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ON $(k; r)$-TYPE FRACTIONAL INTEGRAL INEQUALITIES

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Abstract

Many inequalities involving Riemann-Liouville, Erdélyi-Kober, and k-fractional integrals are available in literature. In this paper, we present certain inequalities associated with the recently introduced generalized $(k; r)$-fractional integrals of Riemann-Liouville type.

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1. Introduction and Preliminaries

Recently inequalities associated with the various fractional integral operators such as Riemann-Liouville fractional integral operator have been actively investigated. In fact, the fractional integral inequalities involving Riemann-Liouville one have proved to be one of the most powerful and far-reaching tools, which are useful in many branches of pure and applied mathematics. These inequalities have gained considerable popularity and importance during the past few decades due to their demonstrated applications in numerical quadrature, transform theory, probability, and statistical problems. Among other things, they are the most useful ones in establishing uniqueness of solutions in fractional boundary value problems and fractional partial differential equations. For details of their various applications, one may refer to the works (for example) [1-6, 8, 9, 11, 14, 20].

Several interesting and useful k-extensions of some familiar fractional integral operators such as Riemann-Liouville one have been investigated (see, e.g., [13, 16-18]). Very recently, motivated by those earlier works, Set et al. [19] established some Grüss type inequalities involving the k-fractional integrals. In the present sequel to the aforementioned investigations, here, in this paper, we establish certain generalized inequalities involving generalized Riemann-Liouville $(k;\, r)$-fractional integrals.

Throughout this paper, $\mathbb{C}, \mathbb{R}, \mathbb{R}^+, \mathbb{N}$ and $\mathbb{Z}_0^-$ are the sets of complex numbers, real and positive real numbers, and positive and non-positive integers, respectively, and $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$. Let $f,\, g,\, p,\, q : [a,\, b] \to \mathbb{R}$ be integrable functions. One defines the following functionals of f and g with one weighted function p and two weighted functions p and q, respectively:

$$T(f,\, g;\, p) := \int_a^b p(x)dx \int_a^b p(x)\, f(x) g(x) dx - \int_a^b p(x)\, f(x) dx \int_a^b p(x) g(x) dx \tag{1.1}$$

and

$$T(f, g; p, q) := \int_a^b q(x)dx \int_a^b p(x) f(x) g(x) dx$$
$$+ \int_a^b p(x)dx \int_a^b q(x) f(x) g(x) dx$$
$$- \int_a^b q(x) f(x) dx \int_a^b p(x) g(x) dx$$
$$- \int_a^b p(x) f(x) dx \int_a^b q(x) g(x) dx. \tag{1.2}$$

It is obvious to see that $T(f, g; p, p) = 2T(f, g; p)$.

Further, if $p : [a, b] \to \mathbb{R}_0^+$ is an integrable function, we have

$$T(f, g; p) \geq 0, \tag{1.3}$$

whose inequality is due to Chebyshev (see [4]).

Two functions *f* and *g* are said to be *synchronous* on $[a, b]$ if

$$(f(x) - f(y))(g(x) - g(y)) \geq 0 \quad (x, y \in [a, b]). \tag{1.4}$$

Two functions *f* and *g* are said to be *asynchronous* on $[a, b]$ if

$$(f(x) - f(y))(g(x) - g(y)) \leq 0 \quad (x, y \in [a, b]). \tag{1.5}$$

If *f*, g are synchronous on $[a, b]$ and $p, q : [a, b] \to \mathbb{R}_0^+$ are integrable on $[a, b]$, then the following inequality holds (see, e.g., [10, 12]):

$$T(f, g; p, q) \geq 0. \tag{1.6}$$

Ostrowski [15] established the following generalization of the Chebyshev inequality: If f and g are two differentiable and synchronous functions on $[a, b]$, p is a positive integrable function on $[a, b]$, and $|f'(x)| \geq \mathbf{m}$ and $|g'(x)| \geq \mathbf{r}$ $(x \in [a, b])$ for some real numbers $\mathbf{m}$ and $\mathbf{r}$, then

$$T(f,\ g;\ p) \geq \mathbf{mr}T(x-a,\ x-a,\ p) \geq 0. \tag{1.7}$$

If f and g are asynchronous on $[a,\ b]$, then the inequality in (1.7) is reversed:

$$T(f,\ g;\ p) \leq \mathbf{mr}T(x-a,\ x-a,\ p) \leq 0. \tag{1.8}$$

If f and g are differentiable functions on $[a,\ b]$, p is a positive integrable function on $[a,\ b]$, and $|\ f'(x)| \leq \mathbf{M}, |\ g'(x)| \leq \mathbf{R}(x \in [a,\ b])$ for some real numbers **M** and **R**, then

$$|\ T(f,\ g,\ p)| \leq \mathbf{MR}T(x-a,\ x-a;\ p) \leq 0. \tag{1.9}$$

The following definitions are also required:

1. A real-valued function $f(t)\ (t \in \mathbb{R}^+)$ is said to be in the space $C_\mu\ (\mu \in \mathbb{R})$ if there exists a real number $p > \mu$ such that $f(t) = t^p \phi(t)$, where $\phi(t) \in C(0,\ \infty)$. A function $f(t)\ (t \in \mathbb{R}^+)$ is said to be in the space $C_\mu^\nu\ (\nu \in \mathbb{R})$ if $f^{(\nu)} \in C_\mu$.

2. The Pochhammer k-symbol $(x)_{n,k}$ and the k-gamma function Γ_k are defined as follows (see [7]):

$$(x)_{n,k} := x(x+k)(x+2k)\cdots(x+(n-1)k) \quad (n \in \mathbb{N};\ k \in \mathbb{R}^+) \tag{1.10}$$

and

$$\Gamma_k(x) := \lim_{n\to\infty} \frac{n!\,k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}} \quad (k \in \mathbb{R}^+;\ x \in \mathbb{C}\backslash k\mathbb{Z}_0^-), \tag{1.11}$$

where $k\mathbb{Z}_0^- := \{kn : n \in \mathbb{Z}_0^-\}$. It is noted that the case $k = 1$ of (1.10) and (1.11) reduces to the familiar Pochhammer symbol $(x)_n$ and the gamma function Γ. The function Γ_k is given by the following integral:

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}}\, dt \quad (\Re(x) > 0). \tag{1.12}$$

The function Γ_k defined on $\mathbb{R}^+$ is characterized by the following three properties: (i) $\Gamma_k(x+k) = x\Gamma_k(x)$; (ii) $\Gamma_k(k) = 1$; (iii) $\Gamma_k(x)$ is logarithmically convex. It is easy to see that

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right) \quad (\Re(x) > 0;\, k \in \mathbb{R}^+). \tag{1.13}$$

3. Mubeen and Habibullah [13] introduced k-fractional integral of the Riemann-Liouville type of order α as follows (see also [19]):

$$R_{a,k}^{\alpha}\{f(t)\} = \frac{1}{k\Gamma_k(\alpha)}\int_a^t (t-\tau)^{\frac{\alpha}{k}-1} f(\tau)d\tau \quad (\alpha \in \mathbb{R}^+;\, t > a), \tag{1.14}$$

which, upon setting $k = 1$, is seen to yield the classical Riemann-Liouville fractional integral of order α:

$$R_a^{\alpha}\{f(t)\} := R_{a,1}^{\alpha}\{f(t)\} = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau \ (\alpha \in \mathbb{R}^+;\, t > a). \tag{1.15}$$

Sarikaya and Karaca [17] presented $(k;\, r)$-fractional integral of the Riemann-Liouville type of order α, which is a generalization of the k-fractional integral (1.14), defined as follows:

$$R_{a,k}^{\alpha,r}\{f(t)\} = \frac{(1+r)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t (t^{r+1} - \tau^{r+1})^{\frac{\alpha}{k}-1}\tau^r f(\tau)d\tau$$

$$(\alpha,\, k \in \mathbb{R}^+;\, t > a;\, r \geq 0), \tag{1.16}$$

which, upon setting $r = 0$, immediately reduces to the k-integral (1.14).

2. Inequalities Involving Generalized Riemann-Liouville $(k;\, r)$-fractional Integral

Here we establish some inequalities involving the generalized Riemann-Liouville $(k;\, r)$-fractional integral $R_{a,k}^{\alpha,r}$ in (1.16). To do this, similarly as in (1.2), we begin by defining the following functional of f and g with two weighted functions u and v with respective to the fractional integral (1.16):

$$
\begin{aligned}
\mathcal{I}(f(t),\, g(t);\, u(t),\, v(t)) := {} & R_{a,k}^{\alpha,r}\{u(t)\} R_{a,k}^{\alpha,r}\{v(t) f(t) g(t)\} \\
& + R_{a,k}^{\alpha,r}\{v(t)\} R_{a,k}^{\alpha,r}\{u(t) f(t) g(t)\} \\
& - R_{a,k}^{\alpha,r}\{u(t) f(t)\} R_{a,k}^{\alpha,r}\{v(t) g(t)\} \\
& - R_{a,k}^{\alpha,r}\{v(t) f(t)\} R_{a,k}^{\alpha,r}\{u(t) g(t)\} \qquad (2.1)
\end{aligned}
$$

provided that $\alpha,\ k \in \mathbb{R}^+,\ a,\ r \in \mathbb{R}_0^+$ and any $t \in \mathbb{R}^+$ with $t > a$; four functions $f,\ g : \mathbb{R}_0^+ \to \mathbb{R}$ and $u,\ v : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are assumed to be so constrained that all involved fractional integrals can exist.

Lemma 1. *In addition to the conditions in* (2.1), *let f and g be synchronous functions on $\mathbb{R}_0^+$. Then the following Chebyshev type inequality holds true*: *For all $t \in \mathbb{R}^+$ with $t > a$,*

$$
\mathcal{I}(f(t),\, g(t);\, u(t),\, v(t)) \geq 0. \qquad (2.2)
$$

Proof. We find that the functional $\mathcal{I}(f(t),\, g(t);\, u(t),\, v(t))$ can be expressed as follows:

$$
\mathcal{I}(f(t),\, g(t);\, u(t),\, v(t)) = \left(\frac{(1+r)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \right)^2 \int_a^t \int_a^t \mathcal{H}(\tau,\, \rho)\, d\tau d\rho, \qquad (2.3)
$$

where

$$
\begin{aligned}
\mathcal{H}(\tau,\, \rho) := {} & (t^{r+1} - \tau^{r+1})^{\frac{\alpha}{k}-1} (t^{r+1} - \rho^{r+1})^{\frac{\alpha}{k}-1} \\
& \times (f(\tau) - f(\rho))(g(\tau) - g(\rho)) u(\tau) v(\rho) \tau^r \rho^r.
\end{aligned}
$$

Since f and g are synchronous functions on $[0,\, \infty)$,

$$
\{(f(\tau) - f(\rho))(g(\tau) - g(\rho))\} \geq 0 \quad (\tau,\, \rho \in [0,\, \infty)) \qquad (2.4)
$$

and so $\mathcal{H}(\tau, \rho)(\tau, \rho \in [0, \infty))$. Then the inequality (2.2) is seen to be proved. □

Theorem 1. *In addition to the conditions in* (2.1), *suppose f and g are bounded functions on* $\mathbb{R}_0^+$. *Let* $\Phi := \sup_{\tau \in [0, \infty)} f(\tau)$, $\varphi := \inf_{\tau \in [0, \infty)} f(\tau)$, $\Psi := \sup_{\tau \in [0, \infty)} g(\tau)$, *and* $\psi := \inf_{\tau \in [0, \infty)} g(\tau)$. *Then the following inequality holds true*: *For all* $t \in \mathbb{R}^+$ *with* $t > a$,

$$|\mathcal{I}(f(t), g(t); u(t), v(t))| \leq (\Phi - \varphi)(\Psi - \psi) R_{a,k}^{\alpha, r}\{u(t)\} R_{a,k}^{\alpha, r}\{v(t)\}. \tag{2.5}$$

Furthermore, *let* $u = c_1$, $v = c_2$ *be non-negative real constants on* $\mathbb{R}_0^+$. *Then we have*

$$|\mathcal{I}(f(t), g(t); u(t), v(t))|$$

$$\leq \frac{c_1 c_2}{(\alpha \Gamma_k(\alpha))^2} \left(\frac{t^{r+1} - a^{r+1}}{r+1} \right)^{\frac{2\alpha}{k}} (\Phi - \varphi)(\Psi - \psi). \tag{2.6}$$

Proof. It is noted that

$$|f(\tau) - f(\rho)| \leq \Phi - \varphi \text{ and } |g(\tau) - g(\rho)| \leq \Phi - \psi \tag{2.7}$$

for all $\tau, \rho \in \mathbb{R}_0^+$. It is easy to compute that

$$R_{a,k}^{\alpha, r}\{c\} = \frac{c}{\alpha \Gamma_k(\alpha)} \left(\frac{t^{r+1} - a^{r+1}}{r+1} \right)^{\frac{\alpha}{k}}, \tag{2.8}$$

where c is a real constant. It follows from (2.3) that

$$|\mathcal{I}(f(t), g(t); u(t), v(t))| \leq \left(\frac{(1+r)^{1-\frac{\alpha}{k}}}{k \Gamma_k(\alpha)} \right)^2 \int_a^t \int_a^t |\mathcal{H}(\tau, \rho)| d\tau d\rho. \tag{2.9}$$

Now applying (2.7) and (2.8) to (2.9) successively is easily seen to yield the desired inequalities (2.5) and (2.6).

Theorem 2. *In addition to the conditions in* (2.1), *suppose f and g are differentiable on* $\mathbb{R}_0^+$ *such that* f' *and* g' *are bounded on* $\mathbb{R}_0^+$. *Let* $\| f' \|_\infty := \sup_{\tau \in [0, \infty)} | f'(\tau) |$ *and* $\| g' \|_\infty := \sup_{\tau \in [0, \infty)} | g'(\tau) |$. *Then the following inequality holds true*: *For all* $t \in \mathbb{R}^+$ *with* $t > a$,

$$| \mathcal{I}(f(t), g(t); u(t), v(t)) |$$
$$\leq (t - a)^2 \| f' \|_\infty \| g' \|_\infty R_{a,k}^{\alpha, r}\{u(t)\} R_{a,k}^{\alpha, r}\{v(t)\}. \tag{2.10}$$

Proof. Let $t, a \in \mathbb{R}_0^+$ such that $t > a$ and $a \leq \tau, \rho \leq t$ $(\tau \neq \rho)$. Then use mean value theorem to get

$$f(\tau) - f(\rho) = f'(\xi)(\tau - \rho)$$

for some ξ between τ and ρ. We thus find that

$$| f(\tau) - f(\rho) | \leq | f'(\xi) || \tau - \rho | \leq \| f' \|_\infty (t - a) \tag{2.11}$$

for all τ, ρ between a and t. Similarly, we have

$$| g(\tau) - g(\rho) | \leq \| f' \|_\infty (t - a) \tag{2.12}$$

for all τ, ρ between a and t. Finally applying (2.11) and (2.12) to the inequality (2.9) is seen to yield the desired inequality (2.10). □

Theorem 3. *Let* $\alpha, k \in \mathbb{R}^+$ *and* $a, r \geq 0$. *Also let f and g be synchronous functions on* $[0, \infty)$ *and l*, *m and n be non-negative real-valued functions on* $[0, \infty)$. *Then the following inequality holds true*: *For all* $t \in \mathbb{R}^+$ *with* $t > a$,

$$2R_{a,k}^{\alpha, r}\{l(t)\}[R_{a,k}^{\alpha, r}\{m(t)\} R_{a,k}^{\alpha, r}\{n(t) f(t) g(t)\}$$
$$+ R_{a,k}^{\alpha, r}\{n(t)\} R_{a,k}^{\alpha, r}\{m(t) f(t) g(t)\}]$$
$$+ 2R_{a,k}^{\alpha, r}\{m(t)\} R_{a,k}^{\alpha, r}\{n(t)\} R_{a,k}^{\alpha, r}\{l(t) f(t) g(t)\}$$
$$\geq R_{a,k}^{\alpha, r}\{l(t)\}[R_{a,k}^{\alpha, r}\{m(t) f(t)\} R_{a,k}^{\alpha, r}\{n(t) g(t)\}$$

$$+ R_{a,k}^{\alpha,r}\{n(t)\,f(t)\}R_{a,k}^{\alpha,r}\{m(t)\,g(t)\}]$$

$$+ R_{a,k}^{\alpha,r}\{m(t)\}[R_{a,k}^{\alpha,r}\{l(t)\,f(t)\}R_{a,k}^{\alpha,r}\{n(t)\,g(t)\}$$

$$+ R_{a,k}^{\alpha,r}\{n(t)\,f(t)\}R_{a,k}^{\alpha,r}\{l(t)\,g(t)\}]$$

$$+ R_{a,k}^{\alpha,r}\{n(t)\}[R_{a,k}^{\alpha,r}\{l(t)\,f(t)\}R_{a,k}^{\alpha,r}\{m(t)\,g(t)\}$$

$$+ R_{a,k}^{\alpha,r}\{m(t)\,f(t)\}R_{a,k}^{\alpha,r}\{l(t)\,g(t)\}] \tag{2.13}$$

provided all of the functions f, g, l, m and n are restricted so that all involved integrals are convergent.

Proof. Setting $u = m$ and $v = n$ in (2.2) gives

$$R_{a,k}^{\alpha,r}\{m(t)\}R_{a,k}^{\alpha,r}\{n(t)\,f(t)\,g(t)\} + R_{a,k}^{\alpha,r}\{n(t)\}R_{a,k}^{\alpha,r}\{m(t)\,f(t)\,g(t)\}$$

$$\geq R_{a,k}^{\alpha,r}\{m(t)\,f(t)\}R_{a,k}^{\alpha,r}\{n(t)\,g(t)\} + R_{a,k}^{\alpha,r}\{n(t)\,f(t)\}R_{a,k}^{\alpha,r}\{m(t)\,g(t)\}. \tag{2.14}$$

Multiplying both sides of (2.14) by $R_{a,k}^{\alpha,r}\{l(t)\}(\geq 0)$, we have

$$R_{a,k}^{\alpha,r}\{l(t)\}[R_{a,k}^{\alpha,r}\{m(t)\}R_{a,k}^{\alpha,r}\{n(t)\,f(t)\,g(t)\}$$

$$+ R_{a,k}^{\alpha,r}\{n(t)\}R_{a,k}^{\alpha,r}\{m(t)\,f(t)\,g(t)\}]$$

$$\geq R_{a,k}^{\alpha,r}\{l(t)\}[R_{a,k}^{\alpha,r}\{m(t)\,f(t)\}R_{a,k}^{\alpha,r}\{n(t)\,g(t)\}$$

$$+ R_{a,k}^{\alpha,r}\{n(t)\,f(t)\}R_{a,k}^{\alpha,r}\{m(t)\,g(t)\}]. \tag{2.15}$$

Similarly, replacing u, v in (2.2) by l, n and l, m, respectively, and multiplying both sides of the resulting inequalities by $R_{a,k}^{\alpha,r}\{m(t)\}$ and $R_{a,k}^{\alpha,r}\{n(t)\}$, respectively, we get the following inequalities:

$$R_{a,k}^{\alpha,r}\{m(t)\}[R_{a,k}^{\alpha,r}\{l(t)\}R_{a,k}^{\alpha,r}\{n(t)f(t)g(t)\}$$

$$+R_{a,k}^{\alpha,r}\{n(t)\}R_{a,k}^{\alpha,r}\{l(t)f(t)g(t)\}]$$

$$\geq R_{a,k}^{\alpha,r}\{m(t)\}[R_{a,k}^{\alpha,r}\{l(t)f(t)\}R_{a,k}^{\alpha,r}\{n(t)g(t)\}$$

$$+R_{a,k}^{\alpha,r}\{n(t)f(t)\}R_{a,k}^{\alpha,r}\{l(t)g(t)\}] \tag{2.16}$$

and

$$R_{a,k}^{\alpha,r}\{n(t)\}[R_{a,k}^{\alpha,r}\{l(t)\}R_{a,k}^{\alpha,r}\{m(t)f(t)g(t)\}$$

$$+R_{a,k}^{\alpha,r}\{m(t)\}R_{a,k}^{\alpha,r}\{l(t)f(t)g(t)\}]$$

$$\geq R_{a,k}^{\alpha,r}\{n(t)\}[R_{a,k}^{\alpha,r}\{l(t)f(t)\}R_{a,k}^{\alpha,r}\{m(t)g(t)\}$$

$$+R_{a,k}^{\alpha,r}\{m(t)f(t)\}R_{a,k}^{\alpha,r}\{l(t)g(t)\}]. \tag{2.17}$$

Finally, by adding (2.15), (2.16) and (2.17) side by side, we arrive at the desired result (2.13). □

To establish our second result, we give the following inequality involving the generalized $(k;\, r)$-fractional integral operator in (1.16), which is asserted by Lemma 2.

Lemma 2. *Let* $\alpha, \beta, k \in \mathbb{R}^+$ *and* $a, r \geq 0$. *Also let* f *and* g *be synchronous functions on* $[0, \infty)$ *and* u *and* v *be non-negative real-valued functions on* $[0, \infty)$. *Then the following inequality holds true*: *For all* $t \in \mathbb{R}^+$ *with* $t > a$,

$$R_{a,k}^{\beta,r}\{v(t)\}R_{a,k}^{\alpha,r}\{u(t)f(t)g(t)\}+R_{a,k}^{\beta,r}\{v(t)f(t)g(t)\}R_{a,k}^{\alpha,r}\{u(t)\}$$

$$\geq R_{a,k}^{\beta,r}\{v(t)g(t)\}R_{a,k}^{\alpha,r}\{u(t)f(t)\}+R_{a,k}^{\beta,r}\{v(t)f(t)\}R_{a,k}^{\alpha,r}\{u(t)g(t)\}, \tag{2.18}$$

provided all of the functions f, g, u *and* v *are restricted so that all involved integrals are convergent.*

Proof. Multiplying both sides of (2.4) by

$$\frac{(1+r)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}(t^{r+1}-\rho^{r+1})^{\frac{\beta}{k}-1}\rho^r \quad (t>\rho),$$

and integrating the resulting inequality with respect to ρ from a to t, in view of (1.16), we are led to the desired result (2.18). □

Theorem 4. *Let* $\alpha, \beta, k \in \mathbb{R}^+$ *and* $a, r \geq 0$. *Also let* f *and* g *be synchronous functions on* $[0, \infty)$ *and* l, m *and* n *be non-negative real-valued functions on* $[0, \infty)$. *Then the following inequality holds true*: *For all* $t \in \mathbb{R}^+$ *with* $t > a$,

$$\begin{aligned}
&R_{a,k}^{\alpha,r}\{l(t)\}[2R_{a,k}^{\alpha,r}\{m(t)\}R_{a,k}^{\beta,r}\{n(t)f(t)g(t)\}\\
&+R_{a,k}^{\alpha,r}\{n(t)\}R_{a,k}^{\beta,r}\{m(t)f(t)g(t)\}\\
&+R_{a,k}^{\beta,r}\{n(t)\}R_{a,k}^{\alpha,r}\{m(t)f(t)g(t)\}]\\
&+R_{a,k}^{\alpha,r}\{l(t)f(t)g(t)\}[R_{a,k}^{\alpha,r}\{m(t)\}R_{a,k}^{\beta,r}\{n(t)\}+R_{a,k}^{\alpha,r}\{n(t)\}R_{a,k}^{\beta,r}\{m(t)\}]\\
\geq\ &R_{a,k}^{\alpha,r}\{l(t)\}[R_{a,k}^{\alpha,r}\{m(t)f(t)\}R_{a,k}^{\beta,r}\{n(t)g(t)\}\\
&+R_{a,k}^{\alpha,r}\{m(t)g(t)\}R_{a,k}^{\beta,r}\{n(t)f(t)\}]\\
&+R_{a,k}^{\alpha,r}\{m(t)\}[R_{a,k}^{\alpha,r}\{l(t)f(t)\}R_{a,k}^{\beta,r}\{n(t)g(t)\}\\
&+R_{a,k}^{\alpha,r}\{l(t)g(t)\}R_{a,k}^{\beta,r}\{n(t)f(t)\}]\\
&+R_{a,k}^{\alpha,r}\{n(t)\}[R_{a,k}^{\alpha,r}\{l(t)f(t)\}R_{a,k}^{\beta,r}\{m(t)g(t)\}\\
&+R_{a,k}^{\alpha,r}\{l(t)g(t)\}R_{a,k}^{\beta,r}\{m(t)f(t)\}],
\end{aligned} \tag{2.19}$$

provided all of the functions f, g, l, m *and* n *are restricted so that all involved integrals are convergent.*

Proof. Here, using the result in Lemma 2, a similar argument as in the proof of Theorem 3 will establish the inequality (2.19). So the detailed account of the proof is omitted. □

Remark 1. The inequalities in Lemmas 1 and 2, and Theorems 3 and 4 are reversed if the functions f and g are asynchronous on $[0, \infty)$. The case $\alpha = \beta$ of (2.19) reduces to the inequality in Theorem 3.

All results presented here can yield those involving *k*-fractional integral of the Riemann-Liouville type of order α in (1.14) and, further, the classical Riemann-Liouville fractional integral operator in (1.15).

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