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NEW UNIFORM ORDER EIGHT HYBRID THIRD DERIVATIVE BLOCK METHOD FOR SOLVING SECOND ORDER INITIAL VALUE PROBLEMS

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Abstract

A number of authors have considered the solution of second order initial value problems (IVPs) and the adoption of block methods of order eight has been seen to be widely applied. However, these previously developed block methods have considered non-hybrid grid points. Hence, this article presents a new hybrid block method of order eight for solving second order IVPs with an improved level of accuracy when compared to previously existing order eight block methods in terms of error. The methodology employed involves a new generalized algorithm for developing the hybrid block method which is another novel contribution existing in this work. Hence, not only this article presents a new block method that can be adopted when solving real life problems modelled as second order IVPs, it also gives a more convenient algorithm for developing hybrid block methods.

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1. Introduction

Real life mathematical models come in various forms and differential equations is one vastly adopted way of interpreting certain real life situations such as beam deflection and deformation, transmission of heat, temperature distribution across a rod, mathematically [6, 12]. Second order initial value problem which is a subset of differential equations generally, is likewise used to model real life problems. However, sometimes, there is a need to numerically approximate the solution of certain IVPs either as a result of the exact solution not being in existence or to compare the accuracy of new numerical methods to previously existing methods in literature. The numerical solution of second order initial value problems dates quite far back in literature, growing from the approach of reduction of second order initial value problems to systems of first order initial value problems [7], to direct predictor-corrector methods [3], and now to the vastly adopted block methods in diverse literature [1, 5, 11].

Recently, in literature, certain researchers ventured into the derivation of block methods of uniform order eight for the solution of second order initial value problems [10, 2, 4]. Hence, this article tends to push the mark by presenting a new block method, also of order eight with superior accuracy in terms of comparison in error to previously exiting methods. In addition, the algorithm presented in this article is a new introduction of an alternate approach for developing hybrid block methods.

2. Methodology

The development of block methods to solve second initial value problems of the following form is considered:

$$y'' = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0. \tag{1}$$

The algorithm adopted to develop the hybrid block method is given below which is a modification of the general higher derivative algorithm in [8] to accommodate the presence of hybrid points

$$y_{n+r\xi} = \sum_{i=0}^{1} \frac{(r\xi h)^i}{i!} y_n^{(i)} + \sum_{i=0}^{k} (\phi_{\xi i} f_{n+ri} + \tau_{\xi i} g_{n+ri}), \quad \xi = 1, 2, ..., k \tag{2}$$

with first derivative

$$y'_{n+r\xi} = y'_n + \sum_{i=0}^{k} (\omega_{\xi i} f_{n+ri} + \varphi_{\xi i} g_{n+ri}), \quad \xi = 1, 2, ..., k, \tag{3}$$

where $f_{n+ri} = f(x_{n+ri}, y_{n+ri}, y'_{n+ri})$, $g_{n+ri} = g(x_{n+ri}, y_{n+ri}, y'_{n+ri}) = \frac{df(x_{n+ri}, y_{n+ri}, y'_{n+ri})}{dx}$, $(\phi_{\xi 0}, \phi_{\xi 1}, ..., \phi_{\xi k}, \tau_{\xi 0}, \tau_{\xi 1}, ..., \tau_{\xi k})^T = A^{-1}B$ and $(\omega_{\xi 0}, \omega_{\xi 1}, ..., \omega_{\xi k}, \varphi_{\xi 0}, \varphi_{\xi 1}, ..., \tau_{\xi k})^T = A^{-1}D,$ with

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & rh & 2rh & \cdots & krh & 1 & 1 & 1 & \cdots & 1 \\ 0 & \frac{(rh)^2}{2!} & \frac{(2rh)^2}{2!} & \cdots & \frac{(krh)^2}{2!} & 0 & h & 2rh & \cdots & krh \\ 0 & \frac{(rh)^3}{3!} & \frac{(2rh)^3}{3!} & \cdots & \frac{(krh)^3}{3!} & 0 & \frac{(rh)^2}{2!} & \frac{(2rh)^2}{2!} & \cdots & \frac{(krh)^2}{2!} \\ 0 & \frac{(rh)^4}{4!} & \frac{(2rh)^4}{4!} & \cdots & \frac{(krh)^4}{4!} & 0 & \frac{(rh)^3}{3!} & \frac{(2rh)^3}{3!} & \cdots & \frac{(krh)^3}{3!} \\ . & . & . & \cdots & . & . & . & . & \cdots & . \\ . & . & . & \cdots & . & . & . & . & \cdots & . \\ . & . & . & \cdots & . & . & . & . & \cdots & . \\ 0 & \frac{(rh)^{(2kr+3)}}{(2kr+3)!} & \frac{(2rh)^{(2kr+3)}}{(2kr+3)!} & \cdots & \frac{(krh)^{(2kr+3)}}{(2kr+3)!} & 0 & \frac{(rh)^{(2kr+2)}}{(2kr+2)!} & \frac{(2rh)^{(2kr+2)}}{(2kr+3)!} & \cdots & \frac{((k-1)rh)^{(2kr+2)}}{(2kr+2)!} \end{pmatrix},$$

$$B = \begin{pmatrix} \frac{(\xi rh)^2}{2!} \\ \frac{(\xi rh)^{(3)}}{(3)!} \\ \frac{(\xi rh)^{(4)}}{(4)!} \\ . \\ . \\ . \\ \frac{(\xi rh)^{((2kr+3)+k)}}{((2kr+3)+k)!} \end{pmatrix}, \qquad D = \begin{pmatrix} \frac{(\xi rh)^{(1)}}{(1)!} \\ \frac{(\xi rh)^2}{2!} \\ \frac{(\xi rh)^{(3)}}{(3)!} \\ . \\ . \\ . \\ \frac{(\xi rh)^{((2kr+3)+k-1)}}{((2kr+3)+k-1)!} \end{pmatrix}.$$

Taking $r = \frac{2}{3}$ and $k = 3$ gives the algorithm from equations (2) and (3) in the form

$$y_{n+\frac{2}{3}\xi} = \sum_{i=0}^{1} \frac{\left(\frac{2}{3}\xi h\right)^{i}}{i!} y_n^{(i)} + \sum_{i=0}^{3} (\phi_{\xi i} f_{n+\frac{2}{3}i} + \tau_{\xi i} g_{n+\frac{2}{3}i}), \quad \xi = 1, 2, 3 \tag{4}$$

with first derivative

$$y'_{n+\frac{2}{3}\xi} = y'_n + \sum_{i=0}^{3} (\omega_{\xi i} f_{n+\frac{2}{3}i} + \varphi_{\xi i} g_{n+\frac{2}{3}i}), \quad \xi = 1, 2, 3, \tag{5}$$

$(\phi_{\xi 0}, \phi_{\xi 1}, \phi_{\xi 2}, \phi_{\xi 3}, \tau_{\xi 0}, \tau_{\xi 1}, \tau_{\xi 2}, \tau_{\xi 3})^T = A^{-1}B$ and $(\omega_{\xi 0}, \omega_{\xi 1}, \omega_{\xi 2}, \omega_{\xi 3}, \varphi_{\xi 0}, \varphi_{\xi 1}, \varphi_{\xi 2}, \varphi_{\xi 3})^T = A^{-1}D$, where

$$A = \begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & \left(\frac{2}{3}\right)h & 2\left(\frac{2}{3}\right)h & 3\left(\frac{2}{3}\right)h & 1 & 1 & 1 & 1 \\
0 & \frac{\left(\left(\frac{2}{3}\right)h\right)^2}{2!} & \frac{\left(2\left(\frac{2}{3}\right)h\right)^2}{2!} & \frac{\left(3\left(\frac{2}{3}\right)h\right)^2}{2!} & 0 & \left(\frac{2}{3}\right)h & 2\left(\frac{2}{3}\right)h & 3\left(\frac{2}{3}\right)h \\
0 & \frac{\left(\left(\frac{2}{3}\right)h\right)^3}{3!} & \frac{\left(2\left(\frac{2}{3}\right)h\right)^3}{3!} & \frac{\left(3\left(\frac{2}{3}\right)h\right)^3}{3!} & 0 & \frac{\left(\left(\frac{2}{3}\right)h\right)^2}{2!} & \frac{\left(2\left(\frac{2}{3}\right)h\right)^2}{2!} & \frac{\left(3\left(\frac{2}{3}\right)h\right)^2}{2!} \\
0 & \frac{\left(\left(\frac{2}{3}\right)h\right)^4}{4!} & \frac{\left(2\left(\frac{2}{3}\right)h\right)^4}{4!} & \frac{\left(3\left(\frac{2}{3}\right)h\right)^4}{4!} & 0 & \frac{\left(\left(\frac{2}{3}\right)h\right)^3}{3!} & \frac{\left(2\left(\frac{2}{3}\right)h\right)^3}{3!} & \frac{\left(3\left(\frac{2}{3}\right)h\right)^3}{3!} \\
0 & \frac{\left(\left(\frac{2}{3}\right)h\right)^5}{5!} & \frac{\left(2\left(\frac{2}{3}\right)h\right)^5}{5!} & \frac{\left(3\left(\frac{2}{3}\right)h\right)^5}{5!} & 0 & \frac{\left(\left(\frac{2}{3}\right)h\right)^4}{4!} & \frac{\left(2\left(\frac{2}{3}\right)h\right)^4}{4!} & \frac{\left(3\left(\frac{2}{3}\right)h\right)^4}{4!} \\
0 & \frac{\left(\left(\frac{2}{3}\right)h\right)^6}{6!} & \frac{\left(2\left(\frac{2}{3}\right)h\right)^6}{6!} & \frac{\left(3\left(\frac{2}{3}\right)h\right)^6}{6!} & 0 & \frac{\left(\left(\frac{2}{3}\right)h\right)^5}{5!} & \frac{\left(2\left(\frac{2}{3}\right)h\right)^5}{5!} & \frac{\left(3\left(\frac{2}{3}\right)h\right)^5}{5!} \\
0 & \frac{\left(\left(\frac{2}{3}\right)h\right)^7}{7!} & \frac{\left(2\left(\frac{2}{3}\right)h\right)^7}{7!} & \frac{\left(3\left(\frac{2}{3}\right)h\right)^7}{7!} & 0 & \frac{\left(\left(\frac{2}{3}\right)h\right)^6}{6!} & \frac{\left(2\left(\frac{2}{3}\right)h\right)^6}{6!} & \frac{\left(3\left(\frac{2}{3}\right)h\right)^6}{6!}
\end{pmatrix},$$

$$B = \begin{pmatrix} \frac{\left(\xi \frac{2}{3} h\right)^2}{2!} \\ \frac{\left(\xi \frac{2}{3} h\right)^{(3)}}{(3)!} \\ \frac{\left(\xi \frac{2}{3} h\right)^{(4)}}{(4)!} \\ \frac{\left(\xi \frac{2}{3} h\right)^{(5)}}{(5)!} \\ \frac{\left(\xi \frac{2}{3} h\right)^{(6)}}{(6)!} \\ \frac{\left(\xi \frac{2}{3} h\right)^{(7)}}{(7)!} \\ \frac{\left(\xi \frac{2}{3} h\right)^{(8)}}{(8)!} \\ \frac{\left(\xi \frac{2}{3} h\right)^{(9)}}{(9)!} \end{pmatrix}, \qquad D = \begin{pmatrix} \frac{\left(\xi \frac{2}{3} h\right)^1}{1!} \\ \frac{\left(\xi \frac{2}{3} h\right)^2}{2!} \\ \frac{\left(\xi \frac{2}{3} h\right)^{(3)}}{(3)!} \\ \frac{\left(\xi \frac{2}{3} h\right)^{(4)}}{(4)!} \\ \frac{\left(\xi \frac{2}{3} h\right)^{(5)}}{(5)!} \\ \frac{\left(\xi \frac{2}{3} h\right)^{(6)}}{(6)!} \\ \frac{\left(\xi \frac{2}{3} h\right)^{(7)}}{(7)!} \\ \frac{\left(\xi \frac{2}{3} h\right)^{(8)}}{(8)!} \end{pmatrix}.$$

Expanding (4) and (5) gives the block method

$$y_{n+\frac{2}{3}} = \frac{\left(\frac{2}{3} h\right)^0}{0!} y_n + \frac{\left(\frac{2}{3} h\right)^1}{1!} y'_n + (\phi_{10} f_n + \phi_{11} f_{n+\frac{2}{3}} + \phi_{12} f_{n+\frac{2}{3}(2)}$$

$$+ \phi_{13} f_{n+\frac{2}{3}(3)} + \tau_{10} g_n + \tau_{11} g_{n+\frac{2}{3}} + \tau_{12} g_{n+\frac{2}{3}(2)} + \tau_{13} g_{n+\frac{2}{3}(3)}),$$

$$y_{n+\frac{2}{3}(2)} = \frac{\left(\frac{2}{3}(2) h\right)^0}{0!} y_n + \frac{\left(\frac{2}{3}(2) h\right)^1}{1!} y'_n + (\phi_{20} f_n + \phi_{21} f_{n+\frac{2}{3}} + \phi_{22} f_{n+\frac{2}{3}(2)}$$

$$+\phi_{23}f_{n+\frac{2}{3}(3)}+\tau_{20}g_n+\tau_{21}g_{n+\frac{2}{3}}+\tau_{22}g_{n+\frac{2}{3}(2)}+\tau_{23}g_{n+\frac{2}{3}(3)}),$$

$$y_{n+\frac{2}{3}(3)}=\frac{\left(\frac{2}{3}(3)h\right)^0}{0!}y_n+\frac{\left(\frac{2}{3}(3)h\right)^1}{1!}y'_n+(\phi_{30}f_n+\phi_{31}f_{n+\frac{2}{3}}+\phi_{32}f_{n+\frac{2}{3}(2)}$$

$$+\phi_{33}f_{n+\frac{2}{3}(3)}+\tau_{30}g_n+\tau_{31}g_{n+\frac{2}{3}}+\tau_{32}g_{n+\frac{2}{3}(2)}+\tau_{33}g_{n+\frac{2}{3}(3)}), \quad (6)$$

with first derivative

$$y'_{n+\frac{2}{3}}=y'_n+(\omega_{10}f_n+\omega_{11}f_{n+\frac{2}{3}}+\omega_{12}f_{n+\frac{2}{3}(2)}+\omega_{13}f_{n+\frac{2}{3}(3)}$$

$$+\varphi_{10}g_n+\varphi_{11}g_{n+\frac{2}{3}}+\varphi_{12}g_{n+\frac{2}{3}(2)}+\varphi_{13}g_{n+\frac{2}{3}(3)}),$$

$$y'_{n+\frac{2}{3}(2)}=y'_n+(\omega_{20}f_n+\omega_{21}f_{n+\frac{2}{3}}+\omega_{22}f_{n+\frac{2}{3}(2)}+\omega_{23}f_{n+\frac{2}{3}(2)}$$

$$+\varphi_{20}g_n+\varphi_{21}g_{n+\frac{2}{3}}+\varphi_{22}g_{n+\frac{2}{3}(2)}+\varphi_{23}g_{n+\frac{2}{3}(3)})$$

$$y'_{n+\frac{2}{3}(3)}=y'_n+(\omega_{30}f_n+\omega_{31}f_{n+\frac{2}{3}}+\omega_{32}f_{n+\frac{2}{3}(3)}+\omega_{33}f_{n+\frac{2}{3}(3)}$$

$$+\varphi_{30}g_n+\varphi_{31}g_{n+\frac{2}{3}}+\varphi_{32}g_{n+\frac{2}{3}(2)}+\varphi_{33}g_{n+\frac{2}{3}(3)}) \quad (7)$$

and coefficients

$$(\phi_{10},\phi_{11},\phi_{12},\phi_{13},\tau_{10},\tau_{11},\tau_{12},\tau_{13})^T,$$

$$(\phi_{20},\phi_{21},\phi_{22},\phi_{23},\tau_{20},\tau_{21},\tau_{22},\tau_{23})^T,$$

$$(\phi_{30},\phi_{31},\phi_{32},\phi_{33},\tau_{30},\tau_{31},\tau_{32},\tau_{33})^T,$$

$$(\omega_{10},\omega_{11},\omega_{12},\omega_{13},\varphi_{10},\varphi_{11},\varphi_{12},\varphi_{13})^T,$$

$$(\omega_{20},\omega_{21},\omega_{22},\omega_{23},\varphi_{20},\varphi_{21},\varphi_{22},\varphi_{23})^T$$

and $(\omega_{30}, \omega_{31}, \omega_{32}, \omega_{33}, \varphi_{30}, \varphi_{31}, \varphi_{32}, \varphi_{33})^T$ are obtained as follows:

$$(\phi_{10}, \phi_{11}, \phi_{12}, \phi_{13}, \tau_{10}, \tau_{11}, \tau_{12}, \tau_{13})^T$$

$$= \left(\frac{19519h^2}{153090}, \frac{1301h^2}{22680}, \frac{181h^2}{5670}, \frac{3329h^2}{612360}, \frac{371h^3}{43740}, -\frac{313h^3}{8505}, -\frac{89h^3}{6804}, -\frac{137h^3}{153090}\right)^T,$$

$$(\phi_{20}, \phi_{21}, \phi_{22}, \phi_{23}, \tau_{20}, \tau_{21}, \tau_{22}, \tau_{23})^T$$

$$= \left(\frac{22924h^2}{76545}, \frac{1184h^2}{2835}, \frac{436h^2}{2835}, \frac{1376h^2}{76545}, \frac{1648h^3}{76545}, -\frac{160h^3}{1701}, -\frac{416h^3}{8505}, -\frac{32h^3}{10935}\right)^T,$$

$$(\phi_{30}, \phi_{31}, \phi_{32}, \phi_{33}, \tau_{30}, \tau_{31}, \tau_{32}, \tau_{33})^T$$

$$= \left(\frac{67h^2}{140}, \frac{243h^2}{280}, \frac{81h^2}{140}, \frac{3h^2}{40}, \frac{h^3}{28}, -\frac{9h^3}{70}, -\frac{9h^3}{140}, -\frac{h^3}{105}\right)^T,$$

$$(\omega_{10}, \omega_{11}, \omega_{12}, \omega_{13}, \varphi_{10}, \varphi_{11}, \varphi_{12}, \varphi_{13})^T$$

$$= \left(\frac{6893h}{27216}, \frac{313h}{1008}, \frac{89h}{1008}, \frac{397h}{27216}, \frac{1283h^2}{68040}, -\frac{851h^2}{7560}, -\frac{269h^2}{7560}, -\frac{163h^2}{68040}\right)^T,$$

$$(\omega_{20}, \omega_{21}, \omega_{22}, \omega_{23}, \varphi_{20}, \varphi_{21}, \varphi_{22}, \varphi_{23})^T$$

$$= \left(\frac{446h}{1701}, \frac{40h}{63}, \frac{26h}{63}, \frac{40h}{1701}, \frac{172h^2}{8505}, -\frac{64h^2}{945}, -\frac{76h^2}{945}, -\frac{32h^2}{8505}\right)^T$$

$$(\omega_{30}, \omega_{31}, \omega_{32}, \omega_{33}, \varphi_{30}, \varphi_{31}, \varphi_{32}, \varphi_{33})^T$$

$$= \left(\frac{31h}{112}, \frac{81h}{112}, \frac{81h}{112}, \frac{31h}{112}, \frac{19h^2}{840}, -\frac{9h^2}{280}, \frac{9h^2}{280}, -\frac{19h^2}{840}\right)^T$$

(see Appendix for details).

Substituting these coefficients back in equations (6) and (7) gives the desired block method

$$y_{n+\frac{2}{3}} = y_n + \frac{2}{3}hy'_n$$

$$+ \frac{h^2}{612360}\left(78076f_n + 35127f_{n+\frac{2}{3}} + 19548f_{n+\frac{4}{3}} + 3329f_{n+2}\right)$$

$$+ \frac{h^3}{306180}\left(2597g_n - 11268g_{n+\frac{2}{3}} - 4005g_{n+\frac{4}{3}} - 274g_{n+2}\right),$$

$$y_{n+\frac{4}{3}} = y_n + \frac{4}{3}hy'_n$$

$$+ \frac{h^2}{76545}\left(22924f_n + 31968f_{n+\frac{2}{3}} + 11772f_{n+\frac{4}{3}} + 1376f_{n+2}\right)$$

$$+ \frac{h^3}{76545}\left(1648g_n - 7200g_{n+\frac{2}{3}} - 3744g_{n+\frac{4}{3}} - 224g_{n+2}\right),$$

$$y_{n+2} = y_n + 2hy'_n + \frac{h^2}{840}\left(402f_n + 729f_{n+\frac{2}{3}} + 486f_{n+\frac{4}{3}} + 63f_{n+2}\right)$$

$$+ \frac{h^3}{840}\left(30g_n - 108g_{n+\frac{2}{3}} - 54g_{n+\frac{4}{3}} - 8g_{n+2}\right),$$

$$y'_{n+\frac{2}{3}} = y'_n + \frac{h}{27216}\left(6893f_n + 8451f_{n+\frac{2}{3}} + 2403f_{n+\frac{4}{3}} + 397f_{n+2}\right)$$

$$+ \frac{h^2}{68040}\left(1283g_n - 7659g_{n+\frac{2}{3}} - 2421g_{n+\frac{4}{3}} - 163g_{n+2}\right),$$

$$y'_{n+\frac{4}{3}} = y'_n + \frac{h}{1701}\left(446f_n + 1080f_{n+\frac{2}{3}} + 702f_{n+\frac{4}{3}} + 40f_{n+2}\right)$$

$$+ \frac{h^2}{8505}\left(172g_n - 576g_{n+\frac{2}{3}} - 684g_{n+\frac{4}{3}} - 32g_{n+2}\right),$$

$$y'_{n+2} = y'_n + \frac{h}{112}\left(31f_n + 81f_{n+\frac{2}{3}} + 81f_{n+\frac{4}{3}} + 31f_{n+2}\right)$$

$$+ \frac{h^2}{840}\left(19g_n - 27g_{n+\frac{2}{3}} + 27g_{n+\frac{4}{3}} - 19g_{n+2}\right). \quad (8)$$

3. Analysis of the Block Method

3.1. Order of the block method

Adopting the approach for obtaining the order of multistep methods as stated by Lambert in Kuboye and Omar [10], the individual terms of the correctors of the block method (8) are expanded using Taylor series expansion and after simplification, the correctors are of uniform order $[8, 8, 8]^T$ with error constant $[1.2254 \times 10^{-7}, 3.7139 \times 10^{-7}, 7.4653 \times 10^{-7}]^T$.

3.2. Zero-stability of the block method

To analyze the block method (8) for zero-stability, the correctors of the block method are normalized to give the first characteristic polynomial $\rho(R)$ as

$$\rho(R) = \det(RA^0 - A^1) = R^2(R - 1),$$

where A^0 is the identity matrix of dimension 3 and A^1 is a matrix of dimension 3 given by

$$A^1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The roots of $\rho(R) = 0$ satisfy $| R_j | \leq 1, \quad j = 1, \ldots, 3$ (see [8]). Hence, the block method is said to be *zero-stable*.

3.3. Convergence of the block method

The block method has order greater than one and is also zero-stable, hence the block method is convergent.

3.4. Error accuracy test with numerical examples

Consider the following second order initial value problems:

(1) $y'' - 100y = 0,\ y(0) = 1,\ y'(0) = -10,\ h = 0.01,$

$$y = e^{-10x}.$$

Source: Kuboye and Omar [10].

(2) $y'' + \frac{6}{x}y' + \frac{4}{x^2}y = 0,\ y(1) = 1,\ y'(1) = 1,\ h = \frac{0.1}{32},$

$$y = \frac{5}{3x} - \frac{2}{3x^4}.$$

Source: Badmus [4].

(3) $y'' + y = 0,\ y(0) = 1,\ y'(0) = 1,\ h = 0.1,$

$$y = \cos x + \sin x.$$

Source: Kuboye and Omar [10].

(4) $y'' = -y + 2\cos x,\ y(0) = 1,\ y'(0) = 0,$

$$y = \cos x + x\sin x.$$

Source: Kuboye and Omar [10].

(5) Bessel's IVP:

$$t^2 y'' + ty' + (t^2 - 0.25)y = 0,\ y(1) = \sqrt{\frac{2}{\pi}}\sin 1 \simeq 0.6713967071418031,$$

$$y'(1) = (2\cos 1 - \sin 1)/\sqrt{2\pi} \simeq 0.0954005144474746,$$

$$y(t) = J_{1/2}(t) = \sqrt{\frac{2}{\pi t}}\sin t.$$

Source: Jator and Lee [9].

All the problems stated above have been solved in previous literature by authors who also developed block methods of order eight. The numerical results will be compared in terms of error and the maximum error over the displayed interval of solution will also be investigated to show the superiority of the new hybrid block method in comparison to previously existing methods.

In addition, for the Bessel's IVP (numerical example 5), the theoretical solution was computed at $t = 8$ which is $y(8) = \sqrt{\frac{2}{8\pi}} \sin 8 \simeq 0.279092789108058969$ as stated in [9]. Hence, the errors in the solution as shown in Table 5 were obtained at $t = 8$ using the hybrid block method of order 8. Comparison in terms of errors was made with previously existing literature; [13] method of order 8 and [9] method of order 7. The basis for comparison with these authors is chosen because the order of the numerical methods presented in the separate works is either equal or very close to that of the new hybrid block method.

Table 1. Comparison of results with Kuboye and Omar [10] for solving problem (1)

x	Exact solution	Computed solution	Error (Kuboye and Omar [10])	Error (new method)
0.0067	0.935506985031617738	0.935506985031617727	-	1.124E-17
0.0100	0.904837418035959573	-	5.744294E-13	-
0.0133	0.875173319042947454	0.875173319042947420	-	3.391E-17
0.0200	0.818730753077981859	0.818730753077981791	1.225396E-10	6.811E-17
0.0267	0.765928338364648693	0.765928338364648570	-	1.229E-16
0.0300	0.740818220681717866	-	2.179856E-10	-
0.0333	0.716531310573789250	0.71653131057378906291	-	1.875E-16
0.0400	0.670320046035639301	0.670320046035639039	3.139226E-10	2.623E-16
0.0467	0.627089085273056128	0.627089085273055773	-	3.549E-16
0.0500	0.606530659712633424	-	4.196442E-10	-
0.0533	0.627089085273056128	0.627089085273055773	-	3.549E-16
0.0600	0.548811636094026433	0.548811636094025865	5.896942E-10	5.680E-16
0.0667	0.513417119032592027	0.513417119032591331	-	6.955E-16
0.0700	0.496585303791409515	-	2.036323E-10	-
0.0733	0.480305301089799372	0.480305301089798539	-	8.323E-16
0.0800	0.449328964117221591	0.449328964117220612	1.847891E-10	9.791E-16
0.0867	0.420350384508681923	0.420350384508680782	-	1.141E-15
0.0900	0.406569659740599112	-	1.677546E-10	-
0.0933	0.393240720868598260	0.393240720868596946	-	1.314E-15
0.1000	0.367879441171442322	0.367879441171440824	1.523623E-10 (MAXE)	1.497E-15 (MAXE)

Table 2. Comparison of results with Badmus [4] for solving problem (2)

x	Exact solution	Computed solution	Error (Badmus [4]) $p = 8$	Error (new method) $p = 8$
1.00208333	1.0020617370305455294	1.0020617370964731665	-	6.593E-11
1.00312500	1.0030765258576962262	-	8.30000E-08	-
1.00416667	1.0040806985680117885	1.0040806987850003993	-	2.170E-10
1.00625000	1.0060575030835162830	1.0060575039914360574	1.16000E-06	9.079E-10
1.00833333	1.0079927595815356024	1.0079923283822239948	-	4.312E-07
1.00937500	1.0089449950888375792	-	6.63000E-06	-
1.01041667	1.0098870677601007513	1.0098862098657828366	-	8.579E-07
1.01250000	1.0117410181679885288	1.0117397393515215468	9.49100E-06	1.279E-06
1.01458333	1.0135551923589710213	1.0135530860873990001	-	2.106E-06
1.01562500	1.0144475426864138744	-	1.95350E-06	-
1.01666667	1.0153301630431838703	1.0153272395606502234	-	2.923E-06
1.01875000	1.0170664942356726084	1.0170627640614517225	9.41600E-06	3.730E-06
1.02083333	1.0187647414021750274	1.0187598227038787095	-	4.919E-06
1.02187500	1.0195997547562875920	-	4.65050E-05	-
1.02291667	1.0204254516021962276	1.0204193589985889787	-	6.093E-06
1.02500000	1.0220491636294317413	1.0220419119368141202	4.71220E-05	7.252E-06
1.02708333	1.0236364081495928741	1.0236276387687372771	-	8.769E-06
1.02812500	1.0244165187384026804	-	1.86926E-04	-
1.02916667	1.0251877078356872026	1.0251774393377305134	-	1.027E-05
1.03125000	1.0267035775008059839	1.0266918285840912173	4.43321E-04 (MAXE)	1.175E-05 (MAXE)

Table 3. Comparison of results with Kuboye and Omar [10] for solving problem (3)

x	Exact solution	Computed solution	Error (Kuboye and Omar [10])	Error (new method)
0.067	1.064395895624515304	1.064395895624515318	-	1.340E-17
0.100	1.094837581924853918	-	5.535572E-13	-
0.133	1.124062894656493434	1.124062894656493475	-	4.030E-17
0.200	1.178735908636302847	1.178735908636302927	1.377787E-12	8.110E-17
0.267	1.228172036374098983	1.228172036374099134	-	1.504E-16
0.300	1.250856695786945595	-	2.116307E-12	-
0.333	1.272151643110889909	1.272151643110890143	-	2.343E-16
0.400	1.310479336311535575	1.310479336311535908	1.934453E-12	3.332E-16
0.467	1.342984833754428954	1.342984833754429417	-	4.627E-16
0.500	1.357008100494575716	-	7.957857E-12	-
0.533	1.369523720061111420	1.369523720061112027	-	6.070E-16
0.600	1.389978088304713654	1.389978088304714420	8.164358E-11	7.661E-16
0.667	1.404257063846685009	1.404257063846685964	-	9.556E-16
0.700	1.409059874522179480	-	4.594665E-10	-
0.733	1.412297208074510229	1.412297208074511387	-	1.158E-15
0.800	1.414062800246688183	1.414062800246689557	1.304548E-07	1.374E-15

0.867	1.409545996192792110	1.409545996192793728	-	1.618E-15
0.900	1.404936877898147845	-	2.600661E-07	-
0.933	1.398766863163538995	1.398766863163540868	-	1.873E-15
1.000	1.381773290676036224	1.381773290676038361	5.102239E-07 (MAXE)	2.137E-15 (MAXE)

Table 4. Comparison of results with Kuboye and Omar [10] for solving problem (4)

h-values	Number of steps	Error (Kuboye and Omar [10])	Error (new method)
10^{-2}	54	4.932632E-11	0.000000E+00
	39	2.491618E-11	0.000000E+00
10^{-3}	504	5.595524E-14	0.000000E+00
	339	4.551914E-15	0.000000E+00
10^{-4}	5004	8.104628E-13	0.000000E+00
	3339	1.054712E-14	0.000000E+00
10^{-5}	50004	1.287859E-12	0.000000E+00
	33339	2.704503E-13	0.000000E+00

Table 5. Comparison of results for solving problem (5)

Number of steps [13]	Error [13]	Number of steps [9]	Error [9]	Number of steps (new method)	Error (new method)
67	7.1122E-7	60	2.49E-8	60	1.422544E-10
82	9.2632E-8	80	3.16E-9	80	1.658845E-11
97	8.7834E-9	100	6.04E-10	100	3.003209E-12
112	1.2108E-10	112	2.57E-10	112	1.248113E-12

4. Conclusion

This paper has presented a new uniform order eight numerical method for approximating the solution of second order initial value problems in hybrid block mode. The property of the method has shown its convergence and the numerical problems considered have likewise shown its superiority over previously existing methods in literature. Also, as aforementioned, a new algorithm for developing the hybrid block method is also given.

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Appendix

$$\begin{pmatrix} \phi_{10} \\ \phi_{11} \\ \phi_{12} \\ \phi_{13} \\ \tau_{10} \\ \tau_{11} \\ \tau_{12} \\ \tau_{13} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & (\frac{2}{3})h & 2(\frac{2}{3})h & 3(\frac{2}{3})h & 1 & 1 & 1 & 1 \\ 0 & \frac{((\frac{2}{3})h)^2}{2!} & \frac{(2(\frac{2}{3})h)^2}{2!} & \frac{(3(\frac{2}{3})h)^2}{2!} & 0 & (\frac{2}{3})h & 2(\frac{2}{3})h & 3(\frac{2}{3})h \\ 0 & \frac{((\frac{2}{3})h)^3}{3!} & \frac{(2(\frac{2}{3})h)^3}{3!} & \frac{(3(\frac{2}{3})h)^3}{3!} & 0 & \frac{((\frac{2}{3})h)^2}{2!} & \frac{(2(\frac{2}{3})h)^2}{2!} & \frac{(3(\frac{2}{3})h)^2}{2!} \\ 0 & \frac{((\frac{2}{3})h)^4}{4!} & \frac{(2(\frac{2}{3})h)^4}{4!} & \frac{(3(\frac{2}{3})h)^4}{4!} & 0 & \frac{((\frac{2}{3})h)^3}{3!} & \frac{(2(\frac{2}{3})h)^3}{3!} & \frac{(3(\frac{2}{3})h)^3}{3!} \\ 0 & \frac{((\frac{2}{3})h)^5}{5!} & \frac{(2(\frac{2}{3})h)^5}{5!} & \frac{(3(\frac{2}{3})h)^5}{5!} & 0 & \frac{((\frac{2}{3})h)^4}{4!} & \frac{(2(\frac{2}{3})h)^4}{4!} & \frac{(3(\frac{2}{3})h)^4}{4!} \\ 0 & \frac{((\frac{2}{3})h)^6}{6!} & \frac{(2(\frac{2}{3})h)^6}{6!} & \frac{(3(\frac{2}{3})h)^6}{6!} & 0 & \frac{((\frac{2}{3})h)^5}{5!} & \frac{(2(\frac{2}{3})h)^5}{5!} & \frac{(3(\frac{2}{3})h)^5}{5!} \\ 0 & \frac{((\frac{2}{3})h)^7}{7!} & \frac{(2(\frac{2}{3})h)^7}{7!} & \frac{(3(\frac{2}{3})h)^7}{7!} & 0 & \frac{((\frac{2}{3})h)^6}{6!} & \frac{(2(\frac{2}{3})h)^6}{6!} & \frac{(3(\frac{2}{3})h)^6}{6!} \end{pmatrix}^{-1} \begin{pmatrix} \frac{(\frac{2}{3}h)^2}{2!} \\ \frac{(\frac{2}{3}h)^{(3)}}{(3)!} \\ \frac{(\frac{2}{3}h)^{(4)}}{(4)!} \\ \frac{(\frac{2}{3}h)^{(5)}}{(5)!} \\ \frac{(\frac{2}{3}h)^{(6)}}{(6)!} \\ \frac{(\frac{2}{3}h)^{(7)}}{(7)!} \\ \frac{(\frac{2}{3}h)^{(8)}}{(8)!} \\ \frac{(\frac{2}{3}h)^{(9)}}{(9)!} \end{pmatrix},$$

$$\begin{pmatrix} \phi_{20} \\ \phi_{21} \\ \phi_{22} \\ \phi_{23} \\ \tau_{20} \\ \tau_{21} \\ \tau_{22} \\ \tau_{23} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & (\frac{2}{3})h & 2(\frac{2}{3})h & 3(\frac{2}{3})h & 1 & 1 & 1 & 1 \\ 0 & \frac{((\frac{2}{3})h)^2}{2!} & \frac{(2(\frac{2}{3})h)^2}{2!} & \frac{(3(\frac{2}{3})h)^2}{2!} & 0 & (\frac{2}{3})h & 2(\frac{2}{3})h & 3(\frac{2}{3})h \\ 0 & \frac{((\frac{2}{3})h)^3}{3!} & \frac{(2(\frac{2}{3})h)^3}{3!} & \frac{(3(\frac{2}{3})h)^3}{3!} & 0 & \frac{((\frac{2}{3})h)^2}{2!} & \frac{(2(\frac{2}{3})h)^2}{2!} & \frac{(3(\frac{2}{3})h)^2}{2!} \\ 0 & \frac{((\frac{2}{3})h)^4}{4!} & \frac{(2(\frac{2}{3})h)^4}{4!} & \frac{(3(\frac{2}{3})h)^4}{4!} & 0 & \frac{((\frac{2}{3})h)^3}{3!} & \frac{(2(\frac{2}{3})h)^3}{3!} & \frac{(3(\frac{2}{3})h)^3}{3!} \\ 0 & \frac{((\frac{2}{3})h)^5}{5!} & \frac{(2(\frac{2}{3})h)^5}{5!} & \frac{(3(\frac{2}{3})h)^5}{5!} & 0 & \frac{((\frac{2}{3})h)^4}{4!} & \frac{(2(\frac{2}{3})h)^4}{4!} & \frac{(3(\frac{2}{3})h)^4}{4!} \\ 0 & \frac{((\frac{2}{3})h)^6}{6!} & \frac{(2(\frac{2}{3})h)^6}{6!} & \frac{(3(\frac{2}{3})h)^6}{6!} & 0 & \frac{((\frac{2}{3})h)^5}{5!} & \frac{(2(\frac{2}{3})h)^5}{5!} & \frac{(3(\frac{2}{3})h)^5}{5!} \\ 0 & \frac{((\frac{2}{3})h)^7}{7!} & \frac{(2(\frac{2}{3})h)^7}{7!} & \frac{(3(\frac{2}{3})h)^7}{7!} & 0 & \frac{((\frac{2}{3})h)^6}{6!} & \frac{(2(\frac{2}{3})h)^6}{6!} & \frac{(3(\frac{2}{3})h)^6}{6!} \end{pmatrix}^{-1} \begin{pmatrix} \frac{((2)\frac{2}{3}h)^2}{2!} \\ \frac{((2)\frac{2}{3}h)^{(3)}}{(3)!} \\ \frac{((2)\frac{2}{3}h)^{(4)}}{(4)!} \\ \frac{((2)\frac{2}{3}h)^{(5)}}{(5)!} \\ \frac{((2)\frac{2}{3}h)^{(6)}}{(6)!} \\ \frac{((2)\frac{2}{3}h)^{(7)}}{(7)!} \\ \frac{((2)\frac{2}{3}h)^{(8)}}{(8)!} \\ \frac{((2)\frac{2}{3}h)^{(9)}}{(9)!} \end{pmatrix},$$

$$\begin{pmatrix} \phi_{30} \\ \phi_{31} \\ \phi_{32} \\ \phi_{33} \\ \tau_{30} \\ \tau_{31} \\ \tau_{32} \\ \tau_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & (\frac{2}{3})h & 2(\frac{2}{3})h & 3(\frac{2}{3})h & 1 & 1 & 1 & 1 \\ 0 & \frac{((\frac{2}{3})h)^2}{2!} & \frac{(2(\frac{2}{3})h)^2}{2!} & \frac{(3(\frac{2}{3})h)^2}{2!} & 0 & (\frac{2}{3})h & 2(\frac{2}{3})h & 3(\frac{2}{3})h \\ 0 & \frac{((\frac{2}{3})h)^3}{3!} & \frac{(2(\frac{2}{3})h)^3}{3!} & \frac{(3(\frac{2}{3})h)^3}{3!} & 0 & \frac{((\frac{2}{3})h)^2}{2!} & \frac{(2(\frac{2}{3})h)^2}{2!} & \frac{(3(\frac{2}{3})h)^2}{2!} \\ 0 & \frac{((\frac{2}{3})h)^4}{4!} & \frac{(2(\frac{2}{3})h)^4}{4!} & \frac{(3(\frac{2}{3})h)^4}{4!} & 0 & \frac{((\frac{2}{3})h)^3}{3!} & \frac{(2(\frac{2}{3})h)^3}{3!} & \frac{(3(\frac{2}{3})h)^3}{3!} \\ 0 & \frac{((\frac{2}{3})h)^5}{5!} & \frac{(2(\frac{2}{3})h)^5}{5!} & \frac{(3(\frac{2}{3})h)^5}{5!} & 0 & \frac{((\frac{2}{3})h)^4}{4!} & \frac{(2(\frac{2}{3})h)^4}{4!} & \frac{(3(\frac{2}{3})h)^4}{4!} \\ 0 & \frac{((\frac{2}{3})h)^6}{6!} & \frac{(2(\frac{2}{3})h)^6}{6!} & \frac{(3(\frac{2}{3})h)^6}{6!} & 0 & \frac{((\frac{2}{3})h)^5}{5!} & \frac{(2(\frac{2}{3})h)^5}{5!} & \frac{(3(\frac{2}{3})h)^5}{5!} \\ 0 & \frac{((\frac{2}{3})h)^7}{7!} & \frac{(2(\frac{2}{3})h)^7}{7!} & \frac{(3(\frac{2}{3})h)^7}{7!} & 0 & \frac{((\frac{2}{3})h)^6}{6!} & \frac{(2(\frac{2}{3})h)^6}{6!} & \frac{(3(\frac{2}{3})h)^6}{6!} \end{pmatrix}^{-1} \begin{pmatrix} \frac{((3)\frac{2}{3}h)^2}{2!} \\ \frac{((3)\frac{2}{3}h)^{(3)}}{(3)!} \\ \frac{((3)\frac{2}{3}h)^{(4)}}{(4)!} \\ \frac{((3)\frac{2}{3}h)^{(5)}}{(5)!} \\ \frac{((3)\frac{2}{3}h)^{(6)}}{(6)!} \\ \frac{((3)\frac{2}{3}h)^{(7)}}{(7)!} \\ \frac{((3)\frac{2}{3}h)^{(8)}}{(8)!} \\ \frac{((3)\frac{2}{3}h)^{(9)}}{(9)!} \end{pmatrix},$$

$$
\begin{pmatrix} \omega_{10} \\ \omega_{11} \\ \omega_{12} \\ \omega_{13} \\ \varphi_{10} \\ \varphi_{11} \\ \varphi_{12} \\ \varphi_{13} \end{pmatrix}
=
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & (\frac{2}{3})h & 2(\frac{2}{3})h & 3(\frac{2}{3})h & 1 & 1 & 1 & 1 \\
0 & \frac{((\frac{2}{3})h)^2}{2!} & \frac{(2(\frac{2}{3})h)^2}{2!} & \frac{(3(\frac{2}{3})h)^2}{2!} & 0 & (\frac{2}{3})h & 2(\frac{2}{3})h & 3(\frac{2}{3})h \\
0 & \frac{((\frac{2}{3})h)^3}{3!} & \frac{(2(\frac{2}{3})h)^3}{3!} & \frac{(3(\frac{2}{3})h)^3}{3!} & 0 & \frac{((\frac{2}{3})h)^2}{2!} & \frac{(2(\frac{2}{3})h)^2}{2!} & \frac{(3(\frac{2}{3})h)^2}{2!} \\
0 & \frac{((\frac{2}{3})h)^4}{4!} & \frac{(2(\frac{2}{3})h)^4}{4!} & \frac{(3(\frac{2}{3})h)^4}{4!} & 0 & \frac{((\frac{2}{3})h)^3}{3!} & \frac{(2(\frac{2}{3})h)^3}{3!} & \frac{(3(\frac{2}{3})h)^3}{3!} \\
0 & \frac{((\frac{2}{3})h)^5}{5!} & \frac{(2(\frac{2}{3})h)^5}{5!} & \frac{(3(\frac{2}{3})h)^5}{5!} & 0 & \frac{((\frac{2}{3})h)^4}{4!} & \frac{(2(\frac{2}{3})h)^4}{4!} & \frac{(3(\frac{2}{3})h)^4}{4!} \\
0 & \frac{((\frac{2}{3})h)^6}{6!} & \frac{(2(\frac{2}{3})h)^6}{6!} & \frac{(3(\frac{2}{3})h)^6}{6!} & 0 & \frac{((\frac{2}{3})h)^5}{5!} & \frac{(2(\frac{2}{3})h)^5}{5!} & \frac{(3(\frac{2}{3})h)^5}{5!} \\
0 & \frac{((\frac{2}{3})h)^7}{7!} & \frac{(2(\frac{2}{3})h)^7}{7!} & \frac{(3(\frac{2}{3})h)^7}{7!} & 0 & \frac{((\frac{2}{3})h)^6}{6!} & \frac{(2(\frac{2}{3})h)^6}{6!} & \frac{(3(\frac{2}{3})h)^6}{6!}
\end{pmatrix}^{-1}
\begin{pmatrix} \frac{(\frac{2}{3}h)^1}{1!} \\ \frac{(\frac{2}{3}h)^2}{2!} \\ \frac{(\frac{2}{3}h)^{(3)}}{(3)!} \\ \frac{(\frac{2}{3}h)^{(4)}}{(4)!} \\ \frac{(\frac{2}{3}h)^{(5)}}{(5)!} \\ \frac{(\frac{2}{3}h)^{(6)}}{(6)!} \\ \frac{(\frac{2}{3}h)^{(7)}}{(7)!} \\ \frac{(\frac{2}{3}h)^{(8)}}{(8)!} \end{pmatrix},
$$

$$
\begin{pmatrix} \omega_{20} \\ \omega_{21} \\ \omega_{22} \\ \omega_{23} \\ \varphi_{20} \\ \varphi_{21} \\ \varphi_{22} \\ \varphi_{23} \end{pmatrix}
=
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & (\frac{2}{3})h & 2(\frac{2}{3})h & 3(\frac{2}{3})h & 1 & 1 & 1 & 1 \\
0 & \frac{((\frac{2}{3})h)^2}{2!} & \frac{(2(\frac{2}{3})h)^2}{2!} & \frac{(3(\frac{2}{3})h)^2}{2!} & 0 & (\frac{2}{3})h & 2(\frac{2}{3})h & 3(\frac{2}{3})h \\
0 & \frac{((\frac{2}{3})h)^3}{3!} & \frac{(2(\frac{2}{3})h)^3}{3!} & \frac{(3(\frac{2}{3})h)^3}{3!} & 0 & \frac{((\frac{2}{3})h)^2}{2!} & \frac{(2(\frac{2}{3})h)^2}{2!} & \frac{(3(\frac{2}{3})h)^2}{2!} \\
0 & \frac{((\frac{2}{3})h)^4}{4!} & \frac{(2(\frac{2}{3})h)^4}{4!} & \frac{(3(\frac{2}{3})h)^4}{4!} & 0 & \frac{((\frac{2}{3})h)^3}{3!} & \frac{(2(\frac{2}{3})h)^3}{3!} & \frac{(3(\frac{2}{3})h)^3}{3!} \\
0 & \frac{((\frac{2}{3})h)^5}{5!} & \frac{(2(\frac{2}{3})h)^5}{5!} & \frac{(3(\frac{2}{3})h)^5}{5!} & 0 & \frac{((\frac{2}{3})h)^4}{4!} & \frac{(2(\frac{2}{3})h)^4}{4!} & \frac{(3(\frac{2}{3})h)^4}{4!} \\
0 & \frac{((\frac{2}{3})h)^6}{6!} & \frac{(2(\frac{2}{3})h)^6}{6!} & \frac{(3(\frac{2}{3})h)^6}{6!} & 0 & \frac{((\frac{2}{3})h)^5}{5!} & \frac{(2(\frac{2}{3})h)^5}{5!} & \frac{(3(\frac{2}{3})h)^5}{5!} \\
0 & \frac{((\frac{2}{3})h)^7}{7!} & \frac{(2(\frac{2}{3})h)^7}{7!} & \frac{(3(\frac{2}{3})h)^7}{7!} & 0 & \frac{((\frac{2}{3})h)^6}{6!} & \frac{(2(\frac{2}{3})h)^6}{6!} & \frac{(3(\frac{2}{3})h)^6}{6!}
\end{pmatrix}^{-1}
\begin{pmatrix} \frac{((2)\frac{2}{3}h)^1}{1!} \\ \frac{((2)\frac{2}{3}h)^2}{2!} \\ \frac{((2)\frac{2}{3}h)^{(3)}}{(3)!} \\ \frac{((2)\frac{2}{3}h)^{(4)}}{(4)!} \\ \frac{((2)\frac{2}{3}h)^{(5)}}{(5)!} \\ \frac{((2)\frac{2}{3}h)^{(6)}}{(6)!} \\ \frac{((2)\frac{2}{3}h)^{(7)}}{(7)!} \\ \frac{((2)\frac{2}{3}h)^{(8)}}{(8)!} \end{pmatrix},
$$

$$
\begin{pmatrix} \omega_{30} \\ \omega_{31} \\ \omega_{32} \\ \omega_{33} \\ \varphi_{30} \\ \varphi_{31} \\ \varphi_{32} \\ \varphi_{33} \end{pmatrix}
=
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & (\frac{2}{3})h & 2(\frac{2}{3})h & 3(\frac{2}{3})h & 1 & 1 & 1 & 1 \\
0 & \frac{((\frac{2}{3})h)^2}{2!} & \frac{(2(\frac{2}{3})h)^2}{2!} & \frac{(3(\frac{2}{3})h)^2}{2!} & 0 & (\frac{2}{3})h & 2(\frac{2}{3})h & 3(\frac{2}{3})h \\
0 & \frac{((\frac{2}{3})h)^3}{3!} & \frac{(2(\frac{2}{3})h)^3}{3!} & \frac{(3(\frac{2}{3})h)^3}{3!} & 0 & \frac{((\frac{2}{3})h)^2}{2!} & \frac{(2(\frac{2}{3})h)^2}{2!} & \frac{(3(\frac{2}{3})h)^2}{2!} \\
0 & \frac{((\frac{2}{3})h)^4}{4!} & \frac{(2(\frac{2}{3})h)^4}{4!} & \frac{(3(\frac{2}{3})h)^4}{4!} & 0 & \frac{((\frac{2}{3})h)^3}{3!} & \frac{(2(\frac{2}{3})h)^3}{3!} & \frac{(3(\frac{2}{3})h)^3}{3!} \\
0 & \frac{((\frac{2}{3})h)^5}{5!} & \frac{(2(\frac{2}{3})h)^5}{5!} & \frac{(3(\frac{2}{3})h)^5}{5!} & 0 & \frac{((\frac{2}{3})h)^4}{4!} & \frac{(2(\frac{2}{3})h)^4}{4!} & \frac{(3(\frac{2}{3})h)^4}{4!} \\
0 & \frac{((\frac{2}{3})h)^6}{6!} & \frac{(2(\frac{2}{3})h)^6}{6!} & \frac{(3(\frac{2}{3})h)^6}{6!} & 0 & \frac{((\frac{2}{3})h)^5}{5!} & \frac{(2(\frac{2}{3})h)^5}{5!} & \frac{(3(\frac{2}{3})h)^5}{5!} \\
0 & \frac{((\frac{2}{3})h)^7}{7!} & \frac{(2(\frac{2}{3})h)^7}{7!} & \frac{(3(\frac{2}{3})h)^7}{7!} & 0 & \frac{((\frac{2}{3})h)^6}{6!} & \frac{(2(\frac{2}{3})h)^6}{6!} & \frac{(3(\frac{2}{3})h)^6}{6!}
\end{pmatrix}^{-1}
\begin{pmatrix} \frac{((3)\frac{2}{3}h)^1}{1!} \\ \frac{((3)\frac{2}{3}h)^2}{2!} \\ \frac{((3)\frac{2}{3}h)^{(3)}}{(3)!} \\ \frac{((3)\frac{2}{3}h)^{(4)}}{(4)!} \\ \frac{((3)\frac{2}{3}h)^{(5)}}{(5)!} \\ \frac{((3)\frac{2}{3}h)^{(6)}}{(6)!} \\ \frac{((3)\frac{2}{3}h)^{(7)}}{(7)!} \\ \frac{((3)\frac{2}{3}h)^{(8)}}{(8)!} \end{pmatrix}.
$$