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ON NEW FURI-PERA TYPE FIXED POINT THEOREMS IN BANACH ALGEBRAS

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Abstract

In this paper, we establish new Furi-Pera type fixed point theorems for the sum and the product of four nonlinear operators in Banach algebras; two of the operators are $\mathcal{D}$-Lipschitzian and the other two are completely continuous. Our results extend and improve some existing known ones.

1. Introduction

It is known that in many areas of natural sciences, mathematical physics, mechanics and population dynamics, problems are modeled by mathematical equations which may be reduced to perturbed nonlinear equations of the form:

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$$Ax + Bx = x, \quad x \in M,$$

where M is closed, convex subset of a Banach space X, and A, B are two nonlinear operators. A useful tool to deal with such problems is the celebrated fixed point theorem due to Krasnozelskii (see [15, 16]).

Definition 1.1. Let E be a Banach space and $f : E \to E$ be a mapping. Then f is said to be *compact* if $f(E)$ is compact. It is called *totally bounded* whenever $f(K)$ is relatively compact for any subset K of E, and f completely continuous if it is continuous and totally bounded.

Theorem 1.1. *Let K be a nonempty closed convex subset of a Banach space X and $A, B : K \to X$ be two maps such that*

(i) *A is compact and continuous*,

(ii) *B is a contraction*,

(iii) $Ax + By \in K$, *for all* $x, y \in K$.

Then $Ax + Bx = x$ has at least one fixed point in K.

The proof of Theorem 1.1 combines the metric Banach contraction mapping principle both with the topological Schauder's fixed point theorem (see [1, 7, 17, 19]) and uses the fact that if E is a linear vector space, $F \subset E$ a nonempty subset and $g : F \to E$ a contraction, then the mapping $I - g :$ $F \to (I - g)(F)$ is a homeomorphism. In 1998, Burton [6] showed that the Krasnozelskii fixed point theorem remained valid if condition (iii) is replaced by the following less restriction one:

$$x = Ax + By \Rightarrow x \in M, \quad \forall x \in M.$$

However, the study of some integral equations involving the product of operators rather than the sum may be considered only in the framework of Banach algebras for which Dhage proved in [8] the following theorem.

Theorem 1.2 [8]**.** *Let S be a closed, convex and bounded subset of a Banach algebra X and let $A, B : S \to S$ be two operators such that*

(i) *A is Lipschitzian with a Lipschitz constant* α,

(ii) $\left(\frac{I}{A}\right)^{-1}$ *exists on* $B(S)$, *where* I *is an identity operator and the operator* $\frac{I}{A} : X \to X$ *is defined by* $\left(\frac{I}{A}\right)(x) = \frac{x}{Ax}$,

(iii) *B is completely continuous*,

(iv) $AxBy \in S$, *for all* $x, y \in S$.

Then the operator equation $x = AxBx$ *has a solution*, *whenever* $\alpha\mathcal{M} < 1$, *where* $\mathcal{M} := \| B(S) \| = \sup\{\| Bx \| : x \in S\}$.

Note that $\left(\frac{I}{A}\right)^{-1}$ exists if the operator $\frac{I}{A}$ is well defined and is one-to-one. In [9], Dhage improved Theorem 1.2 by removing the restrictive condition (ii). The proof of the improved theorem involved the measure of noncompactness theory. Also, the assumption stating that *A* is Lipschitzian is extended to $\mathcal{D}$-Lipschitzian mappings. Finally, assumption (iv) is weakened to Burton's relaxed condition.

Definition 1.2. A mapping $T : X \to X$ is called $\mathcal{D}$-*Lipschitzian* with $\mathcal{D}$-function ϕ_T if there exists a continuous nondecreasing function $\phi_T : \mathfrak{R}^+ \to \mathfrak{R}^+$ such that $\phi_T(0) = 0$ and

$$\| Tx - Ty \| \leq \phi_T(\| x - y \|), \quad \forall x, y \in X.$$

Moreover, if $\phi_T(r) < r$, $\forall r > 0$, then *T* is called a *nonlinear contraction*. In particular, if $\phi_T(r) = kr$ for some constant $0 < k < 1$, then T is a contraction. *T* is said to be *non-expansive* if $\phi_T(r) = r$, that is

$$\| Tx - Ty \| \leq \| x - y \|, \quad \forall x, y \in X.$$

Lemma 1.1 [13]**.** *Every* $\mathcal{D}$-*Lipschitzian mapping A is bounded*, *i.e. maps bounded sets into bounded sets*.

Next, we state three basic existence results.

Theorem 1.3 [8, Theorem 2.1]**.** *Let S be a closed, convex and bounded subset of a Banach algebra X and let $A, B : S \to X$ be two operators such that*

(i) *A is $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function ϕ_A,*

(ii) *B is completely continuous,*

(iii) $x = AxBy \Rightarrow x \in S$ *for all* $y \in S$.

Then the operator equation $x = AxBx$ has a solution, whenever $\mathcal{M}\phi_A(r) < r$, $r > 0$, where $\mathcal{M} := \| B(S) \|$.

The idea of extending contractions to nonlinear contractions comes from Boyd and Wong [5], and we can see that this theorem generalizes the Banach fixed point principle (see, e.g., [19]).

Theorem 1.4 [5]**.** *Let E be a Banach space and $T : E \to E$ be a nonlinear contraction. Then T has a unique fixed point in E.*

In practice, condition (iii) in Theorem 1.3 is not easy to come by as it is the case in Schauder's fixed point theorem where a compact mapping is asked to map a ball into itself. In 1987, Furi and Pera [14] introduced a new condition instead and proved the following fixed point theorem in the general framework of Frechet spaces:

Theorem 1.5 [14]**.** *Let E be a Frechet space, Q a closed convex subset of E with $0 \in Q$, and let $F : Q \to E$ be a continuous compact mapping. Assume further that*

$$(\mathcal{FP})\begin{cases} \textit{if } \{(x_j, \lambda_j)\}_{j \geq 1} \textit{ is a sequence in } \partial Q \times [0, 1] \\ \textit{converging to } (x, \lambda) \textit{ with } x = \lambda F(x) \textit{ and } 0 \leq \lambda < 1, \\ \textit{then } \lambda_j F(x_j) \in Q \textit{ for } j \textit{ sufficiently large.} \end{cases}$$

Then F has a fixed point in Q.

Recently, Djebali and Hammache [13] proved the existence theorems of

Dhage type with the condition (iii) in Theorem 1.3 replaced by the Furi-Pera condition $(\mathcal{FP})$. More precisely, they considered mappings of the form $F = AB + C$, where B is completely continuous and A, C are $\mathcal{D}$-Lipschitzian while F satisfies the Furi-Pera condition. In this paper, we aim to establish and prove new Furi-Pera type fixed point theorems involving four operators in Banach algebras in a more general setting. More precisely, we investigate the nonlinear operator equations of the form:

$$x = AxBx + AxDx + Cx$$

in Banach algebras.

2. Preliminaries

Definition 2.1. Let E be a Banach space and $\mathcal{B} \subset \mathcal{P}(E)$ be the set of bounded subsets of E. For any subset $A \in \mathcal{B}$, define $\alpha(A) = \inf D$, where

$$D = \left\{ \varepsilon > 0 : A \subset \bigcup_{i=1}^{n} A_i,\ diam(A_i) \leq \varepsilon,\ \forall i = 1, \dots, n \right\}.$$

α is called the *Kuratowski measure of noncompactness*, α-MNC for short.

Hereafter, we gather together its main properties. Fore more details, we refer the readers to [3, 4, 7].

Proposition 2.1. *For any* $A, B \in \mathcal{B}$, *we have*

(i) $0 \leq \alpha(A) \leq diam(A)$,

(ii) $A \subseteq B \Rightarrow \alpha(A) \leq \alpha(B)$ (*i.e.*, α *is nondecreasing*),

(iii) $\alpha(A \cup B) = \max(\alpha(A), \alpha(B))$,

(iv) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$ (*i.e.*, α *is lower-additive*),

(v) $\alpha(convA) = \alpha(\overline{A}) = \alpha(A)$,

(vi) $\alpha(A) = 0 \Rightarrow A$ *is relatively compact*.

Definition 2.2. Let E_1, E_2 be two Banach spaces and $T : E_1 \to E_2$ be a continuous mapping which maps bounded subsets of E_1 into bounded subsets of E_2.

(i) T is called α-*Lipschitz* if there exists some $k \geq 0$ such that

$$\alpha(T(K)) \leq k\alpha(K),$$

for any bounded subset $K \subset E_1$.

(ii) T is a *strictly* α-*contraction* if $k < 1$.

(iii) T is said to be α-*condensing* whenever

$$\alpha(T(K)) \leq \alpha(K),$$

for any bounded subset $K \subset E_1$ with $\alpha(K) \neq 0$.

The following two theorems extend Theorem 1.5 to α-condensing and α-Lipschitz maps in Banach spaces, respectively. We state them without proofs (for the proofs, we refer the readers to [17]). We will use them later to develop our main results.

Theorem 2.2. *Let E be a Banach space and Q be a closed convex bounded subset of E with* $0 \in Q$. *In addition, assume that* $F : Q \to E$ *is an* α-*condensing map which satisfies the Furi-Pera condition. Then F has a fixed point* $x \in Q$.

Theorem 2.3. *Let E be a Banach space and Q be a closed convex bounded subset of E with* $0 \in Q$. *In addition, assume that* $(I - F)(S)$ *is closed*, $F : Q \to E$ *is an* α-*Lipschitz map with* $k = 1$ *and satisfies the Furi-Pera condition. Then F has a fixed point* $x \in Q$.

3. Main Results

We state the following main results of this paper.

Theorem 3.1. *Let S be a closed convex and bounded subset of a Banach algebra X with* $0 \in S$, *and let* $A, C : X \to X$ *and* $B, D : S \to X$ *be four* *operators such that*

(i) A, C *are* $\mathcal{D}$*-Lipschitzian with* $\mathcal{D}$*-functions* ϕ_A *and* ϕ_C *respectively*,

(ii) B, D *are completely continuous*,

(iii) *the operator* $F : S \to X$ *defined by*

$$F(x) = AxBx + AxDx + Cx$$

satisfies the Furi-Pera condition $(\mathcal{FP})$.

Then the abstract equation $x = AxBx + AxDx + Cx$ *has a solution* $x \in S$ *whenever*

$$\mathcal{M}_1\phi_A(r) + \mathcal{M}_2\phi_A(r) + \phi_C(r) < r, \quad \forall r > 0, \tag{3.1}$$

where $\mathcal{M}_1 = \| B(S)\|$, $\mathcal{M}_2 = \| D(S)\|$.

To prove Theorem 3.1, we need to prove the following Lemma 3.1.

Lemma 3.1. *Under assumptions* (i), (ii) *of Theorem* 3.1 *together with the inequality* (3.1), *the map* $F : S \to X$ *defined by* $F(x) = AxBx + AxDx + Cx$ *is* α*-condensing*.

Proof. Let $K \subset S$ be a bounded subset and $\delta > 0$. There exists a covering $(K_i)_{i=1}^n$ such that $K \subset \bigcup_{i=1}^n K_i$ and $diam(K_i) \leq \alpha(K) + \delta$, for each $i = 1, \ldots, n$. For each $i \in \{1, \ldots, n\}$, let $x_1^i = x_1$, $x_2^i = x_2 \in K_i$ and $E_i = F(K_i)$. Clearly $F(K) \subset \bigcup_{i=1}^n E_i$. In addition, we have

$$\begin{aligned}
&\| F(x_1) - F(x_2)\| \\
&\leq \| Ax_1\| \| Bx_1 - Bx_2\| + \| Bx_2\| \| Ax_1 - Ax_2\| + \| Ax_1\| \| Dx_1 - Dx_2\| \\
&\quad + \| Dx_2\| \| Ax_1 - Ax_2\| + \| Cx_1 - Cx_2\| \\
&\leq \| Ax_1\| \| diam(B(K_i)) + \mathcal{M}_1\phi_A(\| x_1 - x_2\|) + \| Ax_1\| \| diam(D(K_i)) \\
&\quad + \mathcal{M}_2\phi_A(\| x_1 - x_2\|) + \phi_C(\| x_1 - x_2\|)
\end{aligned}$$

$$\leq \| Ax_1 \| \| \alpha(B(K_i)) + \mathcal{M}_1\phi_A(\| x_1 - x_2 \|) + \| Ax_1 \| \|\alpha(D(K_i))$$

$$+ \mathcal{M}_2\phi_A(\| x_1 - x_2 \|) + \phi_C(\| x_1 - x_2 \|).$$

Since B, D are completely continuous, $\alpha(B(K_i)) = 0$ and $\alpha(D(K_i)) = 0$, for each $i \in \{1, ..., n\}$ follows from Proposition 2.1, it follows that

$$\| F(x_1) - F(x_2) \|$$

$$\leq M_1\phi_A(\| x_1 - x_2 \|) + M_2\phi_A(\| x_1 - x_2 \|) + \phi_C(\| x_1 - x_2 \|).$$

Since ϕ_A and ϕ_C are nondecreasing, it follows that

$$diamE_i \leq \mathcal{M}_1\phi_A(\| x_1 - x_2 \|) + \mathcal{M}_2\phi_A(\| x_1 - x_2 \|) + \phi_C(\| x_1 - x_2 \|)$$

$$\leq \mathcal{M}_1\phi_A(diam(K_i)) + \mathcal{M}_2\phi_A(diam(K_i)) + \phi_C(diam(K_i))$$

$$\leq \mathcal{M}_1\phi_A(\alpha(K) + \delta) + \mathcal{M}_2\phi_A(\alpha(K) + \delta) + \phi_C(\alpha(K) + \delta).$$

Therefore

$$\alpha(F(K)) \leq \mathcal{M}_1\phi_A(\alpha(K) + \delta) + \mathcal{M}_2\phi_A(\alpha(K) + \delta) + \phi_C(\alpha(K) + \delta).$$

Since $\delta > 0$ is arbitrary, we deduce that

$$\alpha(F(K)) \leq \mathcal{M}_1\phi_A(\alpha(K)) + \mathcal{M}_2\phi_A(\alpha(K)) + \phi_C(\alpha(K)).$$

By inequality (3.1), we obtain

$$\alpha(F(K)) \leq \alpha(K).$$

This proves our claim. □

Now, we prove Theorem 3.1.

Proof. By Lemma 3.1, the map $F : S \to X$ defined by $F(x) = AxBx + AxDx + Cx$ is α-condensing. Since F satisfies the Furi-Pera condition, it follows from Theorem 2.2 that F has at least one fixed point $x \in S$, that is

$$x = AxBx + AxDx + Cx,$$

ending the proof of the theorem. □

Corollary 3.2 [13, Theorem 3.1]**.** *Let S be a closed convex and bounded subset of a Banach algebra X with $0 \in S$, and let $A, C : X \to X$ and $B : S \to X$ be three operators such that*

(i) *A, C are $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-functions ϕ_A and ϕ_C respectively,*

(ii) *B are completely continuous,*

(iii) *the operator $F : S \to X$ defined by*

$$F(x) = AxBx + Cx$$

satisfies the Furi-Pera condition $(\mathcal{FP})$.

Then the abstract equation $x = AxBx + Cx$ has a solution $x \in S$ whenever

$$\mathcal{M}\phi_A(r) + \phi_C(r) < r, \quad \forall r > 0$$

where $\mathcal{M} = \| B(S) \|$.

Theorem 3.3. *Let S be a closed convex and bounded subset of a Banach algebra X with $0 \in S$, and let $A, C : X \to X$ and $B, D : S \to X$ be four operators such that*

(i) *A, C are $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-functions ϕ_A and ϕ_C respectively,*

(ii) *B, D are completely continuous,*

(iii) *the operator $F : S \to X$ defined by*

$$F(x) = AxBx + AxDx + Cx$$

satisfies the Furi-Pera condition $(\mathcal{FP})$.

Then the abstract equation $x = AxBx + AxDx + Cx$ has a solution $x \in S$, provided $(I - F)(S)$ is closed and the inequality

$$\mathcal{M}_1\phi_A(r) + \mathcal{M}_2\phi_A(r) + \phi_C(r) < r, \quad \forall r > 0, \tag{3.2}$$

holds, where $\mathcal{M}_1 = \| B(S) \|, \mathcal{M}_2 = \| D(S) \|$.

Proof. Since the mapping $F : S \to X$ defined by

$$F(x) = AxBx + AxDx + Cx$$

is α-condensing by Lemma 3.1, it is α-Lipschitz with $k = 1$. Moreover, $(I - F)(S)$ is closed and F satisfies the Furi-Pera condition. Therefore, Theorem 2.3 implies that F has at least one fixed point $x \in S$, that is

$$x = AxBx + AxDx + Cx. \qquad \square$$

Theorem 3.4. *Let S be a closed convex and bounded subset of a Banach algebra X with $0 \in S$, and let $A, C : X \to X$ and $B, D : S \to X$ be four operators such that*

(i) *A, C are $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-functions ϕ_A and ϕ_C respectively*,

(ii) *B, D are completely continuous*,

(iii) *the operator $N : S \to X$ defined by $Ny = x$, where x is the unique solution of the operator equation $x = AxBy + AxDy + Cx$, satisfies the Furi-Pera condition*.

Then the operator equation $x = AxBx + AxDx + Cx$ has a solution $x \in S$ provided

$$(\mathcal{H})\begin{cases} \text{the map } \Phi : [0, +\infty) \to [0, +\infty) \\ r \mapsto \Phi(r) = r - \mathcal{M}_1\phi_A(r) + \mathcal{M}_2\phi_A(r) + \phi_C(r) \\ \text{is increasing to infinity}, \end{cases}$$

where $\mathcal{M}_1 = \| B(S) \|, \mathcal{M}_2 = \| D(S) \|$.

The proof of Theorem 3.4 follows from Theorem 1.5 once we have proved the following two technical lemmas, namely Lemma 3.2 and Lemma 3.3.

Lemma 3.2. *Under assumptions of Theorem* 3.4 *the operator $N : S \to X$ introduced in* (iii) *is well defined and is bounded* (*on bounded subsets of X*).

Proof. For any $y \in S$, let the mapping A_y be defined in X by

$$A_y x = AxBy + AxDy + Cx.$$

Then, for any $x_1, x_2 \in X$,

$$\begin{aligned}
&\| A_y(x_1) - A_y(x_2) \| \\
&\le \| B_y \| \| Ax_1 - Ax_2 \| + \| D_y \| \| Ax_1 - Ax_2 \| + \| Cx_1 - Cx_2 \| \\
&\le \mathcal{M}_1 \phi_A(\| x_1 - x_2 \|) + \mathcal{M}_2 \phi_A(\| x_1 - x_2 \|) + \phi_C(\| x_1 - x_2 \|) \\
&\le \| x_1 - x_2 \|.
\end{aligned}$$

By the Boyd and Wong fixed point theorem [5], A_y has only one fixed point $x \in X$ and so the mapping N is well defined. In addition, let $K \subset X$ be any bounded subset, $y \in K$ and $x = Ny$, where x is the unique solution of the equation

$$x = AxBy + AxDy + Cx.$$

Thus

$$\begin{aligned}
\| x \| &\le \| Ax \| \| By \| + \| Ax \| \| Dy \| + \| Cx \| \\
&\le \mathcal{M}_1 \| Ax \| + \mathcal{M}_2 \| Ax \| + \| Cx \|.
\end{aligned}$$

Let $x_0 \in X$, with assumption $(\mathcal{H})$, we have the following estimates:

$$\begin{aligned}
\| x \| \le\; & \mathcal{M}_1(\| Ax - Ax_0 \| + \| Ax_0 \|) \\
& + \mathcal{M}_2(\| Ax - Ax_0 \| + \| Ax_0 \|) + \| Cx - Cx_0 \| + \| Cx_0 \| \\
\le\; & \mathcal{M}_1(\| Ax - Ax_0 \|) + \mathcal{M}_1 \| Ax_0 \| + \mathcal{M}_2(\| Ax - Ax_0 \|) \\
& + \mathcal{M}_2 \| Ax_0 \| + \| Cx - Cx_0 \| + \| Cx_0 \|.
\end{aligned}$$

Hence

$$\begin{aligned}
\| x - x_0 \| &\le \| x \| + \| x_0 \| \\
&\le \mathcal{M}_1(\| Ax - Ax_0 \|) + \mathcal{M}_1 \| Ax_0 \| + \mathcal{M}_2(\| Ax - Ax_0 \|)
\end{aligned}$$

$$+ \mathcal{M}_2 \| Ax_0 \| + \| Cx - Cx_0 \| + \| Cx_0 \| + \| x_0 \|$$

$$\leq \mathcal{M}_1 \phi_A(\| x - x_0 \|) + \mathcal{M}_2 \phi_A(\| x - x_0 \|) + \phi_C(\| x - x_0 \|)$$

$$+ \mathcal{M}_1 \| Ax_0 \| + \mathcal{M}_2 \| Ax_0 \| + \| Cx_0 \| + \| x_0 \|.$$

It follows that

$$\Phi(\| x - x_0 \|) \leq \| x - x_0 \| - \mathcal{M}_1 \phi_A(\| x - x_0 \|)$$

$$+ \mathcal{M}_2 \phi_A(\| x - x_0 \|) + \phi_C(\| x - x_0 \|)$$

$$\leq \mathcal{M}_1 \| Ax_0 \| + \mathcal{M}_2 \| Ax_0 \| + \| Cx_0 \| + \| x_0 \|.$$

This implies successively

$$\| x - x_0 \| \leq \Phi^{-1}(\mathcal{M}_1 \| Ax_0 \| + \mathcal{M}_2 \| Ax_0 \| + \| Cx_0 \| + \| x_0 \|)$$

and

$$\| x_0 \| \leq \| x - x_0 \| + \| x_0 \|$$

$$\leq \Phi^{-1}(\mathcal{M}_1 \| Ax_0 \| + \mathcal{M}_2 \| Ax_0 \| + \| Cx_0 \| + \| x_0 \|) + \| x_0 \|$$

proving our claim. □

Lemma 3.3. *Under assumptions of Theorem* 3.4, *the operator* $N : S \to X$ *introduced in the condition* (iii) *is compact*.

Proof. First, we show that N is continuous. Let $\{x_n\}$ be a sequence in S converging to some limit x. Since S is closed, $x \in S$. Moreover,

$$\| Nx_n - Nx \|$$

$$\leq \| ANx_n Bx_n - ANxBx_n \| + \| ANxBx_n - ANxBx \|$$

$$+ \| ANx_n Dx_n - ANxDx_n \| + \| ANxDx_n - ANxDx \| + \| CNx_n - CNx \|$$

$$\leq \| Bx_n \| \| ANx_n - ANx \| + \| ANx \| \| Bx_n - Bx \|$$

$$+ \| Dx_n \| \| ANx_n - ANx \| + \| ANx \| \| Dx_n - Dx \| + \| CNx_n - CNx \|$$

$$\leq \mathcal{M}_1\| ANx_n - ANx \| + \| ANx \| \| Bx_n - Bx \|$$

$$+ \mathcal{M}_2\| ANx_n - ANx \| + \| ANx \| \| Dx_n - Dx \| + \| CNx_n - CNx \|$$

$$\leq \mathcal{M}_1\phi_A(\| Nx_n - Nx \|) + \| ANx \| \| Bx_n - Bx \|$$

$$+ \mathcal{M}_2\phi_A(\| Nx_n - Nx \|) + \| ANx \| \| Dx_n - Dx \| + \phi_C(\| Nx_n - Nx \|).$$

Whence

$$\limsup_{n\to\infty}\| Nx_n - Nx \|$$

$$\leq \mathcal{M}_1\phi_A(\limsup_{n\to\infty}\| Nx_n - Nx \|) + \| ANx \|(\limsup_{n\to\infty}\| Bx_n - Bx \|)$$

$$+ \mathcal{M}_2\phi_A(\limsup_{n\to\infty}\| Nx_n - Nx \|) + \| ANx \|(\limsup_{n\to\infty}\| Dx_n - Dx \|)$$

$$+ \phi_C(\limsup_{n\to\infty}\| Nx_n - Nx \|).$$

From assumption (ii), B and D are continuous; hence

$$\limsup_{n\to\infty}\| Nx_n - Nx \| = 0$$

yielding the continuity of N.

Second, we show that N is compact. From Lemma 1.1 and Lemma 3.2, there exists a positive constant k_1 such that $\| ANx \| \leq k_1$, $\forall x \in S$. Let $\varepsilon > 0$ be given. Since S is bounded and B, D are completely continuous, $B(S)$, $D(S)$ are relatively compact. Then there exists a set $\varepsilon = \{x_1, ..., x_n\} \subset S$ such that $B(S) \subset \bigcup_i^n \mathcal{B}_{\delta_1}(w_i^1)$ and $D(S) \subset \bigcup_i^n \mathcal{B}_{\delta_2}(w_i^2)$, where $w_i^1 := B(x_i)$; $\delta_1 = k_2\varepsilon$ and $w_i^2 := D(x_i)$; $\delta_2 = k_3\varepsilon$, for some constant k_2, k_3 to be selected later on.

Therefore, for any $x \in S$, there exists some $x_i \in \varepsilon$ such that

$$0 \leq \| Bx - Bx_i \| \leq k_2\varepsilon$$

and

$$0 \le \| Dx - Dx_i \| \le k_3 \varepsilon.$$

We have that

$$\begin{aligned}
&\| Nx_i - Nx \| \\
&= \| ANx_i Bx_i + ANx_i Dx_i + CNx_i - ANxBx - ANxDx - CNx \| \\
&\le \| ANx_i Bx_i - ANxBx_i \| + \| ANxBx_i - ANxBx \| \\
&\quad + \| ANx_i Dx_i - ANxDx_i \| + \| ANxDx_i - ANxDx \| + \| CNx_i - CNx \| \\
&\le \| Bx_i \| \| ANx_i - ANx \| + \| ANx \| \| Bx_i - Bx \| \\
&\quad + \| Dx_i \| \| ANx_i - ANx \| + \| ANx \| \| Dx_i - Dx \| + \| CNx_i - CNx \| \\
&\le \mathcal{M}_1 \| ANx_i - ANx \| + \| ANx \| \| Bx_i - Bx \| \\
&\quad + \mathcal{M}_2 \| ANx_i - ANx \| + \| ANx \| \| Dx_i - Dx \| + \| CNx_i - CNx \| \\
&\le \mathcal{M}_1 \phi_A(\| Nx_i - Nx \|) + \| ANx \| \| Bx_i - Bx \| \\
&\quad + \mathcal{M}_2 \phi_A(\| Nx_i - Nx \|) + \| ANx \| \| Dx_i - Dx \| + \phi_C(\| Nx_i - Nx \|) \\
&\le \mathcal{M}_1 \phi_A(\| Nx_i - Nx \|) + k_1 k_2 \varepsilon \\
&\quad + \mathcal{M}_2 \phi_A(\| Nx_i - Nx \|) + k_1 k_3 \varepsilon + \phi_C(\| Nx_i - Nx \|).
\end{aligned}$$

Hence

$$\begin{aligned}
\Phi(\| Nx_i - Nx \|) &\le \| Nx_i - Nx \| - \mathcal{M}_1 \phi_A(\| Nx_i - Nx \|) \\
&\quad + \mathcal{M}_2 \phi_A(\| Nx_i - Nx \|) + \phi_C(\| Nx_i - Nx \|) \\
&\le k_1 k_2 \varepsilon + k_1 k_3 \varepsilon \\
&= k_1 (k_2 + k_3) \varepsilon.
\end{aligned}$$

From assumption $(\mathcal{H})$, it follows that

$$\| Nx_i - Nx \| \le \Phi^{-1}(k_1(k_2 + k_3)\varepsilon).$$

Choosing $0 < k_2 + k_3 \leq \frac{\Phi(\varepsilon)}{k_1 \varepsilon}$, we obtain

$$\| Nx_i - Nx \| \leq \varepsilon.$$

We have shown that

$$N(S) \subset \bigcup_{i=1}^{n} \mathcal{B}_{\varepsilon}(Nx_i)$$

showing that $N(S)$ is totally bounded. Since N is continuous, it is compact operator, and ending the proof of Lemma 3.3. □

We now have the following corollary.

Corollary 3.5 [13, Theorem 3.3]**.** *Let S be a closed convex and bounded subset of a Banach algebra X with $0 \in S$, and let $A : X \to X$ and $B : S \to X$ be two operators such that*

(i) *A is $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-functions ϕ_A,*

(ii) *B is completely continuous,*

(iii) *the operator $N : S \to X$ defined by $Ny = x$, where x is the unique solution of the operator equation $x = AxBy$, satisfies the Furi-Pera condition.*

Then the operator equation $x = AxBx$ has a solution $x \in S$ provided

$$(\mathcal{H}) \begin{cases} \text{the map } \Phi : [0, +\infty) \to [0, +\infty) \\ r \mapsto \Phi(r) = r - \mathcal{M}\phi_A(r) \\ \text{is increasing to infinity}, \end{cases}$$

where $\mathcal{M} = \| B(S) \|$.

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References

[1] R. P. Agarwal, M. Meehan and D. O'Regan, Fixed point theory and applications, Cambridge Tracts in Mathematics, Vol. 141, Cambridge University Press, 2001.

[2] D. Bainov and P. Simeonov, Integral Inequalities and Applications, Kluwer Academic Publishers, 1992.

[3] J. Banaś and K. Goebel, Measure of noncompactness in Banach spaces, Lecture Notes in Pure and Applied Mathematics, Vol. 57, Marcel Dekker, New York, 1980.

[4] J. Banas and Z. Knap, Measure of noncompactness and nonlinear integral equations of convolution type, J. Math. Anal. 146 (1990), 353-362.

[5] D. W. Boyd and J. S. W. Wong, On nonlinear contractions, Proc. Amer. Math. Soc. 20 (1969), 458-464.

[6] T. A. Burton, A fixed point theorem of Krasnozelskii, Appl. Math. Lett. 11 (1998), 58-88.

[7] K. Deimling, Nonlinear Functional Analysis, Springer Verlag, Berlin, Tokyo, 1985.

[8] B. C. Dhage, On some variants of Schauder's fixed point principle and applications to nonlinear integral equations, J. Math. Phys. Sci. 2 (1988), 603-611.

[9] B. C. Dhage, On a fixed point theorem in Banach algebras with applications, Appl. Math. Lett. 18 (2005), 273-280.

[10] B. C. Dhage, Some nonlinear alternatives in Banach algebras with applications I, Nonlinear Studies 12(3) (2005), 271-281.

[11] B. C. Dhage and S. K. Ntouyas Existence results for nonlinear functional integral equations via a fixed point theorem of Krasnozelskii-Schaefer type, Nonlinear Studies 9(3) (2002), 307-317.

[12] B. Dhage and D. O'Regan, A fixed point theorem in Banach algebras with applications to functional integral equations, Funct. Differ. Equ. 7(3-4) (2000), 259-267.

[13] S. Djebali and K. Hammache, Furi-Pera fixed point theorems in Banach algebras with applications, Acta Univ. Palacki. Olomuc., Fac. Rer. Nat. Math. 47 (2008), 55-75.

[14] M. Furi and P. Pera, A continuation method on locally convex spaces and applications to ordinary differential equations on noncompact intervals, Ann. Polon. Math. 47 (1987), 331-346.

[15] M. A. Krasnozelskii, Integral Operators in Space of Summable Functions, Noordhoff, Leyden, 1976.

[16] M. A. Krasnozelskii, Positive Solutions of Operators Equations, Noordhoff, Groningen, 1964.

[17] D. O'Regan, Fixed-point theory for the sum of two operators, Appl. Math. Lett. 9 (1996), 1-8.

[18] D. R. Smart, Fixed Point Theorems, Cambridge University Press, 1974.

[19] E. Zeidler, Nonlinear Functional Analysis and its Applications, Vol. I: Fixed Point Theorems, Springer Verlag, New York, 1986.