Far East Journal of Mathematical Sciences (FJMS)

Published: December 2016
http://dx.doi.org/10.17654/MS100111745
Volume 100, Number 11, 2016, Pages 1745-1754 **ISSN: 0972-0871**

A COMPARISON OF THE PERFORMANCE OF *t* CONTROL CHARTS IN SHORT RUNS PROCESSES

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Abstract

The main objective of this paper is to study an efficiency of control charts using median run length (MRL). The performance of control charts for quickest detection of change in mean process with exponentially weighted moving average $(\text{EWMA}\ \overline{X})$, t exponentially weighted moving average (EWMA t) and t cumulative sum (CUSUM t) charts when observation is from non-central t distribution is studied. We compare the performances between EWMA $\overline{X}$, EWMA t and CUSUM t charts by using MRL criterion. The numerical results indicate that EWMA t chart is an effective alternative to the EWMA $\overline{X}$ and CUSUM t control charts for small shifts.

Received: May 27, 2016; Revised: July 23, 2016; Accepted: September 10, 2016
2010 Mathematics Subject Classification: 62F25.
Keywords and phrases: median run length (MRL), EWMA $\overline{X}$, EWMA t, CUSUM t.

1. Introduction

Statistical process control (SPC) charts are often used to monitor processes for the purpose of detecting, monitoring and improving for a change in a process. A variety of statistical methods have been developed in many areas of interest including, industrial, engineering, epidemiology and health care and others. Examples of SPC charts include the Shewhart control chart proposed by Shewhart [5], the cumulative sum (CUSUM) control chart first presented by Page [3], and the exponentially weighted moving average $(\text{EWMA}\ \bar{X})$ control chart was introduced by Roberts [4]. These charts are used to monitor product quality and detect the occurrence of special causes that may indicate out-of-control situations. The Shewhart chart is still widely used in many applications as it is useful for detecting large changes in process means. However, the Shewhart chart has been found to be inadequate for detecting small shifts in parameters. The CUSUM and EWMA charts have been proposed as good alternatives to the Shewhart chart for detecting small shifts. Borror et al. [1] compared the average run length (ARL) performance between Shewhart and EWMA charts for the case of non-normal distributions using Markov chain approach and have shown that EWMA chart is more robust to the assumption of normality. Recently, Zhang et al. [6] proposed the EWMA *t* chart and showed that the EWMA *t* chart is more robust than the EWMA $\bar{X}$ chart in detecting changes in the process standard deviation. Later, Celano et al. [2] studied economic design with CUSUM *t* control chart for monitoring short production runs and compared the performances of CUSUM *t* and CUSUM $\bar{X}$ charts.

In most situations, the average run length (ARL) or the median run length (MRL) is used to measure a chart's performance. The ARL and MRL are defined as the average and median numbers of sample points that are plotted on a chart before an out-of-control signal is issued, namely, ARL_0 and MRL_0, respectively. The second important characteristic for SPC charts is the expected number or median of observations taken from an out-of-control process until the control chart signals that the process is out-of-control denoted by ARL_1 and MRL_1. The ARL_0 and MRL_0 of an acceptable chart should be large when the process is in-control and the ARL_1 and MRL_1 should be small when the process goes out-of-control. Interpretations based on ARL alone can be misleading, as the in-control run length distribution of a control chart is highly skewed. The interpretations become more difficult as the shape of the run length distribution changes according to the mean shifts. However, when using the MRL, this interpretation problem will not occur.

In this paper, the efficiency of median run length (MRL) and average run length (ARL) are studied by using Monte Carlo simulation. The performance of control charts for quickest detection of change in mean processes of exponentially weighted moving average $(\text{EWMA}\,\overline{X})$, t exponentially weighted moving average (EWMA *t*) and *t* cumulative sum (CUSUM *t*) charts when observation is from non-central *t* distribution is presented. Additionally, we compared the performance between $\text{EWMA}\,\overline{X}$, EWMA *t* and CUSUM *t* charts by using MRL criterion.

2. Control Charts and their Properties

Let $X_1, X_2, ..., X_t$, $t = 1, 2, ...$ be sequentially observed independent random variables with a distribution function $F(x, \mu, \nu)$, where μ is a control parameter. It is assumed that $\mu = \mu_0$ while the process is in-control and $\mu = \mu_1 > \mu_0$ when the process goes out-of-control. It is assumed that there is a change-point time $\theta \leq \infty$ at which the parameter changes from $\mu = \mu_0$ to $\mu = \mu_1$. Note that $\theta = \infty$ means that the process always remains in the in-control state.

Assume that samples $\{X_{i,1}, X_{i,2}, ..., X_{i,n}\}$ are taken at time to times. The subgroup mean $\overline{X}_i$ and standard deviation S_i are computed as follows:

$$\overline{X}_i = \frac{\sum_{j=1}^{n} X_{i,j}}{n}, \quad S_i = \frac{\sqrt{\sum_{j=1}^{n} (X_{i,j} - \overline{X}_i)^2}}{n-1}.$$

A. EWMA $\overline{X}$ **chart**

In 1959, Roberts [4] proposed an exponentially weighted moving average $(\text{EWMA } \overline{X})$ control chart which is an effective alternative to the traditional Shewhart control chart for detecting small shifts in a process mean. The EWMA $\overline{X}$ statistics can be written as follows:

$$Z_i = \lambda \overline{X}_i + (1-\lambda) Z_{i-1}, \quad i = 1, 2, \ldots,$$

where λ is a smoothing constant that satisfies $0 < \lambda < 1.$ The control limits of EWMA $\overline{X}$ control chart can be written as follows:

$$UCL_{EWMA} = \mu_0 + k\sqrt{\frac{\lambda}{n(2-\lambda)}} \quad \text{and} \quad LCL_{EWMA} = \mu_0 - k\sqrt{\frac{\lambda}{n(2-\lambda)}},$$

where k is a width of control limit based on desired in-control ARL (ARL_0) or in-control MRL (MRL_0).

The control limit of EWMA control chart is given by:

$$\tau_{UCL_{EWMA}} = \inf\{t > 0 : Z_i > UCL_{EWMA}\}.$$

B. EWMA *t* chart

In 2009, Zhang et al. [6] proposed the *t* exponentially weighted moving average (EWMA *t*) control chart which is an effective alternative to the traditional EWMA $\overline{X}$ control chart for detecting small shifts in a process mean. The EWMA *t* statistics can be written as follows:

$$Y_i = \lambda T_i + (1-\lambda) Y_{i-1}, \quad i = 1, 2, \ldots,$$

and the corresponding statistic T_i is defined as: $T_i = \dfrac{\overline{X}_i - \mu_0}{S_i/\sqrt{n}}$, λ is weight of past information, $0 < \lambda < 1$, $Y_0 = T_0 = 0$. Note that T_i follows a non-central *t* distribution with $n-1$ degrees of freedom and non-centrality parameter μ. The control limits of EWMA *t* control chart are

$$UCL_{EWMA\,t} = F_t^{-1}(1-\alpha/2 \mid n-1) \quad \text{and} \quad LCL_{EWMA\,t} = -UCL_{EWMA\,t},$$

where $F_t^{-1}(\cdot \mid n-1)$ is the inverse distribution function of the Student's *t* distribution with $n-1$ degrees of freedom, and α is the false alarm rate such as $\alpha = 0.0027$.

The control limit of EWMA *t* control chart is given by:

$$\tau_{UCL_{EWMA\,t}} = \inf\{t > 0 : Y_i > UCL_{EWMA\,t}\}.$$

C. CUSUM *t* chart

For the short run process, the T_i statistic is monitored by means of a one-sided CUSUM *t* control chart for detecting upward shifts in the process mean. The CUSUM *t* statistic is

$$C_i = \max(0,\ C_{i-1} + T_i - E(T) - a),$$

where $C_0 = 0$ is an initial value, *a* is a constant recall reference value of CUSUM control chart and $E(T) = 0$ is the mean of the Student's *t* distribution function with $n-1$ degrees of freedom.

The control limit of CUSUM control chart is given by:

$$\tau_b = \inf\{t > 0 : C_i > b\},$$

where *b* is a constant parameter known as the upper control limit.

3. Numerical Results

In this section, we compare the efficiency of median run length (MRL) and average run length (ARL) with the results of Monte Carlo simulations as shown in Tables 1-3. In Table 1, For EWMA $\overline{X}$ chart when $ARL_0 = MRL_0 = 370$, $\lambda = 0.01, 0.05$, $n = 5$, the MRL is slightly less than ARL for all magnitudes of change. In Table 2, For EWMA t chart when ARL_0 = MRL_0 = 370, $\lambda = 0.01$, $n = 5$, the MRL is slightly less than ARL for small otherwise the ARL is slightly less than MRL while for $\lambda = 0.05$, the MRL is slightly less than ARL for all magnitudes of change. In Table 3, for CUSUM t chart when $ARL_0 = MRL_0 = 370$, $a = 0.05$, $n = 5$, the MRL is slightly less than ARL for all magnitudes of change. In Tables 4 to 6, the performances of EWMA $\overline{X}$, EWMA t and CUSUM t are compared when given $MRL_0 = 370$, $a = 0.05$, $n = 3, 5, 10$ by using MRL criteria. The results show that EWMA t control chart performs better than the EWMA $\overline{X}$ and CUSUM t charts for small to moderate values of change otherwise the CUSUM t chart is superior to the EWMA $\overline{X}$ and EWMA t charts.

Table 1. ARL and MRL of EWMA $\overline{X}$ chart when $ARL_0 = 370$, MRL_0 = 370, $n = 5$, $\lambda = 0.01, 0.05$

μ	$\lambda = 0.01$		$\lambda = 0.05$	
	ARL $UCL = 0.0546$	MRL $UCL = 0.0685$	ARL $UCL = 0.1562$	MRL $UCL = 0.1614$
0.0	370.733 ± 1.305*	370	370.683 ± 1.288	370
0.1	62.192 ± 0.924	59	59.841 ± 0.903	54
0.2	30.084 ± 0.817	29	23.148 ± 0.843	21
0.3	21.432 ± 0.779	19	17.795 ± 0.731	16
0.4	14.676 ± 0.654	14	15.504 ± 0.608	14
0.5	12.937 ± 0.524	11	12.178 ± 0.501	11
0.6	9.574 ± 0.489	9	10.671 ± 0.462	9
0.7	8.261 ± 0.317	8	8.654 ± 0.318	8
0.8	7.574 ± 0.227	7	6.924 ± 0.210	6
0.9	6.556 ± 0.115	6	5.344 ± 0.105	5
1.0	5.982 ± 0.092	5	3.933 ± 0.098	3
1.5	3.369 ± 0.053	3	2.686 ± 0.051	2
2.0	2.989 ± 0.021	2	1.107 ± 0.026	1

*standard deviation

Table 2. ARL and MRL of EWMA *t* chart when $ARL_0 = 370$, $MRL_0 = 370$, $n = 5$, $\lambda = 0.01, 0.05$

μ	$\lambda = 0.01$		$\lambda = 0.05$	
	ARL $UCL = 0.0472$	MRL $UCL = 0.046$	ARL $UCL = 0.1365$	MRL $UCL = 0.1423$
0.0	370.748 ± 1.263*	370	370.844 ± 1.317	370
0.1	58.342 ± 0.975	56	55.639 ± 0.963	51
0.2	27.329 ± 0.892	28	25.602 ± 0.832	24
0.3	17.496±0.748	19	15.628 ± 0.786	15
0.4	12.697 ± 0.602	14	10.453 ± 0.650	14
0.5	9.871 ± 0.596	11	9.419 ± 0.573	9
0.6	8.065 ± 0.437	9	8.502 ± 0.487	8
0.7	6.759 ± 0.349	8	7.434 ± 0.349	7
0.8	5.826 ± 0.204	7	6.862 ± 0.292	6
0.9	5.116 ± 0.173	6	5.167 ± 0.186	5
1.0	4.502 ± 0.083	5	3.058 ± 0.097	3
1.5	2.785 ± 0.058	3	2.966 ± 0.058	2
2.0	1.967 ± 0.023	2	1.428 ± 0.017	1

*standard deviation

Table 3. ARL and MRL of CUSUM *t* chart when $ARL_0 = 370$, $MRL_0 = 370$, $n = 5$ and $\alpha = 0.5$

μ	ARL $UCL = 1.3485$	MRL $UCL = 1.46$
0.0	370.956 ± 1.204*	370
0.1	207.671 ± 0.917	191
0.2	106.809 ± 0.836	92
0.3	52.157 ± 0.722	43
0.4	27.404 ± 0.864	22
0.5	15.491 ± 0.733	13
0.6	9.189 ± 0.658	8
0.7	6.159 ± 0.552	5
0.8	3.384 ± 0.472	4
0.9	4.519 ± 0.355	3
1.0	2.704 ± 0.296	2
1.5	1.271 ± 0.061	1
2.0	1.021 ± 0.033	1

*standard deviation

Table 4. MRL of EWMA $\overline{X}$, EWMA t and CUSUM t charts when MRL_0 = 370, $n = 3$, $a = 0.5$

μ	EWMA $\overline{X}$	EWMA t	CUSUM t
	$UCL = 0.088$	$UCL = 0.0502$	$UCL = 1.5511$
0.0	370	370	370
0.1	73	**72**	232
0.2	37	**34**	138
0.3	24	**22**	79
0.4	18	**18**	45
0.5	14	**14**	27
0.6	12	**12**	16
0.7	10	**10**	11
0.8	9	9	**8**
0.9	8	8	**6**
1.0	7	7	**5**
1.5	4	4	**2**
2.0	3	3	**1**

Table 5. MRL of EWMA $\overline{X}$, EWMA t and CUSUM t charts when MRL_0 = 370, $n = 5$, $a = 0.5$

μ	EWMA $\overline{X}$	EWMA t	CUSUM t
	$UCL = 0.0685$	$UCL = 0.046$	$UCL = 1.46$
0.0	370	370	370
0.1	59	**56**	191
0.2	29	**28**	92
0.3	19	**19**	43
0.4	14	**14**	22
0.5	11	**11**	13
0.6	9	9	**8**
0.7	8	8	**5**
0.8	7	7	**4**
0.9	6	6	**3**
1.0	5	5	**2**
1.5	3	3	**1**
2.0	2	2	**1**
3.0	1	1	**1**

Table 6. MRL of EWMA $\overline{X}$, EWMA t and CUSUM t charts when $MRL_0 = 370$, $n = 10$, $a = 0.5$

μ	EWMA $\overline{X}$	EWMA *t*	CUSUM *t*
	$UCL = 0.0557$	$UCL = 0.0375$	$UCL = 1.36242$
0.0	370	370	370
0.1	40	**39**	134
0.2	18	**17**	45
0.3	12	**11**	17
0.4	8	**7**	8
0.5	6	5	**5**
0.6	5	4	**3**
0.7	4	4	**2**
0.8	3	3	**2**
0.9	3	3	**1**
1.0	3	3	**1**
1.5	2	2	**1**
2.0	2	1	**1**
3.0	1	1	**1**

Acknowledgements

The author would like to express his gratitude to King Mongkut's University of Technology North Bangkok, Thailand for supporting research grant NO: KMUTNB-GEN-59-34.

The author thanks the anonymous referees for their valuable suggestions and constructive criticism which improved the presentation of the paper.

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