.e.,  $A = PJP^{-1}$ . Now, we proceed according to each item:

(1) By item (3) of Proposition 2.12, we have that  $\mathcal{R}(A) = M_{\mathcal{R}_c} A M_{\mathcal{R}_r}$ , that is  $\mathcal{R}(A) = M_{\mathcal{R}_c} PJP^{-1} M_{\mathcal{R}_r} = \mathcal{R}_c(P) J \mathcal{R}_r(P^{-1})$ . Now, by properties (32) to (35), we get  $\mathcal{R}(A) = QJQ^{-1}$ , where  $Q = \mathcal{R}_c(P)$ .

(2) By hypothesis and properties (34) and (35) we have that  $\mathcal{R}(A) = \mathcal{R}(PJP^{-1}) = \mathcal{R}(P)\mathcal{R}(J)(\mathcal{R}(P))^{-1} = PJP^{-1} = A$ . Similarly, assuming  $\mathcal{R}(J) = -J$  and  $\mathcal{R}(P) = -P$ , we have that  $\mathcal{R}(A) = \mathcal{R}(PJP^{-1}) = \mathcal{R}(P)\mathcal{R}(J) \times (\mathcal{R}(P))^{-1} = -PJP^{-1} = -A$ .  $\square$

**Remark 2.23.** In general, reversing of a Jordan form is not a Jordan form. Moreover, the converse of item (2) in general is not true.

**Theorem 2.24.** Consider  $f \in \mathbb{R}[x]$  and the  $n \times n$  matrices  $A$  and  $\mathcal{R}_r(A)$  with eigenvalues  $\lambda_i$  and  $\tilde{\lambda}_i$ , respectively, where  $1 \leq i \leq m \leq n$ . The following statements hold:

- (1)  $f(\lambda_i)$  is eigenvalue of  $f(\mathcal{R}(A))$ ,
- (2)  $f(\tilde{\lambda}_i)$  is eigenvalue of  $f(\mathcal{R}_c(A))$ .

**Proof.** For basic linear algebra, we know that  $f(\lambda_i)$ ,  $1 \leq i \leq m \leq n$ , are the eigenvalues of  $f(A)$ . Now we proceed according to each item:

(1) By Theorem 2.19, we have that eigenvalues of  $A$  are the same of  $\mathcal{R}(A)$ . Thus,  $f(\lambda_i)$  is eigenvalue of  $f(\mathcal{R}(A))$  for  $1 \leq i \leq m \leq n$ .

(2) By Theorem 2.19, we have that eigenvalues of  $\mathcal{R}_r A$  are the same of  $\mathcal{R}_c(A)$ . Thus,  $f(\tilde{\lambda}_i)$  is eigenvalue of  $f(\mathcal{R}_c(A))$  for  $1 \leq i \leq m \leq n$ .

The proof is done.  $\square$

**Theorem 2.25.** Assume  $f : \mathfrak{A} \subset \mathbb{R} \rightarrow \mathfrak{B} \subset \mathbb{R}$  is analytic and let  $A$  be an  $n \times n$  matrix. Then the following statements hold:

$$(1) f(\mathcal{R}_r(A)) = \mathcal{R}(f(\mathcal{R}_c(A))),$$

$$(2) f(\mathcal{R}_c(A)) = \mathcal{R}(f(\mathcal{R}_r(A))),$$

$$(3) f(\mathcal{R}(A)) = \mathcal{R}(f(A)).$$

**Proof.** Due to  $f$  is analytic, we have that

$$f(A) = \sum_{k=0}^{\infty} a_k A^k.$$

Using the property  $M_{\mathcal{R}}^2 = I_n$ , we proceed according to each item:

(1) We see that

$$\begin{aligned} (\mathcal{R}_r(A))^k &= (AM_{\mathcal{R}})^k = M_{\mathcal{R}} M_{\mathcal{R}} A M_{\mathcal{R}} A M_{\mathcal{R}} \cdots A M_{\mathcal{R}} A M_{\mathcal{R}} \\ &= M_{\mathcal{R}} B M_{\mathcal{R}} = \mathcal{R}(B), \end{aligned}$$

where  $B = M_{\mathcal{R}} A M_{\mathcal{R}} A \cdots M_{\mathcal{R}} A M_{\mathcal{R}} A = (\mathcal{R}_c(A))^k$ , therefore  $f(\mathcal{R}_r(A)) = \mathcal{R}(f(\mathcal{R}_c(A)))$ .

(2) We see that

$$\begin{aligned} (\mathcal{R}_c(A))^k &= (M_{\mathcal{R}} A)^k = M_{\mathcal{R}} A M_{\mathcal{R}} A \cdots M_{\mathcal{R}} A M_{\mathcal{R}} A M_{\mathcal{R}} \\ &= M_{\mathcal{R}} B M_{\mathcal{R}} = \mathcal{R}(B), \end{aligned}$$

where  $B = A M_{\mathcal{R}} A M_{\mathcal{R}} \cdots A M_{\mathcal{R}} A M_{\mathcal{R}} = (\mathcal{R}_r(A))^k$ , therefore  $f(\mathcal{R}_c(A)) = \mathcal{R}(f(\mathcal{R}_r(A)))$ .

(3) By property (34), taking  $A = B$ , and proceeding inductively, we obtain  $(\mathcal{R}(A))^k = \mathcal{R}(A^k)$ , therefore  $f(\mathcal{R}(A)) = \mathcal{R}(f(A))$ .  $\square$

In this paper, we presented original results concerning to pasting and reversing in the framework of matrix theory. Such results can be implemented in undergraduate and graduate courses of linear algebra. Currently, there are some research projects involving pasting and reversing over other mathematical structures, some of them include applications to orthogonal polynomials, differential equations, difference equations, quantum mechanics, topology, group theory, algebraic varieties, Ore rings, combinatorial dynamics, numerical analysis, graph theory, coding theory, statistics, among others.

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