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UNIQUENESS OF MEROMORPHIC FUNCTIONS THAT SHARE FIXED POINTS

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Abstract

In this paper, we study the uniqueness of meromorphic functions that share fixed points. The result obtained in this paper extends the result due to Lei et al. [3].

1. Introduction and Main Results

Let f be a non-constant meromorphic function in the whole complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions: (see [1, 2])

$$T(r, f), m(r, f), N(r, f), \overline{N}(r, f), N(r, 1/f), \ldots.$$

By $S(r, f)$, we denote any quantity satisfying $S(r, f) = o(T(r, f))$, as $r \to \infty$, possibly outside a set of r with finite linear measure.

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Let a be a finite complex number and k be a positive integer. We denote by $N_{k)}\left(r, \frac{1}{f-a}\right)$ the counting function for zeros of $f(z)-a$ in $|z| \leq r$ with multiplicity $\leq k$ and by $\overline{N}_{k)}\left(r, \frac{1}{f-a}\right)$ the corresponding one for which multiplicity is not counted. Let $N_{(k}\left(r, \frac{1}{f-a}\right)$ be the counting function for zeros of $f(z)-a$ in $|z| \leq r$ with multiplicity $\geq k$ and $\overline{N}_{(k}\left(r, \frac{1}{f-a}\right)$ the corresponding one for which multiplicity is not counted.

Set

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + \cdots + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right).$$

Let f and g be two non-constant meromorphic functions. We say that f, g share the value a CM (counting multiplicities) if f, g have the same a-points with the same multiplicities and we say that f, g share the value a IM (ignoring multiplicities) if we do not consider the multiplicities. We denote by $\overline{N}_L\left(r, \frac{1}{f-a}\right)$ the counting function for a-points of both f and g about which f has larger multiplicity than g, where multiplicity is not counted. Similarly, we have notation $\overline{N}_L\left(r, \frac{1}{g-a}\right)$.

Fang and Hua [4] and Yang and Hua [5] obtained the following unicity theorem:

Theorem A. *Let* $f(z)$, $g(z)$ *be two non-constant meromorphic*(*entire*) *functions*, *let* $n \geq 11 (\geq 6)$ *be a positive integer. If* $f^n f'$ *and* $g^n g'$ *share* 1 *CM*, *then either* $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, *where* c_1, c_2 *and* c *are three non-zero constants satisfying* $(c_1 c_2)^{n+1} c^2 = -1$ *or* $f(z) = tg(z)$ *for a constant* t *such that* $t^{n+1} = 1$.

In 2000, Fang and Qiu [6] proved the following result:

Theorem B. *Let* $f(z)$, $g(z)$ *be two non-constant meromorphic functions, let* $n \geq 11$ *be a positive integer. If* $f^n f'$ *and* $g^n g'$ *share z CM, then either* $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, *where* c_1, c_2 *and c are three non-zero constants satisfying* $4(c_1 c_2)^{n+1} c^2 = -1$ *or* $f(z) = tg(z)$ *for a constant t such that* $t^{n+1} = 1$.

In 2002, Fang [7] proved the following result:

Theorem C. *Let* $f(z)$, $g(z)$ *be two non-constant entire functions, let n, k be two positive integers with* $n > 2k + 4$. *If* $(f^n)^{(k)}$ *and* $(g^n)^{(k)}$ *share* 1 *CM, then either* $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, *where* c_1, c_2 *and c are three non-zero constants satisfying* $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$ *or* $f(z) = tg(z)$ *for a constant t such that* $t^n = 1$.

Recently, Xu et al. [8] proved the following theorem:

Theorem D. *Let* $f(z)$, $g(z)$ *be two non-constant meromorphic functions, let n, k be two positive integers with* $n \geq 3k + 11$. *If* $(f^n)^{(k)}$ *and* $(g^n)^{(k)}$ *share z CM;* $f(z)$ *and* $g(z)$ *share* ∞ *IM, then either* $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, *where* c_1, c_2 *and c are three non-zero constants satisfying* $4n^2 (c_1 c_2)^n c^2 = -1$ *or* $f(z) = tg(z)$ *for a constant t such that* $t^n = 1$.

Recently, Lei et al. [3] improved Theorem D as follows:

Theorem E. *Let* $f(z)$, $g(z)$ *be two non-constant meromorphic functions, let n, k be two positive integers with* $n \geq 3k + 7$. *If* $(f^n)^{(k)}$ *and* $(g^n)^{(k)}$ *share z CM;* $f(z)$ *and* $g(z)$ *share* ∞ *IM, then:* (1) $f(z) = tg(z)$

for $k \geq 2$; (2) *either* $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$ *or* $f(z) = tg(z)$ *for* $k = 1$, *where* c_1, c_2 *and* c *are three non-zero constants satisfying* $4n^2(c_1c_2)^n c^2 = -1$ *and t is a constant such that* $t^n = 1$.

In this paper, we define

$$P(w) = \begin{cases} a_m w^m + a_{m-1}w^{m-1} + \cdots + a_1 w + a_0, & m > 0, \\ a_0, & m = 0, \end{cases} \tag{1}$$

where m is a non-negative integer, $a_0 \neq 0, a_1, ..., a_{m-1}, a_m \neq 0$ are complex constants and hence we extend Theorem E by obtaining the following result:

Theorem 1. *Let* $f(z)$, $g(z)$ *be two non-constant meromorphic functions, let* n, k *and* m *be three positive integers with* $n \geq 3k + m + 8$, $P(f)$ *be defined as in* (1). *If* $(f^n P(f))^{(k)}$ *and* $(g^n P(g))^{(k)}$ *share z CM*; $f(z)$ *and* $g(z)$ *share* ∞ *IM, then one of the following two cases holds*:

(i) $f(z) \equiv tg(z)$ *for a constant* t *such that* $t^d = 1$, *where* $d = GCD(n+m, ..., n+m-i, ..., n)$, $a_{m-i} \neq 0$ *for some* $i = 0, 1, ..., m$;

(ii) *f and g satisfy the algebraic equation* $R(f, g) \equiv 0$, *where*

$$R(w_1, w_2) = w_1^n(a_m w_1^m + a_{m-1}w_1^{m-1} + \cdots + a_1 w_1 + a_0) - w_2^n(a_m w_2^m + a_{m-1}w_2^{m-1} + \cdots + a_1 w_2 + a_0).$$

2. Preliminary Lemmas

Lemma 2.1 (See [9]). *Let* $f(z)$ *be a non-constant meromorphic function satisfying* $f^{(k)}(z) \not\equiv 0$, *let k be a positive integer. Then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f).$$

Lemma 2.2 (See [3]). *Let* $f(z)$ *and* $g(z)$ *be two non-constant meromorphic functions. If* $f(z)$ *and* $g(z)$ *share* 1 *CM*; $f(z)$ *and* $g(z)$ *share* ∞ *IM, then one of the following cases must occur*:

(i) $$T(r, f) + T(r, g) \le 2\left\{N_2\left(r, \frac{1}{f}\right) + N_2\left(r, \frac{1}{g}\right)\right\}$$

$$+ 4\overline{N}(r, f) + 2\{\overline{N_L}(r, f) + \overline{N_L}(r, g)\} + S(r, f) + S(r, g);$$

(ii) *either* $f(z)g(z) \equiv 1$ *or* $f(z) \equiv g(z)$.

By using the same method as in Lemma 5 of [8], we obtain the following lemma:

Lemma 2.3 (See [8]). *Let* $f(z)$ *and* $g(z)$ *be two non-constant meromorphic functions*, $P(f)$ *be defined as in* (1), $n > 0$, $k > 0$ *and* $m \ge 0$ *be three integers with* $n > 2k + m + 1$. *If* $[f^n P(f)]^{(k)} = [g^n P(g)]^{(k)}$, *then* $f^n P(f) = g^n P(g)$.

Lemma 2.4 (See [10]). *Let* $f(z)$ *and* $g(z)$ *be two non-constant meromorphic functions*, $n(\ge 1)$, $k(\ge 1)$, $m(\ge 1)$ *be three integers. Then* $[f^n P(f)]^{(k)} \cdot [g^n P(g)]^{(k)} \ne z^2$.

3. Proof of Theorem 1

Let $P(f)$ be defined as in (1). Set $F = f^n P(f)$, $G = g^n P(g)$. Thus, $\frac{F^{(k)}}{z}$ and $\frac{G^{(k)}}{z}$ share 1 CM; $\frac{F^{(k)}}{z}$ and $\frac{G^{(k)}}{z}$ share ∞ IM. Suppose that

$$T\left(r, \frac{F^{(k)}}{z}\right) + T\left(r, \frac{G^{(k)}}{z}\right)$$

$$\le 2\left\{N_2\left(r, \frac{z}{F^{(k)}}\right) + N_2\left(r, \frac{z}{G^{(k)}}\right)\right\} + 4\overline{N}\left(r, \frac{F^{(k)}}{z}\right)$$

$$+ 2\left\{\overline{N}_L\left(r, \frac{F^{(k)}}{z}\right) + \overline{N}_L\left(r, \frac{G^{(k)}}{z}\right)\right\} + S(r, f) + S(r, g). \tag{2}$$

We note that

$$N_2\left(r, \frac{1}{F^{(k)}}\right) + N_2\left(r, \frac{1}{G^{(k)}}\right)$$

$$\leq N\left(r, \frac{1}{F^{(k)}}\right) - \left(N_{(3}\left(r, \frac{1}{F^{(k)}}\right) - 2\overline{N}_{(3}\left(r, \frac{1}{F^{(k)}}\right)\right)$$

$$+ N\left(r, \frac{1}{G^{(k)}}\right) - \left(N_{(3}\left(r, \frac{1}{G^{(k)}}\right) - 2\overline{N}_{(3}\left(r, \frac{1}{G^{(k)}}\right)\right). \tag{3}$$

If z_0 is a zero of $f(z)$ with multiplicity p, then z_0 is a zero of $[f^n P(f)]^{(k)}$ with multiplicity $np - k$, we have

$$N_{(3}\left(r, \frac{1}{F^{(k)}}\right) - 2\overline{N}_{(3}\left(r, \frac{1}{F^{(k)}}\right) \geq (n - k - 2) N(r, 1/f). \tag{4}$$

Similarly,

$$N_{(3}\left(r, \frac{1}{G^{(k)}}\right) - 2\overline{N}_{(3}\left(r, \frac{1}{G^{(k)}}\right) \geq (n - k - 2) N(r, 1/g). \tag{5}$$

By equations (2)-(5), we have

$$T\left(r, \frac{1}{F^{(k)}}\right) + T\left(r, \frac{1}{G^{(k)}}\right)$$

$$\leq T\left(r, \frac{z}{F^{(k)}}\right) + T\left(r, \frac{z}{G^{(k)}}\right) + 2 \log r$$

$$\leq T\left(r, \frac{F^{(k)}}{z}\right) + T\left(r, \frac{G^{(k)}}{z}\right) + 2 \log r + O(1)$$

$$\leq 2\left(N\left(r, \frac{1}{F^{(k)}}\right) + N\left(r, \frac{1}{G^{(k)}}\right)\right) + 4\overline{N}(r, f)$$

$$+ 2(k + 2 - n)\left(N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right)\right)$$

$$+ 6\log r + 2\left\{\overline{N}_L\left(r, \frac{F^{(k)}}{z}\right) + \overline{N}_L\left(r, \frac{G^{(k)}}{z}\right)\right\} + S(r, f) + S(r, g). \quad (6)$$

Note that

$$(n + m)m\left(r, \frac{1}{f}\right) = m\left(r, \frac{1}{F}\right) \leq m\left(r, \frac{1}{F^{(k)}}\right) + S(r, f)$$

$$\leq T\left(r, \frac{1}{F^{(k)}}\right) - N\left(r, \frac{1}{F^{(k)}}\right) + S(r, f). \quad (7)$$

Similarly, we have

$$(n + m)m\left(r, \frac{1}{g}\right) \leq T\left(r, \frac{1}{G^{(k)}}\right) - N\left(r, \frac{1}{G^{(k)}}\right) + S(r, g). \quad (8)$$

From equations (6)-(8) and Lemma 2.1, we have

$$(n + m)[T(r, f) + T(r, g)]$$

$$\leq N\left(r, \frac{1}{F^{(k)}}\right) + N\left(r, \frac{1}{G^{(k)}}\right) + (2k + 4 + m - n)\left(N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right)\right)$$

$$+ 6\log r + 4\overline{N}(r, f) + 2\left\{\overline{N}_L\left(r, \frac{F^{(k)}}{z}\right) + \overline{N}_L\left(r, \frac{G^{(k)}}{z}\right)\right\}$$

$$+ S(r, f) + S(r, g)$$

$$\leq 2(k + m + 2)\left(N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right)\right) + (2k + 4)\overline{N}(r, f)$$

$$+ 6\log r + 2\left\{\overline{N}_L\left(r, \frac{F^{(k)}}{z}\right) + \overline{N}_L\left(r, \frac{G^{(k)}}{z}\right)\right\} + S(r, f) + S(r, g). \quad (9)$$

Noting that $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share z CM; $f(z)$ and $g(z)$ share ∞ IM, we have

$$2\left\{\overline{N}_L\left(r, \frac{F^{(k)}}{z}\right) + \overline{N}_L\left(r, \frac{G^{(k)}}{z}\right)\right\} \leq \overline{N}(r, f) + \overline{N}(r, g).$$

From (9), we have

$$(n + m)(T(r, f) + T(r, g))$$
$$\leq 2(k + m + 2)\left(N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right)\right) + (k + 3)(\overline{N}(r, f) + \overline{N}(r, g))$$
$$+ 6\log r + S(r, f) + S(r, g). \tag{10}$$

Next, we consider two cases:

Case 1. Either $f(z)$ or $g(z)$ is a transcendental meromorphic function. If $n > 3k + m + 8,$ then it follows from (10) that

$$T(r, f) + T(r, g) \leq 6\log r + S(r, f) + S(r, g),$$

a contradiction. If $n = 3k + m + 8,$ then from (10), we get

$$N_{(2}(r, f) = S(r, f), \quad N_{(2}(r, g) = S(r, g).$$

Thus,

$$\overline{N}_L\left(r, \frac{F^{(k)}}{z}\right) = S(r, f), \quad \overline{N}_L\left(r, \frac{G^{(k)}}{z}\right) = S(r, g).$$

It follows from (9) that

$$T(r, f) + T(r, g) \leq 6\log r + S(r, f) + S(r, g),$$

a contradiction.

Case 2. Both $f(z)$ and $g(z)$ are two non-constant rational functions. If $f(z)$ is a polynomial, then $g(z)$ is a polynomial. Thus, from (9),

$$8\log r \leq (k + 3)(T(r, f) + T(r, g)) \leq 6\log r + O(1),$$

a contradiction. Thus, both $f(z)$ and $g(z)$ are non-polynomial rational functions. By (10), we have

$$2(k+m+2)\left(m\left(r, \frac{1}{f}\right)+m\left(r, \frac{1}{g}\right)\right)$$

$$+(k+3)\left(N_{(2}(r, f)+N_{(2}(r, g)-\overline{N}_{(2}(r, f)-\overline{N}_{(2}(r, g)\right)$$

$$+(k+3)(m(r, f)+m(r, g)) \leq 6 \log r+O(1). \tag{11}$$

Set

$$f(z)=\frac{p_2(z)}{p_1(z)}; \quad g(z)=\frac{q_2(z)}{q_1(z)},$$

where both $p_1(z), p_2(z)$ and $q_1(z), q_2(z)$ are co-prime polynomials.

If $\deg p_2>\deg p_1$, then $m(r, f)=(\deg p_2-\deg p_1) \log r$. It follows from (11) that

$$N_{(2}(r, f)=0, \quad N_{(2}(r, g)=0.$$

Thus,

$$\overline{N}_L\left(r, \frac{F^{(k)}}{z}\right)=0, \quad \overline{N}_L\left(r, \frac{G^{(k)}}{z}\right)=0.$$

It follows from (9) that

$$6 \log r \leq T(r, f)+T(r, g)+(2k+2m+4)\left(m\left(r, \frac{1}{f}\right)+m\left(r, \frac{1}{g}\right)\right)$$

$$+(k+2)(m(r, f)+m(r, g))$$

$$\leq 6 \log r+O(1).$$

Hence,

$$f(z)=\frac{a_2 z^2+a_1 z+a_0}{(z-z_1)}; \quad g(z)=\frac{b_1 z+b_0}{(z-z_1)}, \tag{12}$$

where a_2, a_1, a_0, b_1, b_0 are constants with $a_2 b_1 \neq 0$. From (12), we have

$$(f^n P(f))^{(k)} = \frac{P(z)}{(z - z_1)^{n+m+k}}; \quad (g^n P(g))^{(k)} = \frac{Q(z)}{(z - z_1)^{n+m+k}},$$

where $P(z), Q(z)$ are polynomials with $\deg P = 2(n + m)$ and $\deg Q = n + m - 1$. Thus, $(f^n P(f))^{(k)} - z$ has $2(n + m)$ zeros (counting multiplicity) but $(g^n P(g))^{(k)} - z$ has only $(n + m + k + 1)$ zeros (counting multiplicity). This contradicts $(f^n P(f))^{(k)}$ and $(g^n P(g))^{(k)}$ share z CM. Thus, $\deg p_2 \leq \deg p_1$. If $\deg p_2 < \deg p_1$, then $m(r, 1/f) = (\deg p_1 - \deg p_2) \log r$. It follows from (11) that

$$2(k + m + 2) m(r, 1/f) \leq 6 \log r + O(1)$$

and $N_{(2}(r, f) = 0$, $N_{(2}(r, g) = 0$. Thus,

$$\overline{N}_L\left(r, \frac{F^{(k)}}{z}\right) = 0, \quad \overline{N}_L\left(r, \frac{G^{(k)}}{z}\right) = 0.$$

From (9),

$$8 \log r \leq T(r, f) + T(r, g) + 2(k + m + 2) m\left(r, \frac{1}{f}\right) \leq 6 \log r + O(1),$$

a contradiction. Thus, $\deg p_2 \geq \deg p_1$. Hence, $\deg p_2 = \deg p_1$. Thus, by (11), we have

$$(k + 3)(N_{(2}(r, f) + N_{(2}(r, g) - \overline{N}_{(2}(r, f) - \overline{N}_{(2}(r, g))$$
$$\leq 6 \log r + O(1). \tag{13}$$

If $f(z)$ has a pole with multiplicity atleast 3, then by (13), we have

$$8 \log r \leq 2(k + 3) \log r \leq 6 \log r + O(1),$$

a contradiction. If $f(z)$ has two multiple poles, then by (13), we have

$$8 \log r \leq 2(k + 3) \log r \leq 6 \log r + O(1),$$

a contradiction. Thus, $f(z)$ has at most one multiple pole and its multiplicity is 2. Similarly, we can get that $g(z)$ has one multiple pole with multiplicity 2. If both $f(z)$ and $g(z)$ have one multiple pole, then by (13), we have

$$8\log r \leq 2(k+3)\log r \leq 6\log r + O(1),$$

a contradiction. If $f(z)$ has single multiple pole and $g(z)$ has only simple poles, then

$$f(z) = \frac{a_t z^t + a_{t-1}z^{t-1} + \cdots + a_0}{(z-z_1)^2(z-z_2)\cdots(z-z_{t-1})},$$

$$g(z) = \frac{b_{t-1}z^{t-1} + \cdots + b_0}{(z-z_1)(z-z_2)\cdots(z-z_{t-1})}, \tag{14}$$

where z_l $(l = 1, 2, ..., t-1)$ are distinct complex numbers and a_i $(i = 0, 1, ..., t)$, b_j $(j = 0, 1, ..., t-1)$ are constants with $a_t b_{t-1} \neq 0$. From (14), we have

$$(f^n P(f))^{(k)} = \frac{P_1(z)}{(z-z_1)^{2n+2m+k}(z-z_2)^{n+m+k}\cdots(z-z_{t-1})^{n+m+k}},$$

$$(g^n P(g))^{(k)} = \frac{Q_1(z)}{(z-z_1)^{n+m+k}(z-z_2)^{n+m+k}\cdots(z-z_{t-1})^{n+m+k}},$$

where $P_1(z)$, $Q_1(z)$ are polynomials with $\deg P_1 \leq nt + kt + mt - 2k - 1$ and $\deg Q_1 \leq nt + kt + mt - n - 2k - 1$. Thus, $(f^n P(f))^{(k)} - z$ has $nt + kt + mt - k + 1$ zeros (counting multiplicity) but $(g^n P(g))^{(k)} - z$ has only $(nt + kt + mt - n - k + 1)$ zeros (counting multiplicity). This contradicts $(f^n P(f))^{(k)}$ and $(g^n P(g))^{(k)}$ share z CM. Similarly, if $g(z)$ has single pole and $f(z)$ has only simple poles, then we get a contradiction. Therefore, both $f(z)$ and $g(z)$ have only simple poles, then we have

$$(f^n P(f))^{(k)} = \frac{h_1(z)}{P_2(z)}; \quad (g^n P(g))^{(k)} = \frac{h_2(z)}{P_2(z)},$$

where both $h_1(z)$, $P_2(z)$ and $h_2(z)$, $P_2(z)$ are co-prime polynomials with $\max\{\deg h_1, \deg h_2\} < \deg P_2$. Since $(f^n P(f))^{(k)}$ and $(g^n P(g))^{(k)}$ share z CM, $h_1(z) \equiv h_2(z)$. Thus, $(f^n P(f))^{(k)} \equiv (g^n P(g))^{(k)}$. Therefore, by Lemma 2.2, we get either

(i) $(f^n P(f))^{(k)} \equiv (g^n P(g))^{(k)}$ or

(ii) $(f^n P(f))^{(k)} \cdot (g^n P(g))^{(k)} \equiv z^2$.

By Lemma 2.4, Case (ii) is impossible. By Lemma 2.3, we get $f^n P(f) \equiv g^n P(g)$ from Case (i).

$$\Rightarrow f^n(a_m f^m + a_{m-1} f^{m-1} + \cdots + a_0)$$

$$\equiv g^n(a_m g^m + a_{m-1} g^{m-1} + \cdots + a_0). \tag{15}$$

Let $h = f/g$, if h is constant. Then substituting $f = gh$ in (15), we get

$$a_m g^{n+m}(h^{n+m} - 1) + a_{m-1} g^{n+m-1}(h^{n+m-1} - 1) + \cdots + a_0 g^n (h^n - 1) = 0$$

which implies $h^d = 1$, where $d = (n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$. Thus, $f = tg$ for a constant t such that $t^d = 1$, where $d = (n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$. If h is not constant, then f and g satisfy the algebraic equation $R(f, g) = 0$, where

$$R(w_1, w_2) = w_1^n(a_m w_1^m + a_{m-1} w_1^{m-1} + \cdots + a_1 w_1 + a_0)$$

$$- w_2^n(a_m w_2^m + a_{m-1} w_2^{m-1} + \cdots + a_1 w_2 + a_0).$$

Hence the proof of Theorem 1.

Note. When $P(w) = a_0$, then the above theorem reduces to Theorem E.

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