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ON THE CRITERIA OF TRANSIENCE AND RECURRENCE FOR THE DISCRETE-TIME BIRTH-DEATH CHAINS DEFINED BY DIRECTED WEIGHTED CIRCUITS IN RANDOM ERGODIC ENVIRONMENTS

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Abstract

By using the cycle-circuit representation theory of Markov processes, we investigate suitable criteria regarding the properties of the transience and recurrence for the discrete-time birth-death chains

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defined by directed weighted cycles (especially by directed weighted circuits) in random ergodic environments.

1. Introduction

It is known that the classical *birth-death chain* is a special case of homogeneous, aperiodic, irreducible Markov chain (discrete-time or continuous-time) on the set of non-negative integers, where state changes can only happen between neighboring states. This means that the state transitions are of only two types: "*births*" which increase the state variable by one and "*deaths*" which decrease the state variable by one, that is, if the current state at time instant n (discrete-time) is $X_n = i$, then the state at the next time instant $(n+1)$ can only be $X_{n+1} = i+1$ or $(i-1)$ (Nowak [10] and Wilkinson [12]).

The model's name comes from a common application, the use of such models to represent the current size of a population where the transitions are literal births and deaths. In general, the birth-death processes have many applications in demography, epidemiology or in biology since they may be used to study the evolution of bacteria of the number of people with a disease within a population.

By using the cycle-circuit representation theory of Markov processes, the present work arises as an attempt to investigate suitable criteria regarding positive/null recurrence and transience for the corresponding "adjoint" Markov chains describing uniquely the discrete-time birth-death chains by directed circuits and weights in random ergodic environments. (Kalpazidou [7] and Derriennic [1]). This will give us the possibility to study specific problems associated with birth-death chains in another way through cycle-circuit representations. (For the study of cycle-circuit representation of discrete-time birth-death chains in fixed ergodic environments as special cases of random walks we refer the reader to Ganatsiou [2-5]).

The paper is organized as follows: In Section 2, we give a brief account of certain concepts of cycle-circuit representation theory of Markov processes that we shall need throughout the paper. In Section 3, we present some auxiliary results regarding the study of cycle-circuit and weight representations of discrete-time birth-death chains in fixed and random ergodic environments in order to make the presentation of the paper more comprehensible. These results will give us the motivation to study the equivalent proper criteria regarding transience and recurrence of the corresponding Markov chains, describing uniquely the classical discrete-time birth-death chains in random ergodic environments, as it is given in Section 4.

In the next we need the following notation:

$$\mathbb{N} = \{0, 1, 2, ...\}, \quad \mathbb{N}^* = \{1, 2, 3, ...\}.$$

2. Preliminaries

Let S be a denumerable set. The directed sequence $c = (i_1, i_2, ..., i_v, i_1)$ modulo the cyclic permutations, where $i_1, i_2, ..., i_v \in S, \ v > 1,$ completely defines a *directed circuit* in S. The ordered sequence $\hat{c} = (i_1, i_2, ..., i_v)$ associated with the given directed circuit c is called a *directed cycle* in S. A directed circuit may be considered as

$$c = (c(m), c(m+1), ..., c(m+v-1), c(m+v)),$$

if there exists an $m \in Z,$ such that

$$i_1 = c(m+0), i_2 = c(m+1), ..., i_v = c(m+v-1), i_1 = c(m+v),$$

that is a periodic function from Z (the set of integers) to S.

The smallest integer $p \equiv p(c) \geq 1$ satisfying the equation $c(m+p) = c(m),$ for all $m \in Z,$ is the period of c. A directed circuit c such that $p(c) = 1$ is called a *loop*. (In the present work, we shall use directed circuits with distinct point elements).

Let a directed circuit c (or a directed cycle $\hat{c}$) with period $p(c) > 1.$ Then we may define by

$$J_c^{(n)}(i,\ j) = \begin{cases} 1, & \text{if there exists an } m \in Z, \text{ such that } i = c(m),\ j = c(m+n), \\ 0, & \text{otherwise,} \end{cases}$$

the *n-step passage function* associated with the directed circuit *c*, for any $i,\ j \in S,\ n \geq 1.$ We may also define by

$$J_c(i) = \begin{cases} 1, & \text{if there exists an } m \in Z \text{ such that } i = c(m), \\ 0, & \text{otherwise,} \end{cases}$$

the passage function associated with the directed circuit *c*, for any $i \in S.$ The above definitions are due to MacQueen [9] and Kalpazidou [7].

Given a denumerable set *S* and an infinite denumerable class *C* of overlapping directed circuits (or directed cycles) with distinct points (except for the terminals) in *S* such that all the points of *S* can be reached from one another following paths of circuit-edges, that is, for each two distinct points *i* and *j* of *S* there exists a sequence $c_1,\ c_2,\ ...,\ c_k,\ k \geq 1,$ of circuits (or cycles) of *C* such that *i* lies on c_1 and *j* lies on c_k and any pair of consecutive circuits $(c_n,\ c_{n+1})$ have at least one point in common. We may also assume that the chain *C* contains, among its elements, circuits (or cycles) with period greater or equal to 2.

With each directed circuit (or directed cycle) $c \in C$ let us associate a strictly positive weight w_c which must be independent of the choice of the representative of *c*, that is, it must satisfy the consistency condition $w_{\text{cot}_k} = w_c,\ k \in Z,$ where t_k is the translation of length *k*.

For a given class *C* of overlapping directed circuits (or cycles) and for a given sequence $(w_c)_{c \in C}$ of weights we may define by

$$p_{ij} = \frac{\sum_{c \in C} w_c \cdot J_c^{(1)}(i,\ j)}{\sum_{c \in C} w_c \cdot J_c(i)} \tag{2.1}$$

the elements of a Markov transition matrix on *S*, if and only if

$\sum_{c \in C} w_c \cdot J_c(i) < \infty,$ for any $i \in S.$ This means that a given Markov transition matrix $P = (p_{ij}),\ i,\ j \in S,$ can be represented by directed circuits (or cycles) and weights if and only if there exists a class of overlapping directed circuits (or cycles) C and a sequence of positive weights $(w_c)_{c \in C}$ such that the abovementioned formula (2.1) holds. In this case, the representations of the distributions of Markov processes (with discrete or continuous parameter) having an invariant measure as decompositions in terms of the cycle (or circuit) passage functions are called *cycle* (or *circuit*) *representations* while the corresponding discrete parameter Markov chains generated by directed cycles (or circuits) are called *cycle* (or *circuit*) *chains* with Markov transition matrix given by (2.1). Furthermore, the Markov transition matrix P has a unique stationary distribution p which is a solution of $p.\ P = p$ defined by

$$p(i) = \sum_{c \in C} w_c \cdot J_c(i), \quad i \in S.$$

It is known that the following classes of Markov chains may be represented uniquely by circuits (or cycles) and weights:

(i) *The recurrent Markov chains* (Minping and Min [11]).

(ii) *The reversible Markov chains*.

3. Circuit and Weight Representations of Discrete-time Birth-death Chains

3.1. Fixed ergodic environments

Let us consider the Markov chain $(X_n)_{n \geq 0}$ on $\mathbb{N}$, which describes a discrete-time birth-death chain. Since the state transitions are of only two types, that is, $k \to (k+1)$ and $k \to (k-1)$, the elements of the corresponding Markov transition matrix are defined by

$$\mathbb{P}(X_{n+1} = k+1 / X_n = k) = p_k,$$
$$\mathbb{P}(X_{n+1} = k-1 / X_n = k) = q_k,$$

such that $p_k + q_k = 1$, $0 < p_k \leq 1$, for every $k \geq 1$, with $p_0 = 1$, as it is shown in the following diagram:

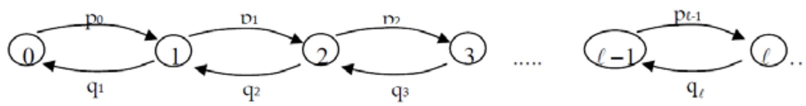

Figure 1

A. Description of the representation of the discrete-time birth-death chain by directed circuits and weights

Let us assume that $(p_k)_{k \geq 0}$ is an arbitrary fixed sequence with $0 < p_k \leq 1$, for every $k \geq 1$, with $p_0 = 1$. Then if we consider the directed circuits $c_k = (k, k+1, k)$, $k \geq 0$, and the collection of weights $(w_{c_k})_{k \geq 0}$ we may obtain that

$$p_k = \frac{w_{c_k}}{w_{c_{k-1}} + w_{c_k}}, \quad p_0 = 1,$$

$$q_k = 1 - p_k = \frac{w_{c_{k-1}}}{w_{c_{k-1}} + w_{c_k}}, \quad k \geq 1.$$

Here the class $C(k)$ contains the directed circuits $c_k = (k, k+1, k)$, $c_{k-1} = (k-1, k, k-1)$. Hence, the transition matrix $P = (p_{ij})$ with

$$p_{ij} = \frac{\sum_{k=0}^{\infty} w_{c_k} \cdot J_{c_k}^{(1)}(i, j)}{\sum_{k=0}^{\infty} w_{c_k} \cdot J_{c_k}(i)}, \quad \text{for } i \neq j, \tag{3.1}$$

$$p_{ii} = 0,$$

where $J_{c_k}^{(1)}(i, j) = 1$, if i, j are consecutive points of the circuit c_k,

$$J_{c_k}(i) = 1, \text{ if } i \text{ is a point of the circuit } c_k,$$

expresses the representation of the Markov chains $(X_n)_{n\geq 0}$ by cycles (especially by directed circuits) and weights.

Furthermore, let us consider the "*adjoint*" Markov chains $(X'_n)_{n\geq 0}$ (of the abovementioned Markov chain $(X_n)_{n\geq 0}$) on $\mathbb{N}$ whose the elements of the corresponding Markov transition matrix are defined by

$$\mathbb{P}(X'_{n+1} = k+1/X'_n = k) = q'_k,$$

$$\mathbb{P}(X'_{n+1} = k-1/X'_n = k) = p'_k,$$

such that $p'_k + q'_k = 1$, $0 < q'_k \leq 1$, for every $k \geq 1$, with $q'_0 = 1$, as it is shown in the following diagram representing the "adjoint" Markov chain of a discrete-time birth-death chain:

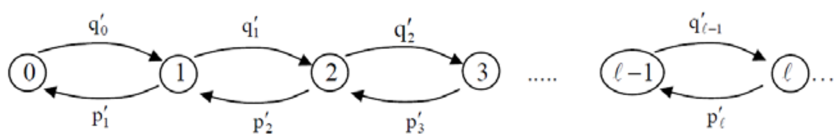

Figure 2

B. Description of the representation of the "adjoint" discrete-time birth-death chain by directed circuits and weights

If we assume that $(q'_k)_{k\geq 0}$ is an arbitrary fixed sequence with $0 < q'_k \leq 1$, for every $k \geq 1$, with $q'_0 = 1$ and if we consider the directed circuits $c'_k = (k+1, k, k+1)$, $k \geq 0$ and the collections of weights $(w_{c'_k})_{k\geq 0}$, then we may have that

$$q'_k = \frac{w_{c'_k}}{w_{c'_{k-1}} + w_{c'_k}}, \quad q'_0 = 1,$$

$$p'_k = 1 - q'_k = \frac{w_{c'_{k-1}}}{w_{c'_{k-1}} + w_{c'_k}}, \quad k \geq 1. \tag{3.2}$$

Here the class $C'(k)$ contains the directed circuits $c'_k = (k+1, k, k+1)$, $c'_{k-1} = (k, k-1, k)$. Hence, the transition matrix $P' = (p'_{ij})$ with elements equivalent to that given by the abovementioned formulas (3.2), expresses also the representation of the "*adjoint*" Markov chain $(X'_n)_{n\geq 0}$ by directed cycles (especially by directed circuits) and weights.

So we have the following:

Proposition 1. *The Markov chain* $(X_n)_{n\geq 0}$ *describes the discrete-time birth-death chain as above has a unique representation by directed cycles* (*especially by directed circuits*) *and weights*.

Proposition 2. *The* "*adjoint*" *Markov chain* $(X'_n)_{n\geq 0}$ *defined as above has a unique representation by directed cycles* (*especially by directed circuits*) *and weights*.

For the proofs of the above propositions, see Ganatsiou [4, 5].

3.2. Random ergodic environments

Let us now consider a discrete-time birth-death chain on $\mathbb{N}$ with transitions $k \to (k-1)$ and $k \to (k+1)$, whose transition probabilities $(p_k)_{k\in N}$ constitute a stationary and ergodic sequence. A realization of this sequence is called a *random environment* for this chain. Regarding the study of the unique cycle and weight representation of this chain in random environments for almost every environment, let us consider a probability space $(\Omega, \mathcal{F}, \mu)$, a measure preserving, ergodic automorphism of this space $\theta : \Omega \to \Omega$ and the measurable function $p : \Omega \to (0, 1)$ such that for every $\omega \in \Omega$, which generates the random environment $p_k \equiv p(\theta^{\kappa}\omega)$, the sequence $(p_k)_{k\in N}$ is a stationary and ergodic sequence of random variables.

Assume that $S = (\mathbb{N})^{\mathbb{N}}$ is an infinite product space with coordinates $(X_n)_{n\geq 0}$. Then we may define a family $(\mathbb{P}^{\omega})_{\omega\in\Omega}$ of probability measures on S such that for every $\omega \in \Omega$, the sequence $(X_n)_{n\geq 0}$ is a Markov chain on $\mathbb{N}$ whose the elements of the corresponding Markov transition matrix are defined by

$$\mathbb{P}^{\omega}(X_0 = 0) = 1, \quad \mathbb{P}^{\omega}(X_{n+1} = k + 1/X_n = k) = p(\theta^k \omega),$$

$$\mathbb{P}^{\omega}(X_{n+1} = k - 1/X_n = k) = 1 - p(\theta^k \omega) \equiv q(\theta^k \omega), \quad \kappa \in \mathbb{N}^*,$$

as it is showed in the following diagram:

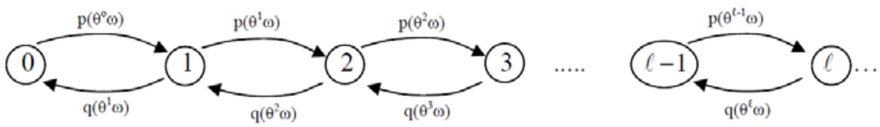

Figure 3

Let us now introduce the “adjoint” discrete-time birth-death chain in random ergodic environments denoted by $(X'_n)_{n\geq 0}$. For every environment $\omega \in \Omega$ let $(\mathbb{P}^{\omega})_{\omega\in\Omega}$ be the family of probability measures on *S*. Then the sequence $(X'_n)_{n\geq 0}$ is a Markov chain on $\mathbb{N}$ whose the elements of the corresponding Markov transition matrix are defined by

$$\mathbb{P}^{\omega}(X'_0 = 0) = 1,$$

$$\mathbb{P}^{\omega}(X'_{n+1} = k - 1/X'_n = k) = p(\theta^k \omega),$$

$$\mathbb{P}^{\omega}(X'_{n+1} = k + 1/X'_n = k) = 1 - p(\theta^k \omega) \equiv q(\theta^k \omega), \quad k \in \mathbb{N}^*,$$

as it is showed in the following diagram:

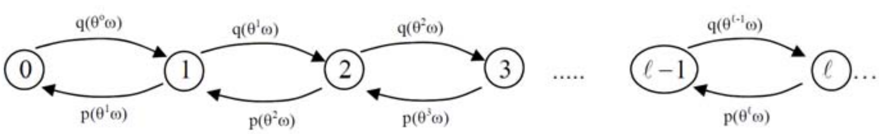

Figure 4

Hence, we have the following:

Proposition 3. *For* μ *almost every environment* $\omega \in \Omega$, *the chain* $(X_n)_{n\geq 0}$ *has a unique circuit and weight representation.*

Proposition 4. *For* μ *almost every environment* $\omega \in \Omega$, *the chain* $(X'_n)_{n\geq 0}$ *has a unique circuit and weight representation.*

For the proofs of the above propositions see Ganatsiou et al. [6].

4. Criteria of Recurrence and Transience for the Markov Chains $(X_n)_{n\geq 0}$, $(X'_n)_{n\geq 0}$

4.1. Fixed ergodic environments

We consider that for the Markov chain $(X_n)_{n\geq 0}$, there is a unique invariant measure up to a multiplicative constant factor $\mu_k = w_{k-1} + w_k$, $k \geq 1$, $\mu_0 = w_0$, while for the Markov chain $(X'_n)_{n\geq 0}$, $\mu'_k = w'_{k-1} + w'_k$, $k \geq 1$ with $\mu'_0 = w'_0$. In the case that an irreducible chain is recurrent, there is only and only one invariant measure (finite or not), so we may obtain the following:

Proposition 5. (i) *The Markov chain* $(X_n)_{n\geq 0}$ *defined as above is positive recurrent if and only if*

$$\sum_{k=1}^{\infty} (b_1 \cdot b_2 \dots b_k)/p_k < +\infty \left(or \ \frac{1}{w_0} \cdot \sum_{k=1}^{\infty} w_k / p_k < +\infty \right).$$

(ii) *The Markov chain* $(X'_n)_{n\geq 0}$ *defined as above is positive recurrent if and only if*

$$\sum_{k=1}^{\infty} \frac{1}{s_1 \cdot s_2 \dots s_k} < +\infty \left(or \ \frac{1}{w'_0} \cdot \sum_{k=1}^{\infty} w'_k < +\infty \right)$$

and

$$\sum_{k=1}^{\infty} (s_1 \cdot s_2 \dots s_k)/p'_k = +\infty \left(or \ w'_0 \cdot \sum_{k=1}^{\infty} \frac{1}{p'_k \cdot w'_k} = +\infty \right).$$

In order to study the properties of recurrence and transience for the Markov chains $(X_n)_{n\geq 0}$, $(X'_n)_{n\geq 0}$ we shall use the following proposition (Karlin and Taylor [8]).

Proposition 6. *Let us consider a Markov chain on* $\aleph$ *which is irreducible. Then if there exists a strictly increasing function that is harmonic on the complement of a finite interval and that is bounded, then the chain is transient. In the case that there exists such a function which is unbounded the chain is recurrent.*

Therefore, we get (see Ganatsiou et al. [5]):

Proposition 7. *The Markov chain* $(X_n)_{n \geq 0}$ *defined as above is transient if and only if the adjoint Markov chain* $(X'_n)_{n \geq 0}$ *is positive recurrent and reciprocally. Moreover, the adjoint Markov chains* $(X_n)_{n \geq 0}, (X'_n)_{n \geq 0}$ *are null recurrent simultaneously. In particular,*

(i) *The Markov chain* $(X_n)_{n \geq 0}$ *defined as above is transient if and only if*

$$\frac{1}{w'_0} \cdot \sum_{k=1}^{\infty} w'_k < +\infty \quad \textit{and} \quad w'_0 \cdot \sum_{k=1}^{\infty} \frac{1}{p'_k \cdot w'_k} = +\infty.$$

(ii) *The Markov chain* $(X'_n)_{n \geq 0}$ *defined as above is transient if and only if*

$$\frac{1}{w_0} \cdot \sum_{k=1}^{\infty} w_k / p_k < +\infty.$$

(iii) *The adjoint Markov chains* $(X_n)_{n \geq 0}, (X'_n)_{n \geq 0}$ *are null recurrent in the case that*

$$\frac{1}{w_0} \cdot \sum_{k=1}^{\infty} w_k / p_k = +\infty \quad \textit{and} \quad \frac{1}{w'_0} \cdot \sum_{k=1}^{\infty} w'_k = +\infty.$$

4.2. Random ergodic environments

Regarding the criteria of recurrence and transience in the case of fixed ergodic environments, we have already proved that the behaviours of recurrence and transience for the Markov chains $(X_n)_{n\geq 0}$, $(X'_n)_{n\geq 0}$ are tied together and depend on the convergence or not of the series

$$\sum_{k=1}^{+\infty} \frac{w_k}{p_k}, \sum_{k=1}^{+\infty} w'_k \quad \text{and} \quad \sum_{k=1}^{+\infty} \frac{1}{w'_k \cdot p'_k}.$$

In the case of random ergodic environments the recurrence and transience are properties which are true for μ almost every environment $\omega \in \Omega$ or for μ almost no environment, because the system $(\Omega, \Im, \mu, \theta)$ is supposed to be ergodic. This is true in general for a random walk in ergodic random environment which is irreducible.

In order to investigate suitable criteria for the transience and recurrence of the corresponding Markov chains representing uniquely by directed circuits and weights a discrete-time birth-death chain in a random ergodic environment, we may use the criteria given in the study for fixed ergodic environments for the chains $(X_n)_{n\geq 0}$, $(X'_n)_{n\geq 0}$ restricted to the half-lines $[j, +\infty)$ with reflection in j.

Therefore, we have the following:

Proposition 8. *The discrete-time birth-death chain* $(X_n)_{n\geq 0}$ *in random ergodic random environments defined as above, is transient, for* μ-*a.e. environment* $\omega \in \Omega$, *if and only if its "adjoint" Markov chain* $(X'_n)_{n\geq 0}$ *is positive recurrent and reciprocally. Moreover, the adjoint Markov chains* $(X_n)_{n\geq 0}$ *and* $(X'_n)_{n\geq 0}$ *are null recurrent simultaneously.*

Proof. Taking into account the Birkhoff's ergodic theorem we have

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \log b_k(\omega) = c < +\infty, \ \mu\text{-a.e.}$$

Taking the following cases we obtain:

(a) $c < 0.$ We may write

$$w_k(\omega) = \prod_{d=1}^{k} b_d(\omega) \sim e^{kc}, \quad k \in \mathbb{N}^*,$$

for the sequence of weights $(w_k(\omega))_{k\in\mathbb{N}^*}$ of the Markov chain $(X_n)_{n\geq 0}$. Therefore, we get $\sum_{k=1}^{+\infty} w_k(\omega) < +\infty,$ μ-a.e.

For the "adjoint" Markov chain $(X'_n)_{n\geq 0}$ we have

$$w'_k(\omega) = \left(\prod_{d=1}^{k} \ell_d(\omega)\right)^{-1} \sim e^{-kc}, \quad k \in \mathbb{N}^*,$$

for the sequence of weights $(w'_k(\omega))_{k\in\mathbb{N}^*}$ of the Markov chain $(X'_n)_{n\geq 0}$. So we get $\sum_{k=1}^{+\infty} w'_k(\omega) = +\infty,$ μ-a.e. By using the criterion given in Subsection 4.1 for the Markov chains $(X_n)_{n\geq 0}$, $(X'_n)_{n\geq 0}$ restricted to the half-line $[j, +\infty)$ with reflection in *j*, we have that the restricted Markov chain $(X_n)_{n\geq 0}$ is positive recurrent on $[j, +\infty),$ while the restricted Markov chain $(X'_n)_{n\geq 0}$ is transient.

(b) $c > 0.$ We get symmetrical results. By using an analogous way given in the case (a), we have that the restricted Markov chain $(X_n)_{n\geq 0}$ is transient on $[j, +\infty),$ while the restricted Markov chain $(X'_n)_{n\geq 0}$ is positive recurrent.

(c) $c = 0.$ Regarding the ergodic theorem, it is well-known that the averages $\frac{1}{n}\sum_{k=0}^{n-1}(fo\theta^k)$ take infinitely many values greater than the limit and infinitely many values smaller than the limit. Therefore, in the sequence of weights $(w_k(\omega))_{k\in\mathbb{N}}$, $(w'_k(\omega))_{k\in\mathbb{N}}$, for a.e. $\omega \in \Omega,$ infinitely many values in both directions are greater than 1. Consequently we may have

$$\sum_{k=1}^{+\infty} w_k(\omega) = \sum_{k=1}^{+\infty} w'_k(\omega) = +\infty, \ \mu\text{-a.e.}$$

By using criteria of null recurrence for the restricted chains $(X_n)_{n\geq 0}$, $(X'_n)_{n\geq 0}$ to the half-line $[j, +\infty)$ with reflection in j, in the case of fixed ergodic environments, we may also have that both chains are null recurrent on $\mathbb{N}$, for μ-a.e. $\omega \in \Omega$.

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