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TRANSIENT ANALYSIS OF AN $M/M/1$ QUEUE WITH A SINGLE WORKING VACATION AND MULTIPLE NONWORKING VACATIONS

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Abstract

An $M/M/1$ queue is considered subject to a policy of single working vacation and multiple nonworking vacations. To be specific, the server has 3 phases: (i) active phase (normal busy period), (ii) slow phase (working vacation period), and (iii) nonworking vacation (full vacation). Customers arrive according to a Poisson process with rate λ in all phases. During the normal busy period and working vacation period, the service rates are, respectively, μ_b and μ_v such that $\mu_v < \mu_b$. When the server is in active phase serving a customer and no further customer is available for service after completion of the present service, the server immediately enters into a working vacation. We assume that the working vacation period has an exponential

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distribution with parameter θ. If, at the expiry of working vacation, no further customer is available in the system, the server begins a full vacation immediately. A full vacation period is exponentially distributed with parameter α. On the other hand, if at the expiry of working vacation, at least one customer is available in the system, the server switches to the active mode. At the expiry of a full vacation, if the server finds no customer in the system, the server takes another full vacation. On the other hand, if at the expiry of a full vacation, at least one customer is available in the system, the server switches to the active mode. Identifying a Markov process, time-dependent state probability distribution of the queueing system is explicitly derived and the steady-state results are deduced.

1. Introduction

Several vacation queue models have been proposed and investigated very extensively for their immediate real-time applications in telecommunication systems and computer networks. In most of the vacation queue models, the server goes on vacation whenever the system becomes empty and resumes work only when at least one customer is present in the system. A huge variety of vacation policies such as single vacation and multiple vacations have been introduced with the aim to optimize the working period of the server. The monographs of Takagi [6] and Tian and Zhang [7] provide excellent and elaborate treatment of such vacation queue models. Servi and Finn [5] have generalized the classical vacation model by introducing a policy called working vacation. They studied an $M/M/1$ queue with multiple working vacations and obtained the probability generating function of the number of customers in the system and the Laplace-Stieltjes transform of the waiting time distribution. Further, their model is a multi-queue system with two speeds mode where they achieved a better performance analysis of a gateway router in fiber communication networks. In working vacation period, the server works at a lower rate. If, at the expiry of a working vacation, no further customer is available in the system, the server begins a full vacation immediately. On the other hand, if at the expiry of working vacation, at least one customer is available in the system, the server switches to the active mode. Liu et al. [3] have considered the working vacation policy of Servi and Finn [5] and obtained simple explicit expressions for the stationary queue length distribution and waiting time. Recently Xu and Tian [10] have investigated an $M/M/1$ working vacation queue with setup times (full vacation). They have used a matrix-solution method (see, for example, Latouche and Ramaswamy [2] and Neuts [4]) and derived the steady-state distribution of the system length. Several authors (for example, see van Doorn [8] and Whitt [9]) have observed that time-dependent analysis is very much essential in several applications of queueing theory. However, time-dependent solutions are not easy to obtain in many situations. In the present paper, we obtain the time-dependent solution for the model of Xu and Tian [10] and deduce their steady-state results.

The organization of the paper is as follows. Section 2 describes the model of the working vacation server with nonworking vacations. In Section 3, we obtain the governing equations for the time-dependent probabilities of the system. In Section 4, we explicitly obtain the time-dependent expressions for the state probabilities of the system. In Section 5, we deduce the stationary results of Xu and Tian [10].

2. Model Description

Xu and Tian [10] studied an $M/M/1$ queue subject to a policy of single working vacation and multiple nonworking vacations. To be specific, the server has 3 phases: (i) active phase (normal busy period), (ii) slow phase (working vacation period), and (iii) nonworking vacation (full vacation). Customers arrive according to a Poisson process with rate λ in all phases. During the normal busy period and working vacation period, the service rates are, respectively, μ_b and μ_v such that $\mu_v < \mu_b$. When the server is in active phase serving a customer and no further customer is available for service after completion of the present service, the server immediately enters into a working vacation. We assume that the working vacation period has an exponential distribution with parameter θ. If, at the expiry of working vacation, no further customer is available in the system, the server begins a full vacation immediately. A full vacation period is exponentially distributed with parameter α. On the other hand, if at the expiry of working vacation, at least one customer is available in the system, the server switches to the active mode. At the expiry of a full vacation, if the server finds no customer in the system, the server takes another full vacation. On the other hand, if at the expiry of a full vacation, at least one customer is available in the system, the server switches to the active mode. For this model, we identify a Markov process and explicitly obtain the time-dependent state probability distribution in the next section.

3. Governing Equations

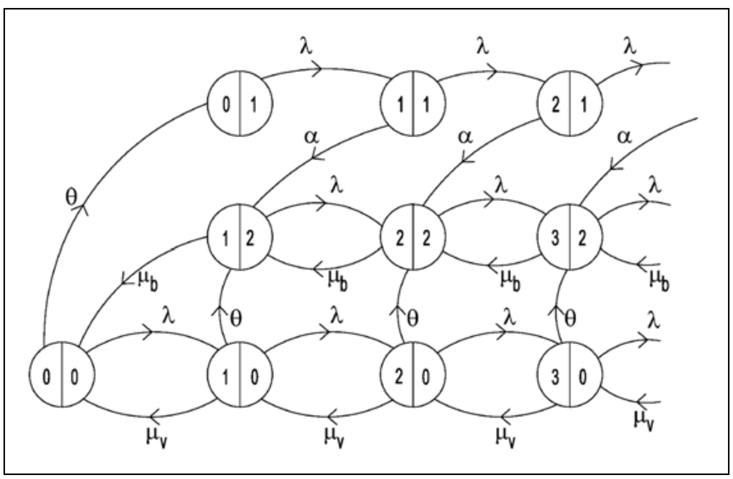
Let $J(t)$ denote the state of the system at time *t*. We set

$$J(t) = \begin{cases} 0 & \text{when the server is in working vacation phase,} \\ 1 & \text{when the server is in the multiple vacation phase,} \\ 2 & \text{when the server is in regular busy period (active phase).} \end{cases}$$

Let $Q(t)$ be the number of customers in the system at time *t*. Then the joint process $\{Q(t),\ J(t) : t \geq 0\}$ is a Markov process. The state-space is given by

$$\{(0,\ 0),\ (0,\ 1)\} \cup \{(k,\ j) : k = 1,\ 2,\ \ldots;\ j = 0,\ 1,\ 2\}.$$

The transition diagram is given below:



At time $t = 0$, we assume that there are no customers in the system and the server is in the working vacation. Then $Q(0) = 0$ and $J(0) = 0$. We define

$$p(i, j, t) = Pr[Q(t) = i, J(t) = j], \tag{3.1}$$

where Pr denotes probability measure. Let the symbol © denote the convolution operator and we follow the notation

$$f(t) © g(t) = \int_0^t f(u) g(t-u) du.$$

Using renewal-type arguments, we obtain

Case 1. $i = 0$ **and** $j = 0$

$$p(0, 0, t) = e^{-(\lambda+\theta)t} + [p(1, 2, t)\mu_b + p(1, 0, t)\mu_v] © e^{-(\lambda+\theta)t}. \tag{3.2}$$

Case 2. $i \geq 1$ **and** $j = 0$

$$p(i, 0, t) = [p(i-1, 0, t)\lambda + p(i+1, 0, t)\mu_v] © e^{-(\lambda+\mu_v+\theta)t}. \tag{3.3}$$

Case 3. $i = 0$ **and** $j = 1$

$$p(0, 1, t) = p(0, 0, t)\theta © e^{-\lambda t}. \tag{3.4}$$

Case 4. $i \geq 1$ **and** $j = 1$

$$p(i, 1, t) = p(i-1, 1, t)\lambda © e^{-(\lambda+\alpha)t}. \tag{3.5}$$

Case 5. $i = 1$ **and** $j = 2$

$$p(1, 2, t) = [p(1, 0, t)\theta + p(2, 2, t)\mu_b + p(1, 1, t)\alpha] © e^{-(\lambda+\mu_b)t}. \tag{3.6}$$

Case 6. $i \geq 2$ **and** $j = 2$

$$p(i, 2, t) = [p(i, 0, t)\theta + p(i-1, 2, t)\lambda$$

$$+ p(i+1, 2, t)\mu_b + p(i, 1, t)\alpha] © e^{-(\lambda+\mu_b)t}. \tag{3.7}$$

Using (3.2) to (3.7), we proceed to obtain $p(i, j, t)$.

4. Transient Analysis

We denote the Laplace transform of $p(i, j, t)$ by

$$\hat{p}(i, j, s) = \int_0^\infty e^{-st} p(i, j, t) dt. \tag{4.1}$$

Using (3.2) to (3.7), we get

$$(s + \lambda + \theta)\hat{p}(0, 0, s) = 1 + \hat{p}(1, 2, s)\mu_b + \hat{p}(1, 0, s)\mu_v, \tag{4.2}$$

$$(s + \lambda + \mu_v + \theta)\hat{p}(i, 0, s) = \hat{p}(i-1, 0, s)\lambda + \hat{p}(i+1, 0, s)\mu_v, \quad i \geq 1, \tag{4.3}$$

$$(s + \lambda)\hat{p}(0, 1, s) = \hat{p}(0, 0, s)\theta, \tag{4.4}$$

$$(s + \lambda + \alpha)\hat{p}(i, 1, s) = \hat{p}(i-1, 1, s)\lambda, \quad i \geq 1, \tag{4.5}$$

$$(s + \lambda + \mu_b)\hat{p}(1, 2, s) = \hat{p}(1, 0, s)\theta + \hat{p}(2, 2, s)\mu_b + \hat{p}(1, 1, s)\alpha, \tag{4.6}$$

$$\begin{aligned}(s + \lambda + \mu_b)\hat{p}(i, 2, s) &= \hat{p}(i, 0, s)\theta + \hat{p}(i+1, 2, s)\mu_b \\ &+ \hat{p}(i, 1, s)\alpha + \hat{p}(i-1, 2, s)\lambda, \quad i \geq 2.\end{aligned} \tag{4.7}$$

We define

$$\hat{G}_j(u, s) = \sum_{i=0}^{\infty} \hat{p}(i, j, s)u^i, \quad j = 0, 1, \tag{4.8}$$

$$\hat{G}_2(u, s) = \sum_{i=1}^{\infty} \hat{p}(i, 2, s)u^i. \tag{4.9}$$

Using (4.5), we get

$$\hat{G}_1(u, s) = \frac{(s + \lambda + \alpha)\hat{p}(0, 1, s)}{\{s + \lambda(1-u) + \alpha\}}. \tag{4.10}$$

From (4.4), we get

$$\hat{p}(0, 1, s) = \frac{\hat{p}(0, 0, s)\theta}{(s + \lambda)}. \tag{4.11}$$

Plugging (4.11) into (4.10), we get

$$\hat{G}_1(u, s) = \frac{(s+\lambda+\alpha)\theta\hat{p}(0, 0, s)}{(s+\lambda)\{s+\lambda(1-u)+\alpha\}}$$

$$= \frac{\theta}{s+\lambda}\sum_{i=0}^{\infty}\left(\frac{\lambda}{s+\lambda+\alpha}\right)^i u^i \hat{p}(0, 0, s). \tag{4.12}$$

From (4.12), we get

$$\hat{p}(i, 1, s) = \frac{\theta}{s+\lambda}\left(\frac{\lambda}{s+\lambda+\alpha}\right)^i \hat{p}(0, 0, s), \quad i \geq 1. \tag{4.13}$$

From (4.3), we get

$$\hat{G}_0(u, s) = \frac{\{(s+\lambda+\theta)u+\mu_v(u-1)\}\hat{p}(0, 0, s) - \mu_v u\hat{p}(1, 0, s)}{(s+\lambda+\mu_v+\theta)u - \lambda u^2 - \mu_v}. \tag{4.14}$$

The denominator of $\hat{G}_0(u, s)$ vanishes at the roots of the quadratic equation

$$(s+\lambda+\mu_v+\theta)u - \lambda u^2 - \mu_v = 0. \tag{4.15}$$

The roots of (4.15) are given by

$$\hat{r}_{0,1} = \frac{(s+\lambda+\mu_v+\theta) - \sqrt{(s+\lambda+\mu_v+\theta)^2 - 4\lambda\mu_v}}{2\lambda}, \tag{4.16}$$

$$\hat{r}_{0,2} = \frac{(s+\lambda+\mu_v+\theta) + \sqrt{(s+\lambda+\mu_v+\theta)^2 - 4\lambda\mu_v}}{2\lambda}. \tag{4.17}$$

The roots (4.16) and (4.17) satisfy the conditions

$$\lambda(u - \hat{r}_{0,1})(\hat{r}_{0,2} - u) = (s+\lambda+\mu_v+\theta)u - \lambda u^2 - \mu_v, \tag{4.18}$$

$$\hat{r}_{0,1} + \hat{r}_{0,2} = \frac{s+\lambda+\mu_v+\theta}{\lambda}, \tag{4.19}$$

$$\hat{r}_{0,1}\hat{r}_{0,2} = \frac{\mu_v}{\lambda}, \tag{4.20}$$

$$|\hat{r}_{0,1}| < 1, \quad |\hat{r}_{0,2}| > 1. \tag{4.21}$$

Due to the analyticity of $\hat{G}_0(u, s)$ inside $|u| < 1$, we get

$$\{(s + \lambda + \theta)\hat{r}_{0,1} + \mu_v(\hat{r}_{0,1} - 1)\}\,\hat{p}(0, 0, s) - \mu_v\hat{r}_{0,1}\hat{p}(1, 0, s) = 0. \quad (4.22)$$

From (4.22), we get

$$\hat{p}(1, 0, s) = \frac{(s + \lambda + \theta)\hat{r}_{0,1} + \mu_v(\hat{r}_{0,1} - 1)}{\mu_v\hat{r}_{0,1}}\,\hat{p}(0, 0, s). \quad (4.23)$$

Plugging (4.23) into (4.14), we get

$$\hat{G}_0(u, s) = \frac{\hat{r}_{0,2}}{\hat{r}_{0,2} - u}\,\hat{p}(0, 0, s) = \sum_{i=0}^{\infty}\left(\frac{\lambda}{\mu_v}\right)^i \hat{r}_{0,1}^i u^i\,\hat{p}(0, 0, s). \quad (4.24)$$

From (4.24), we get

$$\hat{p}(i, 0, s) = \left(\frac{\lambda}{\mu_v}\right)^i \hat{r}_{0,1}^i\hat{p}(0, 0, s), \quad i \geq 1. \quad (4.25)$$

Multiplying both sides of (4.7) by u^i and summing from 2 to ∞, we get

$$\left[(s + \lambda + \mu_b) - \lambda u - \frac{\mu_b}{u}\right]\hat{G}_2(u, s)$$

$$= (s + \lambda + \mu_b)u\hat{p}(1, 2, s) - \theta\hat{p}(0, 0, s)$$

$$- \theta\hat{p}(1, 0, s)u - \mu_b\hat{p}(1, 2, s) - \mu_b\hat{p}(2, 2, s)u$$

$$- \alpha\hat{p}(0, 1, s) - \alpha\hat{p}(1, 1, s)u + \alpha\hat{G}_1(u, s) + \theta\hat{G}_0(u, s). \quad (4.26)$$

Plugging (4.6) into (4.26) and simplifying, we get

$$\hat{G}_2(u, s) = \frac{u}{(s + \lambda + \mu_b)u - \lambda u^2 - \mu_b}[\alpha\hat{G}_1(u, s) + \theta\hat{G}_0(u, s)$$

$$- \theta\hat{p}(0, 0, s) - \mu_b\hat{p}(1, 2, s) - \alpha\hat{p}(0, 1, s)]$$

$$= \frac{u}{(s + \lambda + \mu_b)u - \lambda u^2 - \mu_b}\left[-\mu_b\hat{p}(1, 2, s) + \alpha\sum_{i=1}^{\infty}\hat{p}(i, 1, s)u^i\right.$$

$$\left. + \theta\sum_{i=1}^{\infty}\hat{p}(i, 0, s)u^i\right]. \quad (4.27)$$

The denominator of (4.27) vanishes at the roots of the quadratic equation

$$(s+\lambda+\mu_b)u-\lambda u^2-\mu_b=0. \tag{4.28}$$

The roots of (4.28) are

$$\hat{r}_{2,1}=\frac{(s+\lambda+\mu_b)-\sqrt{(s+\lambda+\mu_b)^2-4\lambda\mu_b}}{2\lambda}, \tag{4.29}$$

$$\hat{r}_{2,2}=\frac{(s+\lambda+\mu_b)+\sqrt{(s+\lambda+\mu_b)^2-4\lambda\mu_b}}{2\lambda}. \tag{4.30}$$

The above roots satisfy the following conditions:

$$|\hat{r}_{2,1}|<1, \quad |\hat{r}_{2,2}|>1, \tag{4.31}$$

$$\lambda(u-\hat{r}_{2,1})(\hat{r}_{2,2}-u)=s+\lambda+\mu_b, \tag{4.32}$$

$$\hat{r}_{2,1}\hat{r}_{2,2}=\frac{\mu_b}{\lambda}. \tag{4.33}$$

Using the analyticity of $\hat{G}_2(u,s)$ in $|u|\le 1$, we get

$$\hat{p}(1,2,s)=\frac{1}{\mu_b}\left[\alpha\sum_{i=1}^{\infty}\hat{p}(i,1,s)\hat{r}_{2,1}^{i}+\theta\sum_{i=1}^{\infty}\hat{p}(i,0,s)\hat{r}_{2,1}^{i}\right]. \tag{4.34}$$

Plugging (4.34) into (4.27) and simplifying, we get

$$\hat{G}_2(u,s)=\frac{\theta}{\mu_b}\left[\frac{\alpha}{s+\lambda}\sum_{j=1}^{\infty}\sum_{k=1}^{j}\sum_{i=k}^{\infty}\left(\frac{\lambda}{\mu_b}\right)^{j-k}\left(\frac{\lambda}{s+\lambda+\alpha}\right)^{i}\hat{r}_{2,1}^{i+j-2k}u^{j}\right.$$
$$\left.+\sum_{j=1}^{\infty}\sum_{k=1}^{j}\sum_{i=k}^{\infty}\left(\frac{\lambda}{\mu_b}\right)^{j-k}\left(\frac{\lambda}{\mu_v}\right)^{i}\hat{r}_{0,1}^{i}\hat{r}_{2,1}^{i+j-2k}u^{j}\right]\hat{p}(0,0,s). \tag{4.35}$$

From (4.35), we get

$$\hat{p}(j,2,s)=\frac{\theta}{\mu_b}\left[\frac{\alpha}{s+\lambda}\sum_{k=1}^{j}\sum_{i=k}^{\infty}\left(\frac{\lambda}{\mu_b}\right)^{j-k}\left(\frac{\lambda}{s+\lambda+\alpha}\right)^{i}\hat{r}_{2,1}^{i+j-2k}\right.$$
$$\left.+\sum_{k=1}^{j}\sum_{i=k}^{\infty}\left(\frac{\lambda}{\mu_b}\right)^{j-k}\left(\frac{\lambda}{\mu_v}\right)^{i}\hat{r}_{0,1}^{i}\hat{r}_{2,1}^{i+j-2k}\right]\hat{p}(0,0,s), \quad j\ge 1. \tag{4.36}$$

To find $\hat{p}(0, 0, s)$, we use the relation

$$\hat{G}_0(1, s) + \hat{G}_1(1, s) + \hat{G}_2(1, s) = \frac{1}{s}. \quad (4.37)$$

From (4.24), we have

$$\hat{G}_0(1, s) = \sum_{i=0}^{\infty}\left(\frac{\lambda}{\mu_v}\right)^i \hat{r}_{0,1}^i \hat{p}(0, 0, s) = \left(\frac{\mu_v}{\mu_v - \lambda\hat{r}_{0,1}}\right)\hat{p}(0, 0, s). \quad (4.38)$$

From (4.12), we have

$$\hat{G}_1(1, s) = \frac{\theta}{s+\lambda}\sum_{i=k}^{\infty}\left(\frac{\lambda}{s+\lambda+\alpha}\right)^i \hat{p}(0, 0, s)$$

$$= \left(\frac{\theta}{s+\lambda}\right)\left(\frac{s+\lambda+\alpha}{s+\alpha}\right)\hat{p}(0, 0, s). \quad (4.39)$$

From (4.35), we have

$$\hat{G}_2(1, s) = \frac{\lambda\theta\hat{r}_{2,1}}{\mu_b - \lambda\hat{r}_{2,1}}\left[\frac{\alpha(s+\lambda+\alpha)}{(s+\lambda)(s+\alpha)(s+\lambda+\alpha-\lambda\hat{r}_{2,1})}\right.$$

$$\left. + \frac{\mu_v\hat{r}_{0,1}}{(\mu_v - \lambda\hat{r}_{0,1}\hat{r}_{2,1})(\mu_v - \lambda\hat{r}_{0,1})}\right]\hat{p}(0, 0, s). \quad (4.40)$$

Consequently, (4.37) gives

$$\hat{p}(0, 0, s) = \frac{1}{s}\left[\left(\frac{\mu_v}{\mu_v - \lambda\hat{r}_{0,1}}\right) + \left(\frac{\theta}{s+\lambda}\right)\left(\frac{s+\lambda+\alpha}{s+\alpha}\right)\right.$$

$$+ \frac{\lambda\theta\alpha(s+\lambda+\alpha)\hat{r}_{2,1}}{(\mu_b - \lambda\hat{r}_{2,1})(s+\lambda)(s+\alpha)(s+\lambda+\alpha-\lambda\hat{r}_{2,1})}$$

$$\left. + \frac{\mu_v\lambda\theta\hat{r}_{2,1}\hat{r}_{0,1}}{(\mu_b - \lambda\hat{r}_{2,1})(\mu_v - \lambda\hat{r}_{0,1}\hat{r}_{2,1})(\mu_v - \lambda\hat{r}_{0,1})}\right]^{-1}. \quad (4.41)$$

Substituting (4.13), (4.25) and (4.34) in (4.2) and solving for $\hat{p}(0, 0, s)$, we get

$$\hat{p}(0,\ 0,\ s)$$

$$= \frac{1}{(s+\lambda+\theta-\lambda\hat{r}_{0,1}) - \left\{\dfrac{\lambda\alpha\theta\hat{r}_{2,1}}{(s+\lambda)(s+\lambda+\alpha-\lambda\hat{r}_{2,1})} + \dfrac{\lambda\alpha\theta\hat{r}_{0,1}\hat{r}_{2,1}}{\mu_v - \lambda\hat{r}_{0,1}\hat{r}_{2,1}}\right\}}. \tag{4.42}$$

From (4.42), we obtain

$$\hat{p}(0,\ 0,\ s) = \sum_{j=0}^{\infty} \frac{\lambda^j \hat{r}_{0,1}^j}{(s+\lambda+\theta)^{j+1}}$$

$$+ \sum_{j=1}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{j}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \binom{-(j+1)}{l}\binom{j}{k}\binom{-(j-k)}{m}\binom{-k}{n}$$

$$\times \frac{\lambda^{j+l+m+n}\alpha^{j-k}\theta^j \hat{r}_{0,1}^{l+k+n}\hat{r}_{2,1}^{j+m+n}}{\mu_v^{k+n}(s+\lambda)^{j-k}(s+\lambda+\theta)^{j+l+1}(s+\lambda+\alpha)^{j+m-k}}. \tag{4.43}$$

To obtain the inverse Laplace transform of (4.43), we make use of the following well known results:

$$L^{-1}\left[\frac{1}{(s+\lambda)^{j-k}}\right] = e^{-\lambda t}\,\frac{t^{j-k-1}}{(j-k-1)!} = \phi_{1,\,j-k}(t), \tag{4.44}$$

$$L^{-1}\left[\frac{1}{(s+\lambda+\theta)^{j+l+1}}\right] = e^{-(\lambda+\theta)t}\,\frac{t^{j+l}}{(j+l)!} = \phi_{2,\,j+l+1}(t), \tag{4.45}$$

$$L^{-1}\left[\frac{1}{(s+\lambda+\alpha)^{j+m-k}}\right] = e^{-(\lambda+\alpha)t}\,\frac{t^{j+m-k-1}}{(j+m-k-1)!} = \phi_{3,\,j+m-k}(t), \tag{4.46}$$

$$L^{-1}[\hat{r}_{0,1}^{l+k+n}] = e^{-(\lambda+\theta+\mu_v)t}\left(\frac{\mu_v}{\lambda}\right)^{(l+k+n)/2}\frac{(l+k+n)}{t}\,I_{l+k+n}(2t\sqrt{\lambda\mu_v})$$

$$= \psi_{1,\,l+k+n}(t), \tag{4.47}$$

$$L^{-1}[\hat{r}_{2,1}^{j+m+n}] = e^{-(\lambda+\mu_b)t}\left(\frac{\mu_b}{\lambda}\right)^{(j+m+n)/2} \frac{(j+m+n)}{t} I_{j+m+n}(2t\sqrt{\lambda\mu_b})$$

$$= \psi_{2,\, j+m+n}(t), \tag{4.48}$$

where $I_v(t)$ is the modified Bessel function (see Abramowitz and Stegun [1]). Now we obtain explicitly

$$p(0,\, 0,\, t) = \sum_{j=0}^{\infty} \lambda^j \phi_{2,\, j+1}(t) © \psi_{1,\, j}(t)$$

$$+ \sum_{j=1}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{j}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \binom{-(j+1)}{l}\binom{j}{k}\binom{-(j-k)}{m}\binom{-k}{n}$$

$$\times \frac{\lambda^{j+l+m+n}\alpha^{j-k}\theta^j}{\mu_v^{k+n}} f_{j,k,l,m,n}(t), \tag{4.49}$$

where © represents the convolution operator and

$$f_{j,k,l,m,n}(t)$$

$$= \phi_{1,\, j-k}(t) © \phi_{2,\, j+l+1}(t) © \phi_{3,\, j+m-k}(t) © \psi_{1,\, l+k+n}(t) © \psi_{2,\, j+m+n}(t).$$

Inverting (4.25), we get

$$p(i,\, 0,\, t) = \left(\frac{\lambda}{\mu_v}\right)^i \psi_{1,\, i}(t) © p(0,\, 0,\, t), \quad i \geq 1. \tag{4.50}$$

Inverting (4.11), we get

$$p(0,\, 1,\, t) = \theta e^{-\lambda t} © p(0,\, 0,\, t). \tag{4.51}$$

Inverting (4.13), we get

$$p(i,\, 1,\, t) = \theta e^{-\lambda t} © \lambda^i \phi_{3,\, i}(t) © p(0,\, 0,\, t), \quad i \geq 1. \tag{4.52}$$

Inverting (4.36), we get

$p(j, 2, t)$

$$= \frac{\theta}{\mu_b}\left[\alpha e^{-\lambda t} © \sum_{k=1}^{j}\sum_{i=k}^{\infty}\left(\frac{\lambda}{\mu_b}\right)^{j-k}\lambda^i \phi_{3,i}(t) © \psi_{2,i+j-2k}(t) © p(0, 0, t)\right.$$

$$\left. + \sum_{k=1}^{j}\sum_{i=k}^{\infty}\left(\frac{\lambda}{\mu_b}\right)^{j-k}\left(\frac{\lambda}{\mu_v}\right)^{i}\psi_{1,i}(t) © \psi_{2,i+j-2k}(t) © p(0, 0, t)\right]. \quad (4.53)$$

Equations (4.49), (4.50), (4.51), (4.52) and (4.53) provide explicit expressions for describing the transient behaviour of the queueing system.

5. Steady-state Distribution

Let $\pi_{m,n} = \lim_{t\to\infty} p(m, n, t)$ be the steady-state distribution of the system. For the existence of the steady-state solution, the denominator of (4.42) should vanish at $s = 0$ and

$$\lambda < \mu_b. \quad (5.1)$$

Plugging $s = 0$ in the denominator of (4.42), we get

$$(\lambda + \theta - \lambda\hat{r}_{0,1}(0)) - \left\{\frac{\lambda\alpha\theta\hat{r}_{2,1}(0)}{\lambda(\lambda + \alpha - \lambda\hat{r}_{2,1}(0))} + \frac{\lambda\theta\hat{r}_{0,1}(0)\hat{r}_{2,1}(0)}{\mu_v - \lambda\hat{r}_{0,1}(0)\hat{r}_{2,1}(0)}\right\} = 0. \quad (5.2)$$

From (4.16) and (4.29), we get

$$\hat{r}_{0,1}(0) = \frac{(\lambda + \mu_v + \theta) - \sqrt{(\lambda + \mu_v + \theta)^2 - 4\lambda\mu_v}}{2\lambda}, \quad (5.3)$$

$$\hat{r}_{2,1}(0) = \frac{(\lambda + \mu_b) - \sqrt{(\lambda + \mu_b)^2 - 4\lambda\mu_b}}{2\lambda} = 1. \quad (5.4)$$

In tune with Xu and Tian [10], we set

$$\hat{r} = \frac{(\lambda + \mu_v + \theta) - \sqrt{(\lambda + \mu_v + \theta)^2 - 4\lambda\mu_v}}{2\mu_v}, \quad (5.5)$$

$$\rho = \frac{\lambda}{\mu_b}. \tag{5.6}$$

Then we have the relation

$$\hat{r}_{0,1}(0) = \frac{\mu_v}{\lambda}\hat{r}. \tag{5.7}$$

Consequently, (5.2) gives

$$\frac{\lambda}{\hat{r}} = \frac{\theta}{1-\hat{r}} + \mu_v. \tag{5.8}$$

The condition (5.8) is in agreement with Xu and Tian [10]. Using (4.41), (5.3), (5.4) and (5.6), we get

$$\pi_{00} = \left[\frac{1}{1-\hat{r}} + \frac{\theta}{\lambda} + \frac{\theta}{\alpha(1-\rho)} + \frac{\theta(1-\hat{r}+\hat{r}^2)}{\mu_b(1-\rho)(1-\hat{r})^2}\right]^{-1} = K. \tag{5.9}$$

Using (4.25), we get

$$\begin{aligned}\pi_{i0} &= \left(\frac{\lambda}{\mu_v}\right)^i \{\hat{r}_{0,1}(0)\}^i \pi_{00} \\ &= \left(\frac{\lambda}{\mu_v}\right)^i \left(\frac{\mu_v}{\lambda}\hat{r}\right)^i K \\ &= K\hat{r}^i, \quad i \geq 1.\end{aligned} \tag{5.10}$$

Using (4.11), we get

$$\pi_{01} = \frac{\theta}{\lambda}\pi_{00} = \frac{K\theta}{\lambda}. \tag{5.11}$$

From (4.13), we get

$$\pi_{i1} = \frac{\theta}{\lambda}\left(\frac{\lambda}{\lambda+\alpha}\right)^i \pi_{00} = \frac{K\theta}{\lambda}\left(\frac{\lambda}{\lambda+\alpha}\right)^i, \quad i \geq 1. \tag{5.12}$$

From (4.36), we get

$$\pi_{j2} = \frac{\theta}{\mu_b}\left[\frac{\alpha}{\lambda}\sum_{k=1}^{j}\left(\frac{\lambda}{\mu_b}\right)^{j-k}\left(\frac{\lambda}{\lambda+\alpha}\right)^{k}\frac{\lambda+\alpha}{\alpha} + \sum_{k=1}^{j}\left(\frac{\lambda}{\mu_b}\right)^{j-k}\frac{\hat{r}^k}{1-\hat{r}}\right]\pi_{00}$$

$$= \frac{\theta}{\mu_b}\left[\frac{\lambda+\alpha}{\lambda}\sum_{k=1}^{j}\rho^{j-k}\left(\frac{\lambda}{\lambda+\alpha}\right)^{k} + \sum_{k=1}^{j}\rho^{j-k}\frac{\hat{r}^k}{1-\hat{r}}\right]K$$

$$= \frac{K\theta}{\mu_b}\left[\sum_{k=1}^{j}\rho^{j-k}\left(\frac{\lambda}{\lambda+\alpha}\right)^{k-1} + \frac{1}{1-\hat{r}}\sum_{k=1}^{j}\hat{r}^k\rho^{j-k}\right], \quad j \ge 1. \tag{5.13}$$

The results (5.9)-(5.13) are in agreement with Xu and Tian [10]. We can also obtain (5.9) from (4.42). Multiplying both sides of (4.42) by s, we get

$$s\hat{p}(0,\ 0,\ s)$$

$$= \frac{s}{(s+\lambda+\theta-\lambda\hat{r}_{0,1}) - \left\{\frac{\lambda\alpha\theta\hat{r}_{2,1}}{(s+\lambda)(s+\lambda+\alpha-\lambda\hat{r}_{2,1})} + \frac{\lambda\theta\hat{r}_{0,1}\hat{r}_{2,1}}{\mu_v - \lambda\hat{r}_{0,1}\hat{r}_{2,1}}\right\}}. \tag{5.14}$$

Both the numerator and denominator of (5.14) become zero at $s = 0$. Applying L'Hospital rule, (5.14) gives

$$\pi_{00} = \frac{1}{1-\lambda\left(\frac{d}{ds}\hat{r}_{0,1}\right)_{s=0} - \left(\frac{d}{ds}\frac{\lambda\alpha\theta\hat{r}_{2,1}}{(s+\lambda)(s+\lambda+\alpha-\lambda\hat{r}_{2,1})}\right)_{s=0} - \left(\frac{d}{ds}\frac{\lambda\theta\hat{r}_{0,1}\hat{r}_{2,1}}{\mu_v-\lambda\hat{r}_{0,1}\hat{r}_{2,1}}\right)_{s=0}}. \tag{5.15}$$

We find that

$$\left(\frac{d}{ds}\hat{r}_{0,1}\right)_{s=0} = -\frac{\mu_v(1-\hat{r})}{\lambda\theta}\hat{r}, \tag{5.16}$$

$$\left(\frac{d}{ds}\hat{r}_{2,1}\right)_{s=0} = -\frac{1}{\mu_v(1-\rho)}. \tag{5.17}$$

Substituting (5.16) and (5.17) in (5.15), we get (5.9).

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