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NUMERICAL SOLUTION OF WAVES GENERATED BY FLOW OVER A BUMP

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Abstract

When a uniform stream on an open channel is disturbed by an existing bump at the bottom of the channel, the free boundary of the stream forms waves that grow, split and propagate, until a steady formation of a solitary-like wave is achieved. The model of that wave generation can be presented in Boussinesq equations, and its solution is able to simulate those processes. For linear model, the solution is a combination of three functions, each represents a wave with different amplitude and wave speed, depending on the strength of the incoming flow, presented as the Froude number. The solution is then used to verify a method of forward-time forward-space for a system of transport equations related to the Boussinesq equation. A good agreement can be obtained for supercritical flow.

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1. Introduction

A 2-D fluid stream is considered, flowing over a bump that is at the bottom of a channel. We assume that the fluid is inviscid and incompressible; and the flow is irrotational, so that the flow can be described in terms of potential function φ. The fluid flows from left to right, with uniform far upstream with velocity U_0 and depth *H*. The bump disturbs the flow, so that the fluid surface can be observed as waves growing and splitting, followed by propagating. The effect of the bump is to generate waves propagating upstream or downstream from the bump, depending on the strength of the uniform flow far upstream, expressed as the Froude number $F = U_0/\sqrt{gH}$, where *g* is the acceleration due to gravity.

Based on small ratio between the fluid depth and the wavelength, the governing equations of the fluid flow are formulated into an fKdV equation for steady case, and Boussinesq equations for unsteady are obtained by involving another small parameter. Wiryanto and Jamhuri [7] derived the fKdV equation, and solved the equation using a shooting method, by assuming the bottom topography and the surface profile are symmetric to the vertical axis. Two solutions with different crest heights are obtained for supercritical flow, that is, the horizontal velocity U_0 is greater than the wave speed $\sqrt{gH}$. This result is also presented by Chardard et al. [3], Camassa and Wu [1], and Camassa and Wu [2]. Wiryanto [4] then solved the fKdV equation using a finite difference method with non-homogeneous grids, and obtained similar result, that is, two solutions and the surface wave in solitary-like profile. Which solution is stable is the question that then was answered by Wiryanto and Mungkasi [8], who modeled the phenomena into Boussinesq-type equations. The numerical solution was obtained by solving the model using the third order Adams-Bashforth, similar method used by Wiryanto [5] for the model of interfacial wave. The numerical procedure is able to simulate the process of the wave generation. The surface elevation above the bump grows and splits into some waves propagating to the left or right direction. For long run, the surface profile tends to a steady solitary-like wave, similar to one of the results for fKdV model with small amplitude.

To analyze the process of the wave generation and to confirm the numerical solution, in this paper, we solve analytically the linear equations of the model. From the steady case of the Boussinesq equations, Wiryanto and Mungkasi [8] obtained the analytical solution, the surface elevation is proportional to the bump. This solution is then used to construct for the unsteady equations, Wiryanto and Mungkasi [9] derived the unsteady solution and compared it with the numerical solution based on predictor-corrector method, used in Wiryanto and Mungkasi [8]. The numerical solution gives different wave speeds and amplitudes. We seek another numerical method to improve the solution.

Since the model is a system of transport equations containing an external force related to the bottom topography, we propose to use the characteristic method. Wiryanto [6] used the same method for different models related to wave propagation over a porous breakwater without external force. As the result, the analytical solution can describe the phenomena. Basically, there are three waves generated by the flow. After splitting, each wave propagates with different wave speed and in different direction. These can be determined from the solution.

2. Boussinesq Equations

The Boussinesq equations are one model that can be seen in many physical phenomena, especially for wave propagation. These equations can also represent wave generation from uniform flow disturbed by a bump. The model that we are interested in is in the form of the surface elevation η and the average depth velocity u. The strength of the incoming uniform flow is presented in the Froude number $F = U_0/\sqrt{gH}$, and the bottom topography is $h(x)$. The model of the wave generation is Boussinesq-type equations given in non-dimensional and scaled variables as

$$\left.\begin{aligned} &\eta_t + F(\eta_x + u_x - h_x) + \varepsilon F(\eta_x u_{xx} + u_x \eta - u_x h) = 0 \\ &u_t + Fu_x + \frac{1}{F}\eta_x + \varepsilon F u u_x = 0 \end{aligned}\right\}. \tag{1}$$

The existence of a small parameter ε is related to non-linear terms. Physically, it is a ratio between the unity of the wave amplitude and the water depth. Wiryanto and Mungkasi [8] derived the model based on a series expansion of potential function. The variables η and u have initial condition $\eta(x, 0) = 0,\quad u(x, 0) = 0.$ They are defined as the perturbation from the situation at the uniform flow, which is in the order of ε.

In case $\varepsilon = 0,$ the linear steady model can be obtained by taking $t \to \infty.$ The model can be solved analytically, giving

$$\eta_0(x) = \frac{F^2}{F^2 - 1} h(x). \tag{2}$$

The profile of the surface is similar to the bottom topography for supercritical flow $F > 1,$ and for subcritical flow $F < 1$ but the surface has opposite amplitude to h. That profile is followed by the velocity of the fluid particle

$$u_0(x) = \frac{-1}{F^2 - 1} h(x). \tag{3}$$

We then call these solutions as the *stationary ones*. For finite t, η and u are added as the transient solution, discussed in the next section.

3. Transient Solution

We denote the transient solutions as η_1 and $u_1,$ so that

$$\eta(x, t) = \eta_0(x) + \eta_1(x, t), \quad u(x, t) = u_0(x) + u_1(x, t)$$

satisfy the linear equations of (1). When these η and u are substituted to (1), we obtain a set of transport equations without forcing term

$$\left.\begin{aligned} \eta_{1_t} + F(\eta_{1_x} + u_{1_x}) &= 0 \\ u_{1_t} + F u_{1_x} + \frac{1}{F}\eta_{1_x} &= 0 \end{aligned}\right\}. \tag{4}$$

The characteristic method can be applied to get the solution. Basically, we determine lines where the functions have the same value. For convenience, we write (4) in the matrix form

$$\bar{s}_t = A\bar{s}x,$$

where

$$\bar{s} = \begin{pmatrix} \eta_1 \\ u_1 \end{pmatrix}, \quad A = \begin{pmatrix} -F & -F \\ -\frac{1}{F} & -F \end{pmatrix}.$$

To solve the system of equations, we first transform the system into an uncoupled one, using pair of eigenvalues and eigenvectors of A, that is $\lambda = 1 - F$ corresponding to $\bar{X} = (-F \quad 1)^T$ and $\lambda = -1 - F$ corresponding to $\bar{X} = (F \quad 1)^T$, so that we have

$$P = \begin{pmatrix} -F & F \\ 1 & 1 \end{pmatrix}.$$

This matrix is used to define a new vector $\bar{s} = P\bar{y}$, so that the system of equations becomes uncoupled $\bar{y}_t = D\bar{y}_x$, where D is a diagonal matrix containing the eigenvalues of A.

If $\bar{y}$ has elements y_1 and y_2, the uncoupled equation has solutions

$$y_1(x,\ t) = f\left(\frac{1}{F-1}x - t\right),$$

$$y_2(x,\ t) = g\left(\frac{1}{F+1}x - t\right)$$

for arbitrary functions f and g, determined from the initial condition, described below. From the previous transformation, we inverse it to get η_1 and u_1, that is,

$$\eta_1(x,\ t) = -Ff\left(\frac{1}{F-1}x - t\right) + Fg\left(\frac{1}{F+1}x - t\right),$$

$$u_1(x,\ t) = f\left(\frac{1}{F-1}x - t\right) + g\left(\frac{1}{F+1}x - t\right).$$

Now, we determine f and g from the initial conditions $\eta(x,\ 0) = 0$ and $u(x,\ 0) = 0$, also involving the stationary solutions (2) and (3), we have the initial conditions

$$\eta_1(x,\ 0) = \frac{-F^2}{F^2 - 1} h(x), \quad u_1(x,\ 0) = \frac{1}{F^2 - 1} h(x). \tag{5}$$

We then use this for η_1 and u_1, which gives relations

$$-Ff\left(\frac{1}{F-1}x\right) + Fg\left(\frac{1}{F+1}x\right) = \frac{-F^2}{F^2-1}h(x),$$

$$f\left(\frac{1}{F-1}x\right) + g\left(\frac{1}{F+1}x\right) = \frac{1}{F^2-1}h(x).$$

These are then solved, giving

$$f(x) = \frac{1}{2(F-1)}h((F-1)x), \quad g(x) = \frac{-1}{2(F+1)}h((F+1)x),$$

we obtain the transient solutions

$$\eta_1(x,\ t) = \frac{-F}{2(F-1)}h(x-(F-1)t) - \frac{F}{2(F+1)}h(x-(F+1)t),$$

$$u_1(x,\ t) = \frac{1}{2(F-1)}h(x-(F-1)t) - \frac{1}{2(F+1)}h(x-(F+1)t).$$

Here η_1 and u_1 are a linear combination of a function having different characteristic lines $x - (F-1)t =$ constant and $x - (F+1)t =$ constant. Along $(x,\ t)$ on the line, the function has the same value. Therefore, η_1 is combination between two waves of form h, where each travels with wave speeds $c = F - 1$ and $c = F + 1$. For supercritical flow $F > 1$, both waves travel to the right. This is different from subcritical flow $0 < F < 1$, where one of the waves propagates to the left. Wiryanto and Mungkasi [8] also simulated both cases from the numerical solution.

4. Characteristic Method

In solving (4), we first discretize the space x by defining small value Δx, so that the observation domain is divided into J sub-intervals with end points $x_j = j\Delta x$, for $j = 0, 1, \ldots, J$. Similarly for t, we define $\Delta t = \Delta x/(1 + F)$ following one of the characteristic line. Therefore, the equation $\bar{y}_t = D\bar{y}_x$ is discretized by finite difference forward-time forward-space, giving

$$\left.\begin{aligned} y_{1j}^{n+1} &= \frac{2}{1+F}\, y_{1j}^{n} - \frac{1-F}{1+F}\, y_{1(j-1)}^{n} \\ y_{2j}^{n+1} &= y_{2(j-1)}^{n} \end{aligned}\right\} \tag{6}$$

for $j = 1, 2, \ldots, J$. We use notation $y_{ij}^{n} \approx y_i(x_j, t_n)$, for $i = 1, 2$. That finite difference is stable unconditionally for supercritical flow $F > 1$. The von Neumann method can be used for the analysis of the stability.

When we obtain y_{ij}^{n+1}, the values of η_1 and u_1 can be evaluated from $\bar{s} = P\bar{y}$, and then we calculate for $\eta_j^{n+1} = \eta_{0j} + \eta_{1j}^{n+1}$, $u_j^{n+1} = u_{0j} + u_{1j}^{n+1}$. As the initial value, we use (5). It is transformed into y_{1j}^{0}, y_{2j}^{0} by $\bar{y} = P^{-1}\bar{s}$. For each time step, (6) needs the values at $j-1$. For $j = 0$, a boundary value is required. We provide by assuming at that position relatively far from the disturbance, so that we can give $y_{i0}^{n} = 0$.

The numerical procedure described above can be used to evaluate the solution of (1). The result can be compared to the analytical solution. We simulate the numerical solution as shown in Figure 1, the stream of $F = 1.6$ flowing over a bottom topography

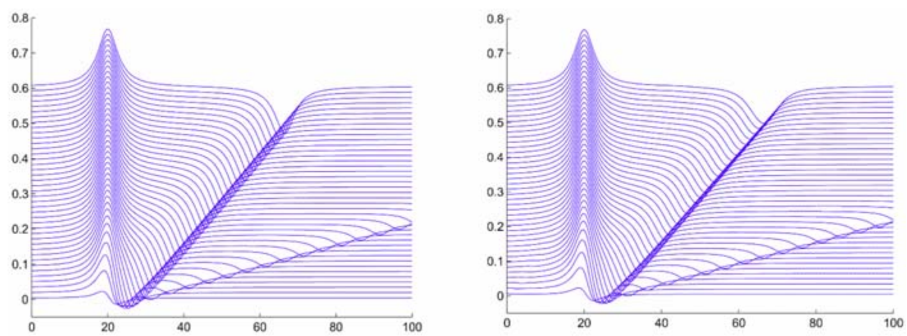
$$h(x) = \frac{0.1}{1 + 0.1(x-20)^2},$$

producing surface waves. For some different times, the surfaces are plotted together in the same coordinate, by shifting upwards. Our numerical calculation is shown in Figure 1(a), and the analytical solution is shown in Figure 1(b). Both plots seem similar. Three waves are generated by disturbance of the flow, each propagates with different amplitude and wave speed. But, if we look closely, then we can see the difference between the numerical and analytical solutions. To do so, we plot the surface η at $t = 26.9$ together between the analytical solution (dash curve) and the numerical one (smooth curve) in Figure 2(a). We choose $t = 2.69$, as at that time we can observe the three waves in the flow domain. Both curves are relatively the same, except at the second wave (in the middle) where we can see that the numerical solution has smaller amplitude than the analytical one, even though they have relatively the same wave speed. We also compare our computation with a result computed by another method that has been used by Wiryanto and Mungkasi [8], that is, the predictor-corrector Adams-Bashforth. We show also in Figure 1(a). The method is able to produce three waves, one stays above the bump and two split and propagate to the right with different wave speeds. The leading wave propagates with smaller wave speed than the same wave from the analytical and our numerical solution. Meanwhile, the predictor-corrector method produces the middle wave propagating with large wave speed, and smaller amplitude, compared to the analytical solution.

For different bottom topography, we obtain a similar result. In Figure 2(b), we show the plot of $\eta(x, 2.69)$ from analytical and characteristic numerical results as well as from the predictor-corrector method. In these results, we use the bottom topography

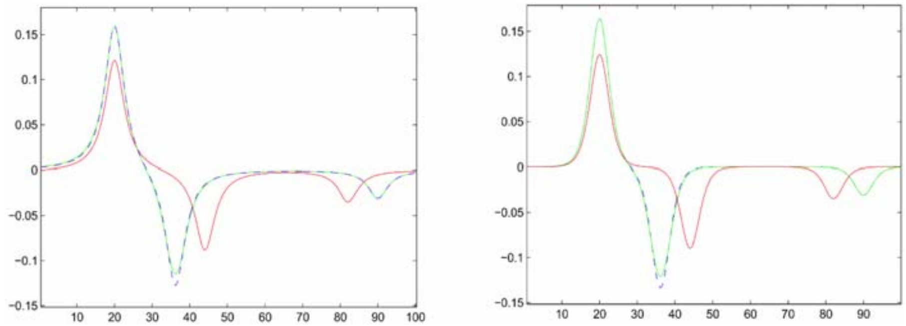
$$h(x) = 0.1\,\mathrm{sech}^2[0.3(x - 20)]$$

disturbing the stream of $F = 1.6$. The effect of this disturbance generates three waves that split each other and propagate with different wave speeds.



(a) (b)

Figure 1. Plot of $\eta(x, t)$ as the result of bottom topography $h(x) = \dfrac{0.1}{1 + 0.1(x - 20)^2}$, disturbing the stream of $F = 1.6$. Here (a) result from the analytical solution (b) result from the characteristic method.



(a) (b)

Figure 2. Comparison between the analytical solution (dash curve (--)) and two numerical solutions (characteristic numerical solution coinciding the analytical one and predictor-corrector numerical solution not well-matching the analytical one) for $\eta(x, 2.69)$. Here we use the same Froude number $F = 1.6$ with the bottom topography (a) $h(x) = \dfrac{0.1}{1 + 0.1(x - 20)^2}$, and (b) $h(x) = 0.1\,\mathrm{sech}^2[0.3(x - 20)]$.

5. Conclusion

We have solved the linear model of wave generation, based on Boussinesq equations. The analytical solution is a linear combination of three functions, which related with the bottom topography. The characters of each wave, amplitude and wave speed, are represented in those functions. This analytical solution is then used to confirm the numerical solution calculated by the characteristic method, which is found in a good agreement for supercritical flow, whereas the predictor-corrector method gives less accurate results for both the amplitude and wave speed.

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