JP Journal of Biostatistics

Published: December 2016
http://dx.doi.org/10.17654/BO013020165
Volume 13, Number 2, 2016, Pages 165-175 ISSN: 0973-5143

CONDITIONS TO HAVE UMVUE FOR THE STAIR NESTED DESIGNS

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Abstract

Stair nesting is a type of nesting that leads to very light models. In this design, the number of treatments is the sum of the number of levels in each factor, instead of being the product of the number of levels in each factor as happens with the usual balanced nesting. In this work, we will present conditions that will allow us to obtain complete sufficient statistics and uniformly minimum variance unbiased estimators (UMVUEs) for the stair nested designs.

Received: July 28, 2016; Accepted: September 20, 2016

Keywords and phrases: uniformly minimum variance unbiased estimator (UMVUE), commutative orthogonal block structure (COBS), stair nested designs.

1. Introduction

Cox and Solomon [1] introduced a new kind of nesting, the stair nesting. The algebraic structure of this new type of nesting was proposed and studied by Fernandes et al. [2] and initially they used the term step nested design instead of stair nested design. However, Fernandes et al. [3-5] did not present conditions to have uniformly minimum variance unbiased estimators (UMVUEs) for the variance component estimators. In this work, we will present conditions that will allow us to obtain complete sufficient statistics and UMVUE for the stair nested designs.

If we have u factors, then we will have $\sum_{i=1}^{u} a_i$ combinations of levels for the stair nested designs in opposition to the usual balanced nested designs for which we have $\prod_{i=1}^{u} a_i$ combinations of levels. This is in fact a big advantage when comparing with balanced nesting because it will allow experiments that will become cheaper, due to the fewer number of observations involved, or with the same resources, we produce more experiments. In Section 2, we present the usual structure of a design with commutative orthogonal block structure (COBS). In Section 3, we integrate the stair nested designs into a class of models with COBS. In Section 4, we will present conditions that will allow us to obtain UMVUE for the stair nested designs.

2. COBS

Let $\boldsymbol{y} = \sum_{h=0}^{w} \boldsymbol{X}_h \boldsymbol{\beta}_h$ be a mixed model with $\boldsymbol{\beta}_0$ fixed and $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_w$ random and independent vectors, with null mean vectors and variance-covariance matrices $\sigma_1^2 \mathbf{I}_{c_1}, \ldots, \sigma_w^2 \mathbf{I}_{c_w}$, where σ_h^2, with $h = 1, \ldots, w$, are the usual variance components and $\mathbf{I}_s$ is the identity matrix of order s.

The observations vector $\boldsymbol{y}$ has mean vector $\boldsymbol{\mu} = \mathbf{X}_0 \boldsymbol{\beta}_0$ and variance-covariance matrix $\mathbf{V} = \sum_{h=1}^{w} \sigma_h^2 \mathbf{M}_h$, where $\mathbf{M}_h = \mathbf{X}_h \mathbf{X}_h'$, with $h = 1, \ldots, w$, and $\mathbf{W}'$ is the transpose matrix of $\mathbf{W}$.

If the orthogonal projection matrix $\mathbf{P}$ on the space spanned by mean vector and the matrices $\mathbf{M}_1, \ldots, \mathbf{M}_w$ commute, then there will exist mutually orthogonal projection matrices (MOPMs) $\mathbf{Q}_1, \ldots, \mathbf{Q}_m$ such that $\mathbf{P} = \sum_{j=1}^{z} \mathbf{Q}_j$ and $\mathbf{M}_h = \sum_{j=1}^{m} b_{h,j}\mathbf{Q}_j$, with $h = 1, \ldots, w$. We point out that $\sum_{j=1}^{m} \mathbf{Q}_j = \mathbf{I}_n$. So we have

$$\mathbf{V} = \sum_{h=1}^{w} \sigma_h^2 \sum_{j=1}^{m} b_{h,j}\mathbf{Q}_j = \sum_{j=1}^{m} \gamma_j \mathbf{Q}_j,$$

where $\gamma_1, \ldots, \gamma_m$ are the canonical variance components and $\gamma_j = \sum_{h=1}^{w} b_{h,j}\sigma_h^2$, with $j = 1, \ldots, m$. So we say that this model will have *COBS*.

Since $\mathbf{P}$ and $\mathbf{V}$ commute, the least square estimators (LSEs) of estimable vectors will be best linear unbiased estimators (BLUEs) (Zmyślony [7]).

For $i = 1, \ldots, z$, we have $R(\mathbf{Q}_j) \subseteq R(\mathbf{P})$, with $R(\mathbf{W})$ the range space of matrix $\mathbf{W}$. So only $\gamma_{z+1}, \ldots, \gamma_m$ are directly estimable. With

$$\mathbf{\Gamma}(1) = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_z \end{bmatrix}; \quad \mathbf{\Gamma}(2) = \begin{bmatrix} \gamma_{z+1} \\ \vdots \\ \gamma_m \end{bmatrix}; \quad \mathbf{\Omega}^2 = \begin{bmatrix} \sigma_1^2 \\ \vdots \\ \sigma_w^2 \end{bmatrix}$$

and matrix $\mathbf{B} = [b_{ij}]$ written as

$$\mathbf{B} = [\mathbf{B}(1) \quad \mathbf{B}(2)],$$

where $\mathbf{B}(1)$ has z columns, we have $\mathbf{\Gamma}(l) = \mathbf{B}'(l)\mathbf{\Omega}^2$, with $l = 1, 2$.

If the row vectors of $\mathbf{B}(2)$ are linearly independent, then the column vectors of $\mathbf{B}'(2)$ are linearly independent, so we have $\mathbf{\Omega}^2 = [\mathbf{B}'(2)]^{+}\mathbf{\Gamma}(2)$ and $\mathbf{\Gamma}(1) = \mathbf{B}'(1)[\mathbf{B}'(2)]^{+}\mathbf{\Gamma}(2)$ and all variance components are estimable, where $\mathbf{W}^{+}$ is the Moore-Penrose inverse of matrix $\mathbf{W}$. Thus, $\mathbf{\Gamma}(2)$ and $\mathbf{\Omega}^2$, the relevant parameters for the random effects part of the model, determine each other and that part segregates as a sub-model, so we say that there is *segregation*.

3. Stair Nested Designs

In a stair nested design, we have a_1 "active" levels for the first factor, combined with a single level of all other factors; then a new single level for the first factor, combined with a_2 new "active" levels of the second factor, combined with a single level of all other factors; and so on. So we will have $c_h = (u - h) + \sum_{k=1}^{h} a_k$ level combinations for the h first factors, with $h = 1, ..., u$. So, if we have $a_1, ..., a_u$ "active" levels for the u factors, then we have $\sum_{i=1}^{u} a_i$ combinations of levels. In Figure 1, we present a stair nested design with $u = 3$ factors, $a_1 = 3$, $a_2 = 2$, $a_3 = 4$ "active" levels and $c_1 = 5$, $c_2 = 6$, $c_3 = 9$ levels. So we have $3 + 2 + 4 = 9$ combinations of levels.

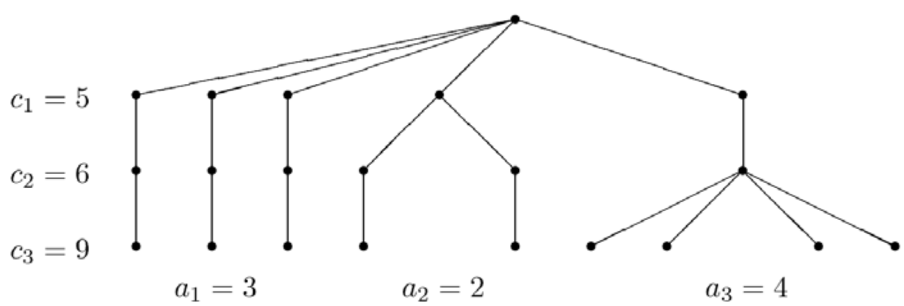

Figure 1. The stair nested design with three factors.

Let us consider a random effects model $\boldsymbol{y} = \sum_{h=0}^{u} \mathbf{X}_h \boldsymbol{\beta}_h$, where the model matrices are block-wise diagonal matrices given by

$$\begin{cases} \mathbf{X}_0 = \mathbf{D}(\mathbf{1}_{a_1}, ..., \mathbf{1}_{a_h}, \mathbf{1}_{a_{h+1}}, ..., \mathbf{1}_{a_u}), \\ \mathbf{X}_h = \mathbf{D}(\mathbf{I}_{a_1}, ..., \mathbf{I}_{a_h}, \mathbf{1}_{a_{h+1}}, ..., \mathbf{1}_{a_u}), \quad h = 1, ..., u, \end{cases}$$

where $\mathbf{1}_s$ is a column vector with s components equal to 1.

We assume that $\boldsymbol{\beta}_0$ is fixed and, for $h = 1, \ldots, u$, $\boldsymbol{\beta}_h$ are normal distributed and independent, with null mean vectors, $\mathbf{0}_{c_h}$, and variance-covariance matrices $\sigma_h^2 \mathbf{I}_{c_h}$.

For the stair nested design presented in Figure 1, we have

$$\begin{cases} \mathbf{X}_0 = \mathbf{D}(\mathbf{1}_3, \mathbf{1}_2, \mathbf{1}_4), \\ \mathbf{X}_1 = \mathbf{D}(\mathbf{I}_3, \mathbf{1}_2, \mathbf{1}_4), \\ \mathbf{X}_2 = \mathbf{D}(\mathbf{I}_3, \mathbf{I}_2, \mathbf{1}_4), \\ \mathbf{X}_3 = \mathbf{D}(\mathbf{I}_3, \mathbf{I}_2, \mathbf{I}_4). \end{cases}$$

Assuming that the observations vector, $\boldsymbol{y}$, is normal distributed with mean vector $\boldsymbol{\mu} = \mathbf{X}_0 \beta_0$ and variance-covariance matrix $\mathbf{V} = \sum_{h=1}^{u} \sigma_h^2 \mathbf{M}_h$, where $\mathbf{M}_h = \mathbf{X}_h \mathbf{X}_h'$, with $h = 1, \ldots, u$, we have

$$\begin{cases} \mathbf{M}_0 = \mathbf{D}(\mathbf{J}_{a_1}, \ldots, \mathbf{J}_{a_h}, \mathbf{J}_{a_{h+1}}, \ldots, \mathbf{J}_{a_u}), \\ \mathbf{M}_h = \mathbf{D}(\mathbf{I}_{a_1}, \ldots, \mathbf{I}_{a_h}, \mathbf{J}_{a_{h+1}}, \ldots, \mathbf{J}_{a_u}), \quad h = 1, \ldots, u, \end{cases}$$

where $\mathbf{J}_s$ is the $s \times s$ matrix with all entries equal to 1.

For the stair nested design presented in Figure 1, we have

$$\begin{cases} \mathbf{M}_0 = \mathbf{D}(\mathbf{J}_3, \mathbf{J}_2, \mathbf{J}_4), \\ \mathbf{M}_1 = \mathbf{D}(\mathbf{I}_3, \mathbf{J}_2, \mathbf{J}_4), \\ \mathbf{M}_2 = \mathbf{D}(\mathbf{I}_3, \mathbf{I}_2, \mathbf{J}_4), \\ \mathbf{M}_3 = \mathbf{D}(\mathbf{I}_3, \mathbf{I}_2, \mathbf{I}_4). \end{cases}$$

For $h = 1, \ldots, u$, we have the MOPMs

$$\begin{cases} \mathbf{Q}_h = \mathbf{D}(\mathbf{Q}_{h,1,1}, \ldots, \mathbf{Q}_{h,u,1}), \\ \mathbf{Q}_{h+u} = \mathbf{D}(\mathbf{Q}_{h,1,2}, \ldots, \mathbf{Q}_{h,u,2}) \end{cases}$$

with $\mathbf{Q}_{h,j,1} = \mathbf{Q}_{h,j,2} = \mathbf{0}_{a_j \times a_j}$, $h \neq j$ and

$$\begin{cases} \mathbf{Q}_{h,h,1} = \dfrac{1}{a_h}\mathbf{J}_{a_h}, \\ \mathbf{Q}_{h,h,2} = \mathbf{K}_{a_h}, \end{cases}$$

where $\mathbf{0}_{s\times s}$ is the $s \times s$ null matrix and $\mathbf{K}_s = \mathbf{I}_s - s^{-1}\mathbf{J}_s$.

For the stair nested design presented in Figure 1, we have

$$\begin{cases} \mathbf{Q}_1 = \mathbf{D}(\mathbf{Q}_{1,1,1}, \mathbf{Q}_{1,2,1}, \mathbf{Q}_{1,3,1}) = \mathbf{D}\left(\dfrac{1}{3}\mathbf{J}_3, \mathbf{0}_{2\times 2}, \mathbf{0}_{4\times 4}\right), \\ \mathbf{Q}_2 = \mathbf{D}(\mathbf{Q}_{2,1,1}, \mathbf{Q}_{2,2,1}, \mathbf{Q}_{2,3,1}) = \mathbf{D}\left(\mathbf{0}_{3\times 3}, \dfrac{1}{2}\mathbf{J}_2, \mathbf{0}_{4\times 4}\right), \\ \mathbf{Q}_3 = \mathbf{D}(\mathbf{Q}_{3,1,1}, \mathbf{Q}_{3,2,1}, \mathbf{Q}_{3,3,1}) = \mathbf{D}\left(\mathbf{0}_{3\times 3}, \mathbf{0}_{2\times 2}, \dfrac{1}{4}\mathbf{J}_4\right), \\ \mathbf{Q}_4 = \mathbf{D}(\mathbf{Q}_{1,1,2}, \mathbf{Q}_{1,2,2}, \mathbf{Q}_{1,3,2}) = \mathbf{D}(\mathbf{K}_3, \mathbf{0}_{2\times 2}, \mathbf{0}_{4\times 4}), \\ \mathbf{Q}_5 = \mathbf{D}(\mathbf{Q}_{2,1,2}, \mathbf{Q}_{2,2,2}, \mathbf{Q}_{2,3,2}) = \mathbf{D}(\mathbf{0}_{3\times 3}, \mathbf{K}_2, \mathbf{0}_{4\times 4}), \\ \mathbf{Q}_6 = \mathbf{D}(\mathbf{Q}_{3,1,2}, \mathbf{Q}_{3,2,2}, \mathbf{Q}_{3,3,2}) = \mathbf{D}(\mathbf{0}_{3\times 3}, \mathbf{0}_{2\times 2}, \mathbf{K}_4). \end{cases}$$

We put

$$\mathbf{P} = \sum_{h=1}^{u}\mathbf{Q}_h = \left(\frac{1}{a_1}\mathbf{J}_{a_1}, \dots, \frac{1}{a_u}\mathbf{J}_{a_u}\right)$$

and we can rewrite the matrices $\mathbf{M}_h$ as

$$\begin{cases} \mathbf{M}_0 = \sum_{h=1}^{u} a_h\mathbf{Q}_h, \\ \mathbf{M}_j = \sum_{h=1}^{j}(\mathbf{Q}_h + \mathbf{Q}_{h+u}) + \sum_{h=j+1}^{u} a_h\mathbf{Q}_h, \quad j = 1, \dots, u-1, \\ \mathbf{M}_u = \sum_{h=1}^{u}(\mathbf{Q}_h + \mathbf{Q}_{h+u}) = \mathbf{I}_n. \end{cases}$$

For the stair nested design presented in Figure 1, we have

$$\mathbf{P} = \mathbf{D}\left(\frac{1}{3}\mathbf{J}_3, \frac{1}{2}\mathbf{J}_2, \frac{1}{4}\mathbf{J}_4\right).$$

So we can write

$$\begin{cases} \mathbf{M}_j = \sum_{h=1}^{2u} b_{j,h}\mathbf{Q}_h, & j = 1, \ldots, u-1, \\ \mathbf{M}_u = \sum_{h=1}^{2u} \mathbf{Q}_h \end{cases}$$

with

$$\begin{cases} b_{j,h} = 1, & j = 1, \ldots, u; \quad h = 1, \ldots, j, \\ b_{j,h} = a_h, & j = 1, \ldots, u-1; \quad h = j+1, \ldots, u, \\ b_{j,h} = 1, & j = 1, \ldots, u; \quad h = u+1, \ldots, u+j, \\ b_{j,h} = 0, & j = 1, \ldots, u-1; \quad h = u+j+1, \ldots, 2u. \end{cases}$$

For the stair nested design presented in Figure 1, we have

$$\mathbf{B} = [\mathbf{B}(1) \quad \mathbf{B}(2)] = \left[\begin{array}{ccc|ccc} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} & b_{1,5} & b_{1,6} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} & b_{2,5} & b_{2,6} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & b_{3,5} & b_{3,6} \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & a_2 & a_3 & 1 & 0 & 0 \\ 1 & 1 & a_3 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right].$$

Thus, $z = u$ and $\mathbf{B}(2)$ is a lower triangular matrix with elements equal to 1. This will be invertible so the models with stair nesting are COBSs since $\boldsymbol{P}$ and $\mathbf{M}_1, \ldots, \mathbf{M}_u$ belong to a quadratic subspace with principal basis $\{\mathbf{Q}_1, \ldots, \mathbf{Q}_{2u}\}$. Furthermore, these COBSs are segregated since matrix $\mathbf{B}(2)$ is invertible.

4. UMVUE

In previous works, Fernandes et al. [3-5] were not able to show that the variance components estimators were UMVUEs. In this section, we will present conditions that will allow us to solve this problem.

We can rewrite the variance-covariance matrix as

$$\mathbf{V} = \sum_{h=1}^{u} \gamma_h \mathbf{Q}_h + \sum_{h=1}^{u} \gamma_{u+h} \mathbf{Q}_{u+h},$$

where the canonical variance components are given by

$$\begin{cases} \gamma_h = \sum_{j=1}^{h-1} a_j a_j^2 + \sum_{j=h}^{u} \sigma_j^2, \\ \gamma_{u+h} = \sum_{j=h}^{u} \sigma_j^2. \end{cases}$$

So $\mathbf{V} = \sum_{h=1}^{2u} \gamma_h \mathbf{Q}_h$ and $\gamma_h = \sum_{j=1}^{u} b_{j,h} \sigma_j^2$, with $h = 1, \ldots, 2u$.

Since $\mathbf{Q}_h = \mathbf{A}'_h \mathbf{A}_h$, with $h = 1, \ldots, 2u$, and

$$\mathbf{A}(1) = \begin{bmatrix} \mathbf{A}_1 \\ \vdots \\ \mathbf{A}_u \end{bmatrix} \quad \text{and} \quad \mathbf{A}(2) = \begin{bmatrix} \mathbf{A}_{u+1} \\ \vdots \\ \mathbf{A}_{2u} \end{bmatrix},$$

we can rewrite the observations vector $\boldsymbol{y}$ as $\boldsymbol{y} = \mathbf{A}'(1)\boldsymbol{y}(1) + \mathbf{A}'(2)\boldsymbol{y}(2)$, where $\boldsymbol{y}(l) = \mathbf{A}(l)\boldsymbol{y}$, with $l = 1, 2$ and we can put, for $h = 0, \ldots, u$, $\mathbf{X}_h(l) = \mathbf{A}(l)\mathbf{X}_h$, with $l = 1, 2$.

For $h = 1, \ldots, u$, matrices $\mathbf{A}_h$ are given by

$$\begin{cases} \mathbf{A}_h = \mathbf{D}(\mathbf{A}_{h,1,1}, \ldots, \mathbf{A}_{h,u,1}), \\ \mathbf{A}_{h+u} = \mathbf{D}(\mathbf{A}_{h,1,2}, \ldots, \mathbf{A}_{h,u,2}) \end{cases}$$

with $\mathbf{A}_{h,j,1} = \mathbf{A}_{h,j,2} = \mathbf{0}'_{a_j}$, $h \neq j$ and

$$\begin{cases} \mathbf{A}_{h,h,1} = \frac{1}{\sqrt{a_h}} \mathbf{1}'_{a_h}, \\ \mathbf{A}_{h,h,2} = \mathbf{T}_{a_h}, \end{cases}$$

where $\mathbf{0}_s$ is a column vector with s components equal to 0, $\mathbf{T}_s$ are orthogonal matrices obtained through the recursive method known as Gram-Schmidt orthogonalization process and $\mathbf{T}'_s \mathbf{T}_s = \mathbf{K}_s$.

For the stair nested design presented in Figure 1, we have

$$\begin{cases} \mathbf{A}_1 = \mathbf{D}(\mathbf{A}_{1,1,1}, \mathbf{A}_{1,2,1}, \mathbf{A}_{1,3,1}) = \mathbf{D}\left(\frac{1}{\sqrt{3}}\mathbf{1}'_3, \mathbf{0}'_2, \mathbf{0}'_4\right), \\ \mathbf{A}_2 = \mathbf{D}(\mathbf{A}_{2,1,1}, \mathbf{A}_{2,2,1}, \mathbf{A}_{2,3,1}) = \mathbf{D}\left(\mathbf{0}'_3, \frac{1}{\sqrt{2}}\mathbf{1}'_2, \mathbf{0}'_4\right), \\ \mathbf{A}_3 = \mathbf{D}(\mathbf{A}_{3,1,1}, \mathbf{A}_{3,2,1}, \mathbf{A}_{3,3,1}) = \mathbf{D}\left(\mathbf{0}'_3, \mathbf{0}'_2, \frac{1}{\sqrt{4}}\mathbf{1}'_4\right), \\ \mathbf{A}_4 = \mathbf{D}(\mathbf{A}_{1,1,2}, \mathbf{A}_{1,2,2}, \mathbf{A}_{1,3,2}) = \mathbf{D}(\mathbf{T}_3, \mathbf{0}'_4, \mathbf{0}'_4), \\ \mathbf{A}_5 = \mathbf{D}(\mathbf{A}_{2,1,2}, \mathbf{A}_{2,2,2}, \mathbf{A}_{2,3,2}) = \mathbf{D}(\mathbf{0}'_3, \mathbf{T}_2, \mathbf{0}'_4), \\ \mathbf{A}_6 = \mathbf{D}(\mathbf{A}_{3,1,2}, \mathbf{A}_{3,2,2}, \mathbf{A}_{3,3,2}) = \mathbf{D}(\mathbf{0}'_3, \mathbf{0}'_2, \mathbf{T}_4), \end{cases}$$

where

$$\mathbf{T}_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix},$$

$$\mathbf{T}_3 = \begin{bmatrix} \frac{2\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

and

$$\mathbf{T}_4 = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{3}}{6} \\ 0 & \frac{\sqrt{6}}{3} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} \\ 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}.$$

Thus, we will have the mean vectors

$$\begin{cases} \boldsymbol{\mu}(1) = \mathbf{A}(1)\boldsymbol{\mu}, \\ \boldsymbol{\mu}(2) = \mathbf{A}(2)\boldsymbol{\mu} = 0 \end{cases}$$

and the variance-covariance matrices $\mathbf{V}(l) = \sum_{h=1}^{u} \sigma_h^2 \mathbf{M}_h(l)$, with $l = 1, 2,$

for both parts of the model. For $h = 1, \ldots, u,$ we can write matrices $\mathbf{M}_h(l),$ with $l = 1, 2,$ as $\mathbf{M}_h(l) = \mathbf{A}(l)\mathbf{M}_h\mathbf{A}'(l).$ Thus, we can obtain the new MOPMs $\mathbf{Q}_h(l),$ with $l = 1, 2,$ as

$$\begin{cases} \mathbf{Q}_h(1) = \mathbf{A}(1)\mathbf{Q}_h\mathbf{A}'(1), & h = 1, \ldots, u, \\ \mathbf{Q}_h(2) = \mathbf{A}(2)\mathbf{Q}_h\mathbf{A}'(2), & h = u+1, \ldots, 2u. \end{cases}$$

Since $\mathbf{M}_j = \sum_{h=1}^{2u} b_{j,h}\mathbf{Q}_h,$ we have, for $j = 1, \ldots, u,$

$$\begin{cases} \mathbf{M}_j(1) = \sum_{h=1}^{u} b_{j,h}\mathbf{Q}_h(1), \\ \mathbf{M}_j(2) = \sum_{h=u+1}^{2u} b_{j,h}\mathbf{Q}_h(2). \end{cases}$$

So we can rewrite the variance-covariance matrix $\mathbf{V}(2)$ as

$$\mathbf{V}(2) = \sum_{j=1}^{u} \sigma_j^2 \mathbf{M}_j(2) = \sum_{h=u+1}^{2u} \gamma_h \mathbf{Q}_h(2)$$

with $\gamma_h = \sum_{j=1}^{u} b_{j,h}\sigma_j^2,$ where $b_{j,h}$ with $j = 1, \ldots, u$ and $h = 1, \ldots, 2u,$ are the elements of matrix $\mathbf{B}(2)$ having $2u - z = 2u - u = u$ columns. Since $\mathbf{B}(2)$ is a lower triangular matrix with elements equal to 1, this matrix will be invertible and we can write $\mathbf{Q}_h(2) = \sum_{j=1}^{u} d_{j,h}\mathbf{M}_j(2),$ with $[d_{j,h}] = \mathbf{B}^{-1}(2).$

Since $\mathbf{Q}_h(2),$ with $h = u+1, \ldots, 2u,$ are MOPMs, we have

$$\mathbf{V}^2(2) = \sum_{h=u+1}^{2u} \gamma_h^2 \mathbf{Q}_h(2),$$

thus the space spanned by the matrices $\mathbf{M}_j(2),$ with $j = 1, \ldots, u$ is a quadratic subspace. For $j = 1, \ldots, u,$ we have $R[\mathbf{M}_j(2)\mathbf{X}_0(2)] \subseteq R[\mathbf{X}_0(2)].$

Then the vector statistic $\mathbf{X}'_0(2)\boldsymbol{y}(2)$, $\boldsymbol{y}'(2)\mathbf{M}_1(2)\boldsymbol{y}(2)$, ..., $\boldsymbol{y}'(2)\mathbf{M}_u(2)\boldsymbol{y}(2)$ is a complete sufficient statistic (Rao and Rao [6]).

Since COBSs with matrix $\mathbf{B}(2)$ invertible are segregated and $\boldsymbol{y}(2)$ are normal distributed, the variance-covariance estimators obtained from $\boldsymbol{y}(2)$ will be UMVUE for the usual and canonical variance components in the family of estimators obtained from $\boldsymbol{y}(2)$.

Acknowledgement

This work was partially supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the project UID/MAT/00297/2013 (Centro de Matemática e Aplicações).

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