**Far East Journal of Applied Mathematics**
© 2016 Pushpa Publishing House, Allahabad, India
Published: October 2016
http://dx.doi.org/10.17654/AM095040265
Volume 95, Number 4, 2016, Pages 265-281 **ISSN: 0972-0960**

# REVISITING THE CLOUGH-TOCHER FINITE ELEMENT

**Koua Brou Jean Claude**, **Jean Stéphane Inpké Haudié and Alain Le Méhauté**

UFR de Mathématiques et Informatique
Université de Cocody 22 BP 582 Abidjan 22
Côte d'Ivoire
e-mail: k_brou@hotmail.com

UFR de Mathématiques et Informatique
Université Félix Houphouet Boigny de Cocody 22 BP 582 Abidjan 22
Côte d'Ivoire
e-mail: inkpehaudie@yahoo.fr

Laboratoire Jean Leray
Département de Mathématiques
Université de Nantes 2
rue de la Houssinière
F-44072, Nantes Cedex
France

## Abstract

In this paper, we provide a new method to construct a piecewise polynomial $C^1$ finite element in $\mathbb{R}^2$ for a triangle. This process re-establishes the $C^1$ Clough-Tocher finite element.

Received: April 21, 2016; Accepted: June 16, 2016
2010 Mathematics Subject Classification: 65-XX.
Keywords and phrases: Clough-Tocher finite element, piecewise polynomial, finite element.

## 1. Introduction

Let $\Omega$ be a connected polygonal bounded domain in $\mathbb{R}^2$ and $\nu = \{\nu_i, i = 1, ..., N_\nu\}$ be a set of $N_\nu$ isolated scattered points in $\overline{\Omega}$.

A collection $\Delta = \{\mathbb{T}_i\}_{i \in I}$ of triangles is said to be a *triangulation of* $\Omega$ provided that:

(1) $\Omega = \bigcup_{i \in I} \mathbb{T}_i$.

(2) The intersection of any two triangles is either empty, a single point or a common edge.

We are aware of the fact that given any set $\nu = \{\nu_i, i = 1, ..., N_\nu\}$ of scattered isolation points in $\overline{\Omega}$, there are many triangulations of $\Omega$ based on $\nu$, i.e., where the set of the vertices of all triangles is exactly the set of all $\nu$. Practically, it is often a Delaunay triangulation which is given, but thereafter, we are not going to use any specific property of any triangulation.

We focus the case where each triangle of $\Delta$ is subdivided into three sub-triangles by joining each vertex to an interior point. Such triangulation is known as Clough-Tocher type [1, 2, 6, 9-11], and we denote this triangulation by $\Delta_{CT}$. We let, in view of simplification, this point to be the centroid.

The paper is organized as follows: In the following section, we introduce some notations and recall some basic results. Then, we present our process for constructing of a finite element of class $C^1$ of HCT type in Section 3.

## 2. Preliminaries

For a function $S \in C^r(\overline{\Omega})$, we denote respectively by $C^p(v_i)$ and $C^q(\sigma_j)$ the fact that $S$ is $C^p$ around the vertex $v_i \in \nu$, and $C^q$ across the edge $\sigma_j \in \Sigma$, the set of edges.

Let us recall from Ciarlet [5] that a finite element is a triplet $(\mathcal{K}, \mathcal{E}, \mathcal{L})$, where $\mathcal{K}$ is a polygon, $\mathcal{E}$ is a space of functions and $\mathcal{L} \in \mathcal{E}^*$ is a finite set of degrees of freedom, defined on $\mathcal{K}$, that enable us to build up functions in $\mathcal{E}$. The degrees of freedom are also called *nodal values* on $\mathcal{K}$.

$\mathcal{L}$ is $\mathcal{E}$-unisolvent iff:

(1) $card\,\mathcal{L} = \dim \mathcal{E}$,

(2) If $\mathcal{L}$ being the set $\mathcal{L} = \{l_i,\ i = 1; \ldots, \dim \mathcal{E}\}$, then given $v \in \mathcal{E}$ such that $l_i(v) = 0,\ i = 1, \ldots, \dim \mathcal{E}$, then $v \equiv 0$.

Moreover, Ciarlet [5], a finite element $(\mathcal{K}, \mathcal{E}, \mathcal{L})$ is of class $C^r$ if, whenever it is used to define the restriction to $\mathcal{K}$ of a (global) function *S*, then *S* has a (global) smoothness *r*.

Spline functions and finite elements approximation are close techniques. Constructing locally supported spline functions is quite easy when dealing with finite elements. In this case, the support of the spline is reduced to an element and its neighbors in the triangulation. Moreover, the explicit construction can be made for each element individually. For this purpose, polynomial finite elements have been heavily studied.

To conclude this section, we give some notations:

Given $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}^d$, where *d* is a nonnegative integer, we let

$$|\beta| = \sum_{i=1}^{d} \beta_i \quad \text{and} \quad \beta! = \prod_{i=1}^{d} \beta_i!.$$

For $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}^3$ and $\mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3$,

$$\mu^\beta = \prod_{i=1}^{3} \mu_i^{\beta_i}.$$

Given $\mathbb{T} = \langle A_1, A_2, A_3 \rangle$ a triangle with vertices $A_1$, $A_2$ and $A_3$. The barycentric coordinates $\mu = (\mu_1, \mu_2, \mu_3)$ of a point *M* relative of $\mathbb{T}$ is the *unique solution* of the system:

$$\begin{cases} \sum_{i=1}^{3} \mu_i A_i = M, \\ |\mu| = 1. \end{cases} \tag{1}$$

Let $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}^3$ be such that $|\beta| = r$ $(r \in \mathbb{N})$. For all functions $f$ very smooth, we define

$$\partial^\beta f(M) = \frac{\partial^{|\beta|} f(M)}{(\partial\mu)^\beta} \quad \text{and} \quad D_\eta^r f(M) = \sum_{|\beta|=r} \frac{r!}{\beta!} \eta^\beta \partial^\beta f(M),$$

with the partial derivative and the $r$th directional derivative of $f$ in the direction $\eta = (\eta_1, \eta_2, \eta_3)$; $|\eta| = 0$. For $i = 1, 2, 3$; $x_i$ and $y_i$ denote respectively the first and the second variable of $A_i$ a point of $\mathbb{R}^2$.

For $r = 1$, we get:

$$D_\eta f(M) = \left(\sum_{i=1}^{3} \eta_i x_i\right) \frac{\partial f}{\partial x}(M) + \left(\sum_{i=1}^{3} \eta_i y_i\right) \frac{\partial f}{\partial y}(M). \tag{2}$$

For $r = 2$, we get:

$$D_\eta^2 f(M) = \left(\sum_{i=1}^{3} \eta_i x_i\right)^2 \frac{\partial^2 f}{\partial x^2}(M)$$
$$+ 2\left(\sum_{i=1}^{3} \eta_i x_i\right)\left(\sum_{i=1}^{3} \eta_i y_i\right) \frac{\partial^2 f}{\partial x \partial y}(M) + \left(\sum_{i=1}^{3} \eta_i y_i\right)^2 \frac{\partial^2 f}{\partial y^2}(M). \tag{3}$$

## 3. Finite Element of Class $C^1$

### 3.1. Introduction

In this section, we revisit the Hiesh-Clough-Tocher (HCT) finite element and prove that it is of class $C^1$.

It is well known that if $\mathcal{K} = \langle A_1, A_2, A_3 \rangle$ is a non-degenerated triangle with vertices $A_1, A_2$ and $A_3$ and $d$ is a positive integer, a polynomial $\mathcal{P}$ of degree $d$ can be written in Bernstein-Bezier form:

$$\mathcal{P}(\mu) = \sum_{|\alpha|=d} C(\alpha) \frac{d!}{\alpha!} \mu^{\alpha}, \tag{4}$$

where $\mu = (\mu_1, \mu_2, \mu_3)$ denotes the barycentrics coordinates of a point $M$ relative to the triangle $\mathcal{K}$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ such that $|\alpha| = d$.

To the coefficients $C(\alpha)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, named Bernstein-Bezier coefficient or $B$-coefficients or ordinates of Bezier, the following are points $P_\alpha$ of the triangle, where

$$P_\alpha = \frac{\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3}{d}; \ |\alpha| = d. \tag{5}$$

The set $\{(P_\alpha; C(\alpha)), |\alpha| = d\}$ is the set of control points in $\mathbb{R}^3$. Representing the points in the triangle leads to the representation of the Bernstein-Bezier of a polynomial and emphases some geometrical properties such as, being collinear for some of them, convex hull property and so on. We refer to [3, 4, 7, 11] for more details on Bernstein-Bezier representation.

Suppose now $\mathcal{K} = \langle A_1, A_2, A_3 \rangle$ with vertices $A_1, A_2$ and $A_3$ be a triangle of $\Delta_{CT}$, subdivided into three triangles $\mathcal{K}_1, \mathcal{K}_2$ and $\mathcal{K}_3$ such as $\mathcal{K}_i = \langle A_{i+2}, A_0, A_{i+1} \rangle$, where $A_0$ is the centroid of $\mathcal{K}$ and $A_i = A_{i+3}$ for $i > 0$.

Consider over $\mathcal{K}$ the degree of freedom defined by the set:

$$\Sigma_K^1 = \left\{ \partial^\alpha f(A_i),\ i = 1, 2, 3; |\alpha| \le 1; \frac{\partial f}{\partial \eta_i}(\overline{A}_i),\ i = 1, 2, 3 \right\}, \tag{6}$$

where $\overline{A}_i$ is a strictly interior point of the edge $\sigma_i = [A_{i-1}, A_{i+1}]$ for

$i = 1, 2$ and 3 with $A_{i+3} = A_i$ for $i \geq 0$ and $\frac{\partial f}{\partial \eta_i}(\overline{A_i})$ is the normal derivative of *f* relative to this edge.

Let $\mathbb{P}_3$ be the space of polynomials of degree 3 over $\mathcal{K}$ and

$$\mathbb{PP}_3(\mathcal{K}) = \{S \in C^0(\mathcal{K}),\ S|_{\mathcal{K}_i} \in \mathbb{P}_3,\ i = 1, 2, 3\}.$$

This triplet $(\mathcal{K}, \mathbb{PP}_3(\mathcal{K}), \Sigma^1_{\mathcal{K}})$ is the well-known HCT finite element which is of class $C^1$ and it is represented by Figure 1 below.

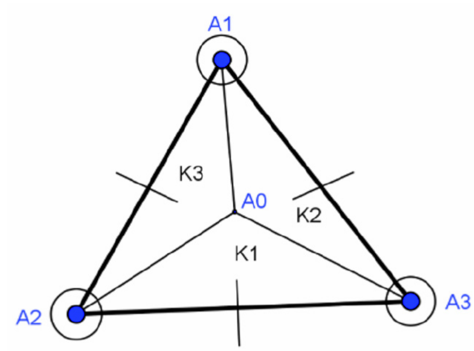


**Figure 1.** Degree of freedom for $k = 1$.

We notice that the degree of freedom available on each sub-triangle $\mathcal{K}_i$ is less that it is needed to construct a polynomial $P_i$ of degree 3 on this sub-triangle. It remains 3 to be able to determine all coefficients of which could be represented in Bernstein-Bezier form:

$$P_i(\mu_i) = \sum_{|\alpha|=3} C_i(\alpha) \frac{3!}{\alpha!} \mu_i^{\alpha}, \tag{7}$$

where $\mu_i = (\mu_{i1}, \mu_{i2}, \mu_{i3})$ denotes the barycentrics coordinates of a point $M_i$ relative to the triangle $\mathcal{K}_i$.

For all points $M_i \in \mathcal{K}_i$, $P_i(\mu_i) = f(M_i)$ and $D_\eta f(M_i) = D_\eta P_i(\mu_i)$, where

$$D_\eta P_i(\mu_i) = 3 \sum_{|\alpha|=2} C_i^1(\alpha)(\eta)\frac{2!}{\alpha!}\mu_i^\alpha$$

with $C_i^1(\alpha)(\eta) = \eta_1 C_i(\alpha + \varepsilon_1) + \eta_1 C_i(\alpha + \varepsilon_2) + \eta_1 C_i(\alpha + \varepsilon_3)$

and $\varepsilon_i$ denotes the $i$th vector of the canonical basis of $\mathbb{R}^3$ for $i = 1, 2, 3$.

### 3.2. Determination of unknown coefficients

The process of determination of the $B$-coefficients is the same on each sub-triangle, so let us consider a generic triangle $\mathcal{T} = \langle A, B, C\rangle$ with vertices $A(x_A; y_A)$, $B(x_B; y_B)$ and $C(x_C; y_C)$ using the following degree of freedom:

$$\partial^\alpha f(A), \partial^\alpha f(C), |\alpha| \le 1, \frac{\partial f}{\partial \eta}(\overline{B}),$$

where $\overline{B}(\bar{x}; \bar{y})$ is the midpoint of the edge $\sigma_{AC}$ and $\frac{\partial f}{\partial \eta}$ is the normal derivative of $f$ relative to this edge.

If $\mathcal{P}$ is a polynomial of degree 3 expressed in Bernstein-Bezier form (4) on $\mathcal{T}$ and $C(\alpha)$ denotes its $B$-coefficients, we get:

$$\begin{cases} C(3, 0, 0) = f(A), \\ C(0, 0, 3) = f(C), \\ C(2, 1, 0) = f(A) + \frac{1}{3}\left[(x_B - x_A)\frac{\partial f}{\partial x}(A) + (y_B - y_A)\frac{\partial f}{\partial y}(A)\right], \\ C(2, 0, 1) = f(A) + \frac{1}{3}\left[(x_C - x_A)\frac{\partial f}{\partial x}(A) + (y_C - y_A)\frac{\partial f}{\partial y}(A)\right], \\ C(1, 0, 2) = f(C) + \frac{1}{3}\left[(x_A - x_C)\frac{\partial f}{\partial x}(C) + (y_A - y_C)\frac{\partial f}{\partial y}(C)\right], \\ C(0, 1, 2) = f(C) + \frac{1}{3}\left[(x_B - x_C)\frac{\partial f}{\partial x}(C) + (y_B - y_C)\frac{\partial f}{\partial y}(C)\right] \end{cases}$$

and

$$C(1, 1, 1)$$
$$= \frac{1}{4}[C(3, 0, 0) + C(0, 0, 3) + 3(C(2, 0, 1) + C(1, 0, 2)) - 2(C(2, 1, 0) + C(0, 1, 2))]$$
$$+ \frac{2}{3}\left[(x_B - \bar{x})\frac{\partial f}{\partial x}(\overline{B}) + (y_B - \bar{y})\frac{\partial f}{\partial y}(\overline{B})\right].$$

The unknown coefficients are in $\mathcal{K}_1$, $C_1(1, 2, 0)$, $C_1(0, 3, 0)$ and $C_1(0, 2, 1)$ and in $\mathcal{K}_2$, $C_2(1, 2, 0)$, $C_2(0, 3, 0)$ and $C_2(0, 2, 1)$, where $C_1(1, 2, 0) = C_2(0, 2, 1)$ and $C_1(0, 3, 0) = C_2(0, 3, 0)$.

By associating on each sub-triangle the $B$-coefficients, it comes on the triangle $\mathcal{K}$, 19 points presented on Figure 2 below. The open circles designate the known coefficients and the full circles those which are unknown.

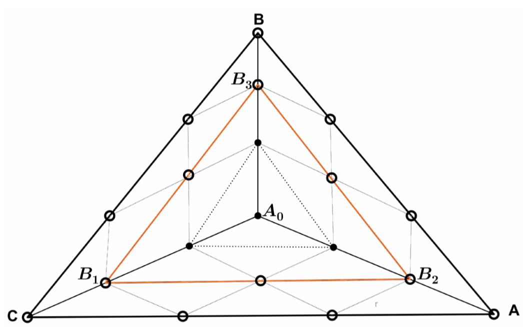


**Figure 2.** Bernstein-Bezier coefficients for $k = 1$.

Let $\mathbb{P}^1_{\mathcal{K}}$ be a set of spline functions defined as follows:

$$\mathbb{P}^1_{\mathcal{K}} = \{S \in C^1(\mathcal{K}),\ S|_{\mathcal{K}_i} \in \mathbb{P}_3,\ i = 1, 2, 3\}. \tag{8}$$

We are now in position to explain the process of determining on each sub-triangle the remaining three coefficients to get a spline of class $C^1$ on $\mathcal{K}$. We begin by considering $\mathcal{K} = \langle A_1, A_2, A_3 \rangle$, the black triangle with vertices $A_1 = B$, $A_2 = C$ and $A_3 = A$ and $\widetilde{\mathcal{K}} = \langle B_1, B_2, B_3 \rangle$, the orange triangle with vertices $B_1$, $B_2$ and $B_3$. It will be associated to the six known *coefficients* of $\widetilde{\mathcal{K}}$, the set of degree of freedom:

$$\Sigma^0_{\widetilde{\mathcal{K}}} = \{f(B_i)\}_{1 \le i \le 3} \cup \{f(B_{ij})\}_{1 \le i < j \le 3}, \text{ where } B_{ij} = \frac{1}{2}(B_i + B_j). \tag{9}$$

The triplet $(\tilde{\mathcal{K}}, \mathbb{P}_2(\tilde{\mathcal{K}}), \Sigma^0_{\tilde{K}})$ is Lagrange's finite element of Type 2. Let us call $\tilde{\mathcal{P}}$ to be the polynomial of degree 2 defined on $\tilde{\mathcal{K}}$ with this degree of freedom. It can be expressed as:

$$\tilde{\mathcal{P}}(\gamma) = \sum_{|\alpha|=2} \tilde{b}(\alpha)\frac{2!}{\alpha!}\gamma^{\alpha}, \tag{10}$$

where $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ denotes the barycentrics coordinates of a point $M$ relative to the triangle $\tilde{\mathcal{K}}$.

Let us consider the splitting of $\tilde{\mathcal{K}}$ derive from those of $\mathcal{K}$ and denote by $\tilde{\mathcal{K}}_i = \langle B_{i+2}, A_0, B_{i+1}\rangle$ the triangle of vertices $B_{i+2}, A_0$ and $B_{i+1}$ for $i = 1, 2$ and $3$ with $B_i = B_{i+3}$, for $i > 0$. The restriction $\tilde{\mathcal{P}}_i$ of $\tilde{\mathcal{P}}$ on a sub-triangle $\tilde{\mathcal{K}}_i$ can be written in the Bernstein-Bezier form like as:

$$\tilde{\mathcal{P}}_i(\gamma_i) = \sum_{|\alpha|=2} \tilde{b}_i(\alpha)\frac{2!}{\alpha!}\gamma_i^{\alpha}, \tag{11}$$

where $\gamma_i = (\gamma_{i1}, \gamma_{i2}, \gamma_{i3})$ denotes the barycentrics coordinates of a point $M$ relative to the triangle $\tilde{\mathcal{K}}_i$.

For all $i = 1, 2, 3,$ the coefficients $\tilde{b}_i(\alpha)$ can be expressed in terms of coefficients $\tilde{b}(\alpha)$. For example, if $i = 3$ (i.e., $\tilde{\mathcal{K}}_3 = \langle B_2 A_0 B_1\rangle$), we get:

$$\begin{cases} \tilde{b}_3(2, 0, 0) = \tilde{b}(0, 2, 0), \\ \tilde{b}_3(1, 1, 0) = \frac{1}{3}[\tilde{b}(1, 1, 0) + \tilde{b}(0, 2, 0) + \tilde{b}(0, 1, 1)], \\ \tilde{b}_3(1, 0, 1) = \tilde{b}(1, 1, 0), \\ \tilde{b}_3(0, 2, 0) = \frac{1}{9}[\tilde{b}(2, 0, 0) + 2\tilde{b}(1, 1, 0) + 2\tilde{b}(1, 0, 1) \\ \qquad + \tilde{b}(0, 2, 0) + 2\tilde{b}(0, 1, 1) + \tilde{b}(0, 0, 2)], \\ \tilde{b}_3(0, 1, 1) = \frac{1}{3}[\tilde{b}(2, 0, 0) + \tilde{b}(1, 1, 0) + \tilde{b}(1, 0, 1)], \\ \tilde{b}_3(0, 0, 2) = \tilde{b}(2, 0, 0). \end{cases} \tag{12}$$

Now, by representing the coefficients of each polynomial $\tilde{\mathcal{P}}_i$ on $\tilde{\mathcal{K}}_i$ for $i = 1, 2, 3,$ we can associate a value to each unknown coefficient in $\tilde{\mathcal{K}}$. Then, we have to examine if with these values, we are able to determine all coefficients of $\tilde{\mathcal{P}}_i$ and if the spline obtained on $\tilde{\mathcal{K}}$ is at least of class $C^1$. Therefore, we must express the restriction $\tilde{\mathcal{P}}_{i+2}$ of $\tilde{\mathcal{P}}_i$ on $\tilde{\mathcal{K}}_{i+2}$ as a polynomial of degree 3. Thus, we need to raise the degree of $\tilde{\mathcal{P}}_{i+2}$ from 2 to 3. This leads to

$$\tilde{\mathcal{P}}_{i+2}(\gamma_{i+2}) = \sum_{|\alpha|=3} \tilde{b}_{i+2}^{(1)}(\alpha)\frac{3!}{\alpha!}\gamma_{i+2}^{\alpha} \tag{13}$$

hence, the coefficients $\tilde{b}_{i+2}^{(1)}(\alpha)$ can be expressed in terms of $\tilde{b}_{i+2}(\alpha)$ as, for example $i = 1,$

$$\tilde{\mathcal{P}}_3(\gamma_3) = \sum_{|\alpha|=3} \tilde{b}_3^{(1)}(\alpha)\frac{3!}{\alpha!}\gamma_3^{\alpha}$$

with

$$\begin{cases} \tilde{b}_3^{(1)}(3, 0, 0) = \tilde{b}_3(2, 0, 0), \\ \tilde{b}_3^{(1)}(2, 1, 0) = \frac{2}{3}\tilde{b}_3(1, 1, 0) + \frac{1}{3}\tilde{b}_3(2, 0, 0), \\ \tilde{b}_3^{(1)}(2, 0, 1) = \frac{2}{3}\tilde{b}_3(1, 0, 1) + \frac{1}{3}\tilde{b}_3(2, 0, 0), \\ \tilde{b}_3^{(1)}(1, 2, 0) = \frac{2}{3}\tilde{b}_3(1, 1, 0) + \frac{1}{3}\tilde{b}_3(0, 2, 0), \\ \tilde{b}_3^{(1)}(1, 1, 1) = \frac{1}{3}\tilde{b}_3(1, 1, 0) + \frac{1}{3}\tilde{b}_3(1, 0, 1) + \frac{1}{3}\tilde{b}_3(0, 1, 1), \\ \tilde{b}_3^{(1)}(1, 0, 2) = \frac{2}{3}\tilde{b}_3(1, 0, 1) + \frac{1}{3}\tilde{b}_3(0, 0, 2), \\ \tilde{b}_3^{(1)}(0, 3, 0) = \tilde{b}_3(0, 2, 0), \\ \tilde{b}_3^{(1)}(0, 2, 1) = \frac{2}{3}\tilde{b}_3(0, 1, 1) + \frac{1}{3}\tilde{b}_3(0, 2, 0), \\ \tilde{b}_3^{(1)}(0, 1, 2) = \frac{2}{3}\tilde{b}_3(0, 1, 1) + \frac{1}{3}\tilde{b}_3(0, 0, 2), \\ \tilde{b}_3^{(1)}(0, 0, 3) = \tilde{b}_3(0, 0, 2). \end{cases} \tag{14}$$

As we consider the restriction of $\mathcal{P}_i$ on $\tilde{\mathcal{K}}_{i+2}$ to be $\tilde{\mathcal{P}}_{i+2}$, the expressions of these two polynomials as elements of $\mathbb{P}_3(\tilde{\mathcal{K}}_{i+2})$ coincide so that, we can express the set of *B*-coefficients $C_i(\beta)$ of $\mathcal{P}_i$ on $\mathcal{K}_i$ in terms of the *B*-coefficients $\tilde{b}_{i+2}(\alpha)$ of $\tilde{\mathcal{P}}_{i+2}$ relative to $\tilde{\mathcal{K}}_{i+2}$. Thus, let $\gamma_{i+2}$ (resp., $\lambda_i$) be the barycentrics coordinates of a point $M \in \tilde{\mathcal{K}}_{i+2}$ (resp., $\mathcal{K}_i$). It comes, for example $i = 1$,

$$\begin{cases} \gamma_{31} = \dfrac{3}{2}\lambda_{11}, \\ \gamma_{32} = \dfrac{3}{2}\left(\lambda_{12} - \dfrac{1}{3}\right), \\ \gamma_{33} = \dfrac{3}{2}\lambda_{13} \end{cases} \tag{15}$$

and $\mathcal{P}_1(\lambda_1) = \tilde{\mathcal{P}}_3(\gamma_3)$ means:

$$\begin{cases} C_1(3,0,0) = \dfrac{1}{4}[9\tilde{b}_3(2,0,0) - 6\tilde{b}_3(1,1,0) + \tilde{b}_3(0,2,0)], \\ C_1(2,1,0) = \dfrac{1}{4}[3\tilde{b}_3(2,0,0) + 2\tilde{b}_3(1,1,0) - \tilde{b}_3(0,2,0)], \\ C_1(2,0,1) = \dfrac{1}{4}[3\tilde{b}_3(2,0,0) - 4\tilde{b}_3(1,1,0) + 6\tilde{b}_3(1,0,1) + \tilde{b}_3(0,2,0) - 2\tilde{b}_3(0,1,1)], \\ C_1(1,2,0) = \tilde{b}_3(1,1,0), \\ C_1(1,1,1) = \dfrac{1}{4}[\tilde{b}_3(1,1,0) + 3\tilde{b}_3(1,0,1) - \tilde{b}_3(0,2,0) + \tilde{b}_3(0,1,1)], \\ C_1(1,0,2) = \dfrac{1}{4}[-2\tilde{b}_3(1,1,0) + 6\tilde{b}_3(1,0,1) + \tilde{b}_3(0,2,0) - 4\tilde{b}_3(0,1,1) + 3\tilde{b}_3(0,0,2)], \\ C_1(0,3,0) = \tilde{b}_3(0,2,0), \\ C_1(0,2,1) = \tilde{b}_3(0,1,1), \\ C_1(0,1,2) = \dfrac{1}{4}[-\tilde{b}_3(0,2,0) + 2\tilde{b}_3(0,1,1) + 3\tilde{b}_3(0,0,2)], \\ C_1(0,0,3) = \dfrac{1}{4}[\tilde{b}_3(0,2,0) - 6\tilde{b}_3(0,1,1) + 9\tilde{b}_3(0,0,2)] \end{cases} \tag{16}$$

but with the subdivision algorithm for example, we can express the $B$-coefficients $C_i(\beta)$ of $\mathcal{P}_i$ relative to $\mathcal{K}_i$ in terms of the $B$-coefficients $C_{i+1}(\beta)$ relative to $\mathcal{K}_{i+1}$. Let $\lambda_i$ be the barycentrics coordinates of a point $M \in \mathcal{K}_i$ and $\mathcal{P}_i$ be the polynomial with $B$-coefficients $C_i(\beta)$, we get:

$$\begin{cases} \lambda_{11} = \lambda_{23} - \lambda_{21}, \\ \lambda_{12} = \lambda_{22} + 3\lambda_{21}, \\ \lambda_{13} = -\lambda_{21} \end{cases} \tag{17}$$

and $\mathcal{P}_1(\lambda_1) = \mathcal{P}_2(\lambda_2)$ means:

$$\begin{cases} C_2(3,0,0) = -C_1(3,0,0) + 9C_1(2,1,0) - 3C_1(2,0,1) - 27C_1(1,2,0) + 18C_1(1,1,1) \\ \qquad\qquad + 27C_1(0,3,0) - 3C_1(1,0,2) - 27C_1(0,2,1) + 9C_1(0,1,2) - C_1(0,0,3), \\ C_2(2,1,0) = C_1(2,1,0) - 6C_1(1,2,0) + 2C_1(1,1,1) + 9C_1(0,3,0) - 6C_1(0,2,1) \\ \qquad\qquad + C_1(0,1,2), \\ C_2(2,0,1) = C_1(3,0,0) - 6C_1(2,1,0) + 2C_1(2,0,1) + 9C_1(1,2,0) - 6C_1(1,1,1) \\ \qquad\qquad + C_1(1,0,2), \\ C_2(1,2,0) = -C_1(1,2,0) + 3C_1(0,3,0) - C_1(0,2,1), \\ C_2(1,1,1) = -C_1(2,1,0) + 3C_1(1,2,0) - C_1(1,1,1), \\ C_2(1,0,2) = -C_1(3,0,0) + 3C_1(2,1,0) - C_1(2,0,1), \\ C_2(0,3,0) = C_1(0,3,0), \\ C_2(0,2,1) = C_1(1,2,0), \\ C_2(0,1,2) = C_1(2,1,0), \\ C_2(0,0,3) = C_1(3,0,0). \end{cases} \tag{18}$$

By reiterating the process on the triangles $\mathcal{K}_1$ and $\mathcal{K}_3$ with the polynomials $\mathcal{P}_1(\lambda_1)$ and $\mathcal{P}_3(\lambda_3)$, we obtain:

$$\begin{cases} C_3(3,0,0) = C_1(0,0,3), \\ C_3(2,1,0) = C_1(0,1,2), \\ C_3(2,0,1) = -C_1(1,0,2) + 3C_1(0,1,2) - C_1(0,0,3), \\ C_3(1,2,0) = C_1(0,2,1), \\ C_3(1,1,1) = -C_1(1,1,1) + 3C_1(0,2,1) - C_1(0,1,2), \\ C_3(1,0,2) = C_1(2,0,1) - 6C_1(1,1,1) + 2C_1(1,0,2) + 9C_1(0,2,1) - 6C_1(0,1,2) \\ \qquad\qquad + C_1(0,0,3), \\ C_3(0,3,0) = C_1(0,3,0), \\ C_3(0,2,1) = -C_1(1,2,0) + 3C_1(0,3,0) - C_1(0,2,1), \\ C_3(0,1,2) = C_1(2,1,0) - 6C_1(1,2,0) + 2C_1(1,1,1) + 9C_1(0,3,0) - 6C_1(0,2,1) \\ \qquad\qquad + C_1(0,1,2), \\ C_3(0,0,3) = -C_1(3,0,0) + 9C_1(2,1,0) - 3C_1(2,0,1) - 27C_1(1,2,0) + 18C_1(1,1,1) \\ \qquad\qquad - 3C_1(1,0,2) + 27C_1(0,3,0) - 27C_1(0,2,1) + 9C_1(0,1,2) - C_1(0,0,3) \\ \qquad\qquad + C_1(0,1,2). \end{cases} \tag{19}$$

At this stage, all of $B$-coefficients of the polynomial in (4) are known since the four remaining coefficients can be expressed according to $\tilde{b}(\alpha)$ as follows:

$$\begin{cases} C_1(1,2,0) = \frac{1}{3}(\tilde{b}(1,1,0) + \tilde{b}(0,2,0) + \tilde{b}(0,1,1)), \\ C_1(0,3,0) = \frac{1}{9}(\tilde{b}(2,0,0) + \tilde{b}(0,2,0) + \tilde{b}(0,0,2) \\ \qquad\qquad + 2\tilde{b}(1,1,0) + 2\tilde{b}(1,0,1) + 2\tilde{b}(0,1,1)), \\ C_1(0,2,1) = \frac{1}{3}(\tilde{b}(2,0,0) + \tilde{b}(1,1,0) + \tilde{b}(1,0,1)), \\ C_2(1,2,0) = \frac{1}{3}(\tilde{b}(1,0,1) + \tilde{b}(0,1,1) + \tilde{b}(0,0,2)). \end{cases} \tag{20}$$

It remains to prove that the spline constructed in this way on $\mathcal{K}$ is of class $C^1$.

### 3.3. Determination of the class

The aim of this section is to prove that our spline is of class $C^1$. To achieve this goal, we consider the triangle $\mathcal{K} = \langle A_1, A_2, A_3 \rangle$ with vertices

$A_1$, $A_2$ and $A_3$; $A_0$ the centroid of $\mathcal{K}$ and the three polynomials $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ with $A_i = A_{i+3}$ and $\mathcal{P}_i = \mathcal{P}_{i+3}$ for $i > 0$. We have to prove that:

$$\partial^\alpha \mathcal{P}_i = \partial^\alpha \mathcal{P}_{i+1}, \text{ along } [A_0; A_{i+2}] \text{ for } i = 1, 2, 3 \tag{21}$$

for all $\alpha \in \mathbb{N}^3$, such as $|\alpha| \le 1$.

We will use the following proposal proved in [10].

**Proposition 3.1.** *Let* $\mathcal{P}(\gamma) = \sum_{|\beta|=d} b_\beta \frac{d!}{\beta!}\gamma^\beta$ *and* $\hat{\mathcal{P}}(\hat{\gamma}) = \sum_{|\beta|=d} \hat{b}_\beta \frac{d!}{\beta!}\hat{\gamma}^\beta$ *be two polynomials of degree d defined respectively on triangles* $\mathcal{K} = \langle A, B, C\rangle$ *and* $\hat{\mathcal{K}} = \langle \hat{C}, B, A\rangle$ *which share the common edge* $\sigma_{AB} = [A; B]$. *Then the* $C^r$ *continuity of* $\mathcal{P}$ *and* $\hat{\mathcal{P}}$ *across* $\sigma_{AB}$ *is satisfied iff, for*

$$\begin{cases} 0 \le k \le r, \text{ and } 0 \le \rho \le d - k, \\ \hat{b}_{(k, \rho, d-k-\rho)} = b^k_{(d-k-\rho, \rho, 0)}(\gamma(\hat{C})), \end{cases} \tag{22}$$

*where* $b^k_\beta(\mu)$ *are defined by*

$$\begin{cases} b^0_\beta(\mu) = b_\beta, \\ b^k_\beta(\mu) = \mu_1 b^{k-1}_{\beta+\varepsilon_1}(\mu) + \mu_2 b^{k-1}_{\beta+\varepsilon_2}(\mu) + \mu_3 b^{k-1}_{\beta+\varepsilon_3}(\mu) \text{ for } k \ge 1, |\beta| = d - k \end{cases}$$

*with* $\mu = \gamma(\hat{C})$ *denotes the barycentrics coordinates of* $\hat{C}$ *over* $\mathcal{K}$ *and* $\varepsilon_i$ *the ith vector of the canonical basis of* $\mathbb{R}^3$.

**Proposition 3.2.** *If the spline function S is defined by* $S|_{\mathcal{K}_i} = P_i$ *and its coefficients are computed with equations* (16), (18) *and* (19), *then* $S \in \mathbb{P}^1_{\mathcal{K}}$.

**Proof.** We have to prove that the spline $S$ is of class $C^1$ across each interior edge of $\mathcal{K}$.

Using Proposition 3.1 with $\mu = (-1, 3, -1)$ for $r = 1$ and $d = 3$:

Across the edge $[A_0;\ A_3]$, we get: $C_2(k,\ \rho,\ 3-k-\rho) = C_1^k(3-k-\rho,\ \rho,\ 0)(-1,\ 3,\ -1)$.

If $k=0$, then

$$C_2(0,\ \rho,\ 3-\rho) = C_1(3-\rho,\ \rho,\ 0),\quad 0 \le \rho \le 3$$

and hence

$$\begin{cases} C_2(0,\ 0,\ 3) = C_1(3,\ 0,\ 0), \\ C_2(0,\ 1,\ 2) = C_1(2,\ 1,\ 0), \\ C_2(0,\ 2,\ 1) = C_1(1,\ 2,\ 0), \\ C_2(0,\ 3,\ 0) = C_1(0,\ 3,\ 0). \end{cases} \tag{23}$$

If $k=1$, then

$$C_2(1,\ \rho,\ 2-\rho) = -C_1(3-\rho,\ \rho,\ 0) + 3C_1(2-\rho,\ \rho+1,\ 0)$$
$$-\ C_1(2-\rho,\ \rho,\ 1),\quad 0 \le \rho \le 2$$

and hence

$$\begin{cases} C_2(1,\ 0,\ 2) = -C_1(3,\ 0,\ 0) + 3C_1(2,\ 1,\ 0) - C_1(2,\ 0,\ 1), \\ C_2(1,\ 1,\ 1) = -C_1(2,\ 1,\ 0) + 3C_1(1,\ 2,\ 0) - C_1(1,\ 1,\ 1), \\ C_2(1,\ 2,\ 0) = -C_1(1,\ 2,\ 0) + 3C_1(0,\ 3,\ 0) - C_1(0,\ 2,\ 1). \end{cases} \tag{24}$$

Across the edge $[A_0;\ A_2]$, we get: $C_3(3-k-\rho,\ \rho,\ k) = C_1^k(0,\ \rho,\ 3-k-\rho)(-1,\ 3,\ -1)$.

If $k=0$, then

$$C_3(3-\rho,\ \rho,\ 0) = C_1(0,\ \rho,\ 3-\rho),\quad 0 \le \rho \le 3$$

and hence

$$\begin{cases} C_3(300) = C_1(003), \\ C_3(210) = C_1(012), \\ C_3(120) = C_1(021), \\ C_3(030) = C_1(030). \end{cases} \tag{25}$$

If $k = 1$, then

$$C_3(2-\rho, \rho, 1) = -C_1(1, \rho, 2-\rho) + 3C_1(0, \rho+1, 2-\rho)$$

$$- C_1(0, \rho, 3-\rho), \quad 0 \leq \rho \leq 2$$

hence

$$\begin{cases} C_3(201) = -C_1(102) + 3C_1(012) - C_1(003), \\ C_3(111) = -C_1(111) + 3C_1(021) - C_1(012), \\ C_3(021) = -C_1(120) + 3C_1(030) - C_1(021). \end{cases} \tag{26}$$

Across the edge $[A_0; A_1]$, we get $C_3(k, \rho, 3-k-\rho) = C_2^k(3-k-\rho, \rho, 0)(-1, 3, -1)$.

If $k = 0$, then

$$C_3(0, \rho, 3-\rho) = C_2(3-\rho, \rho, 0), \quad 0 \leq \rho \leq 3$$

hence

$$\begin{cases} C_3(0, 0, 3) = C_2(3, 0, 0), \\ C_3(0, 1, 2) = C_2(2, 1, 0), \\ C_3(0, 2, 1) = C_2(1, 2, 0), \\ C_3(0, 3, 0) = C_2(0, 3, 0). \end{cases} \tag{27}$$

If $k = 1$, then

$$C_3(1, \rho, 2-\rho) = -C_2(3-\rho, \rho, 0) + 3C_2(2-\rho, \rho+1, 0)$$

$$- C_2(2-\rho, \rho, 1), \quad 0 \leq \rho \leq 2$$

hence

$$\begin{cases} C_3(1, 0, 2) = -C_2(3, 0, 0) + 3C_2(2, 1, 0) - C_2(2, 0, 1), \\ C_3(1, 1, 1) = -C_2(2, 1, 0) + 3C_2(1, 2, 0) - C_2(1, 1, 1), \\ C_3(1, 2, 0) = -C_2(1, 2, 0) + 3C_2(0, 3, 0) - C_2(0, 2, 1). \end{cases} \tag{28}$$

Relations (23), (24), (25), (26), (27) and (28) are checked easily starting from relations (16), (18) and (19).

## 4. Conclusion

This paper gives a new approach for constructing a piecewise polynomial finite element of class $C^1$ of Clough-Tocher type. It can be considered as another proof that the HCT element is of class $C^1$ and gives a method to construct the spline derived from this element. The process can be extended to finite element of class $C^r$, $r \geq 1$. This work is still in progress.

## References

[1] P. Alfred, A bivariate $C^2$ Clough-Tocher scheme, Computer Aided Geometric Design 1 (1984), 257-267.

[2] P. Alfred and L. L. Schumaker, A smooth macro-element based on Clough-Tocher triangle splits, Numer. Math. 90 (2002), 597-616.

[3] W. Böhm, G. Farin and J. Kahmann, A survey of curve and surfaces methods in CAGD, Computer Aided Geometric Design 1 (1984), 1-60.

[4] C. De Boor, *B*-form Basis in Geometric Modeling Algorithms and New Trends, G. E. Farin, ed., SIAM Publications, Philadelphia, 1987, pp. 131-148.

[5] P. G. Ciarlet, On the Finite Element Method for Elliptic Problems, North Holland, 1978.

[6] R. Clough and T. Tocher, Finite element fitness matrices for analysis of plates in bending, Proc. of Conference of Matrix Methods in Structural Analysis, Wright-Paterson Air Force Base, 1965.

[7] G. Farin, Triangular Bernstein-Bezier patches, Computer Aided Geometric Design 3 (1986), 83-127.

[8] P. Sablonnière and Laghchim-Lahlou, Eléments Finis polynômiaux composés de classe $C^r$, C. R. Acad. Sci. Paris, Sér. I Math 316(5) (1993), 503-508.

[9] M. J. Lai and L. L. Schumaker, Macro-elements and stable local bases for splines on Clough-Tocher triangulations Numer. Math. 88 (2001), 105-119.

[10] P. Sablonnière, On composite finite element of classe $C^k$, J. Comp. Appl. Math. 12 (1985), 541-550.

[11] G. Farin, Bezier Polynomials Over Triangles and the Construction of Piecewise $C^r$ Polynomials, England, 1980.