Advances in Fuzzy Sets and Systems

Published Online: October 2016
http://dx.doi.org/10.17654/FS021020121
Volume 21, Number 2, 2016, Pages 121-133 **ISSN: 0973-421X**

FUZZY INCIDENCE GRAPHS

John N. Mordeson

Department of Mathematics
Creighton University
Omaha, Nebraska 68178, U. S. A.
e-mail: mordes@creighton.edu

Abstract

We introduce the notion of the degree of incidence of a vertex and an edge in a fuzzy graph into fuzzy graph theory. We concentrate on incidence, where the edge is adjacent to the vertex. We determine results concerning incidence cutvertices, incidence cutpairs, fuzzy incidence paths, fuzzy incidence trees for fuzzy incidence graphs.

1. Introduction

In [1] and [2], Dinesh introduced the notion of the degree of incidence of a vertex and an edge in fuzzy graph theory. This notion seems to have potential use in a variety of areas involving networks. Basic results concerning fuzzy graphs can be found in [3] and [4].

A fuzzy subset of a set V is a function of V into the closed interval $[0, 1]$, [5]. Let σ be a fuzzy subset of a set V. Define the support of σ, written $\text{Supp}(\sigma)$, to be the set $\{x \in V \mid \sigma(x) > 0\}$. For $t \in [0, 1]$, define the level set σ^t to be the set $\{x \in V \mid \sigma(x) \geq t\}$.

Received: December 24, 2015; Revised: January 1, 2016; Accepted: January 23, 2016
2010 Mathematics Subject Classification: 05C72, 05C05, 05C38, 05C40.
Keywords and phrases: fuzzy graph, fuzzy incidence graph, incidence cutvertices, incidence cutpairs, fuzzy incidence paths, fuzzy incidence tree.

Let V be a set. Define a relation $\sim$ on $V \times V$ by for all $(x, y), (u, v) \in V \times V$, $(x, y)\tilde{}(u, v)$ if and only if $x = u$ and $y = v$ or $x = v$ and $y = u$. Then it is easily shown that $\sim$ is an equivalence relation on $V \times V$. For all $x, y \in V$, let $[(x, y)]$ denote the equivalence class induced by (x, y). Then $[(x, y)] = \{(x, y), (y, x)\}$. Let $\mathcal{E} = \{[(x, y)] \mid x, y \in V, x \neq y\}$. Let $E \subseteq \mathcal{E}$. If we consider the set E to be the set of edges in a graph (V, E), then no element of V has a loop. For $[(x, y)] \in E$, we write xy for $[(x, y)]$. Then $xy = yx$.

2. Preliminaries

We first recall and introduce certain basic notions on incidence for crisp graphs.

Definition 2.1 [2]**.** Let $G = (V, E, I)$, where $I \subseteq V \times E$. Then G is called an *incidence graph*.

We note that if $V = \{u, v\}$, $E = \{uv\}$, and $I = \{(v, uv)\}$, then (V, E, I) is an incidence graph by definition even though $(u, uv) \notin I$.

Definition 2.2 [2]**.** Let $G = (V, E, I)$ be an incidence graph. If $(u, vw) \in I$, then (u, vw) is called an *incidence pair* or simply a *pair*. If (u, uv), (v, uv), (v, vw), $(w, vw) \in I$, then uv and vw are called *adjacent edges*.

Definition 2.3 [2]**.** An *incidence subgraph H* of an incidence graph G is an incidence graph having its vertices, edges, and pairs in G. If H is an incidence subgraph of G, then G is called an *incidence supergraph* of H.

Let $\widetilde{G} = (V, E, I)$ be an incidence graph. Let $V' \subseteq V$, $E' \subseteq E$, and $I' \subseteq I$. Then $\widetilde{G}' = (V', E', I')$ is called a *near incidence subgraph* of $\widetilde{G}$ if: (1) $u'v' \in E' \Rightarrow u' \in V'$ or $v' \in V'$ and (2) $(v', u'v') \in I' \Rightarrow u'v' \in E'$.

Definition 2.4 [2]**.** Let $G = (V, E, I)$ be an incidence graph. A sequence

$$v_0, (v_0, v_0v_1), v_0v_1, (v_1, v_0v_1), v_1, ...,$$

$$v_{n-1}, (v_{n-1}, v_{n-1}v_n), v_{n-1}v_n, (v_n, v_{n-1}v_n), v_n$$

is called a *walk*. It is *closed* if $v_0 = v_n$. If the pairs are distinct, then it is called an *incidence trail*. If the edges are distinct, then it is called a *trail*. If the vertices are distinct, then it is called a *path*. If a path is closed, then it is called a *cycle*.

By the definition of a cycle, all incidence pairs are distinct.

We include the following sequences to be walks:

$v_0, (v_0, v_0v_1), v_0v_1, (v_1, v_0v_1), v_1, ...,$

$v_{n-1}, (v_{n-1}, v_{n-1}v_n), v_{n-1}v_n, (v_n, v_{n-1}v_n), v_n, (v_n, v_nv_{n+1}), v_nv_{n+1};$

$uv_0, (v_0, uv_0), v_0, (v_0, v_0v_1), v_0v_1, (v_1, v_0v_1), v_1, ...,$

$v_{n-1}, (v_{n-1}, v_{n-1}v_n), v_{n-1}v_n, (v_n, v_{n-1}v_n), v_n;$

$uv_0, (v_0, uv_0), v_0, (v_0, v_0v_1), v_0v_1, (v_1, v_0v_1), v_1, ...,$

$v_{n-1}, (v_{n-1}, v_{n-1}v_n), v_{n-1}v_n, (v_n, v_{n-1}v_n), v_n, (v_n, v_nv_{n+1}), v_nv_{n+1}.$

The latter is *closed* if $uv_0 = v_nv_{n+1}$. If the vertices are distinct, then they are called *incidence paths*.

From the definition of a path, if uv is on the path so are $(u, uv), (v, uv)$, but not an incidence pair of the form (u, vw) with $v \neq u \neq w$.

Definition 2.5 [2]**.** An incidence graph in which all pairs of vertices are joined by a path is said to be *connected*.

Definition 2.6 [2]**.** An incidence graph having no cycles is called a *forest*. If it is connected, then it is called a *tree*.

Since a tree is connected, all pairs of vertices are connected by a path. By the definition of a path, if uv is on the path so are $(u, uv), (v, uv),$ but no incidence pair of the form (u, vw) with $v \neq u \neq w$ is on the path.

A *component* in an incidence graph is a maximally connected near incidence subgraph. Recall that the definition of connectedness uses a path which for incidence graphs involves (u, uv) and (v, uv) for every uv in the path. Thus, the removal of a pair (u, uv) can increase the number of components in an incidence graph. For example, consider the incidence graph $G = (\{u, v\}, \{uv\}, \{(u, uv), (v, uv)\})$. Then G is connected, but $H = (\{u, v\}, \{uv\}, (v, uv))$ is not. H has two components, namely: $\{u\}$ and $(\{v\}, \{uv\}, (v, uv))$.

Definition 2.7. If the removal of an edge in an incidence graph increases the number of connected components, then the edge is called a *bridge*.

Definition 2.8. If the removal of a vertex in an incidence graph increases the number of connected components, then the edge is called a *cutvertex*.

Definition 2.9. If the removal of an incidence pair in an incidence graph increases the number of connected components, then the incidence pair is called a *cutpair*.

Consider the incidence graph

$$G = (\{u, v, w\}, \{uv, uw, vw\}, \{(u, uv), (v, uv), (u, uw),$$
$$(w, uw), (v, vw), (w, vw)\}).$$

Then (u, uv) is not a cutpair, since G remains connected since there is a path from u to v going through w.

We next introduce the notion of a fuzzy incidence graph.

Definition 2.10 [2]**.** Let $G = (V, E)$ be a graph and σ be a fuzzy subset of V and μ is a fuzzy subset of $V \times E$. Let Ψ be a fuzzy subset of $V \times E$. If $\Psi(v, e) \leq \sigma(v) \wedge \mu(e)$ for all $v \in V$ and $e \in E$, then Ψ is called a *fuzzy incidence* of G.

Definition 2.11 [2]**.** Let $G = (V, E)$ be a graph and (σ, μ) be a fuzzy subgraph of G. If Ψ is a fuzzy incidence of G, then $\widetilde{G} = (\sigma, \mu, \Psi)$ is called a *fuzzy incidence graph* of G.

Let (V, E) be a graph and (V, E, I) be an incidence graph. Then $I \subseteq V \times E$. We will assume in the following that $I \subseteq \{(u, uv) | uv \in E\}$. Let $E^{(i)} = \{(u, uv) | uv \in E\}$. (Note that since $uv = vu$, $(v, uv) \in E^{(i)}$.) Although not allowed here, incidence pairs of the form (u, vw), where $v \neq u \neq w$ also have potential applications in network theory. For example, (u, vw) might represent u's influence on vw with respect to flow from v to w. The flow might be human trafficking between countries or illicit flow of drugs, arms, or money between countries, [6] and [7].

Let $G = (V, E)$ be a graph and (σ, μ) be a fuzzy subgraph of G. Let (V, E, I) be an incidence graph. Let (σ, μ, Ψ) be a fuzzy incidence graph on G. Define $\sigma \cup \mu : V \cup E \to [0, 1]$ as follows: if $u \in V$, $(\sigma \cup \mu)(u) = \sigma(u)$ and if $uv \in E$, $(\sigma \cup \mu)(uv) = \mu(uv)$. Since $\Psi(u, uv) \leq \sigma(u) \wedge \mu(uv) = (\sigma \cup \mu)(u) \wedge (\sigma \cup \mu)(uv)$, we can consider $(\sigma \cup \mu, \Psi)$ as a fuzzy subgraph of $(V \cup E, E^{(i)})$. That is, the elements of $V \cup E$ can be thought of as the vertices and the elements of $E^{(i)}$ as the edges. This interpretation will aid in the understanding of the proofs to follow.

Definition 2.12 [2]**.** Let $\widetilde{G} = (\sigma, \mu, \Psi)$ be a fuzzy incidence graph and $xy \in \text{Supp}(\mu)$. Then xy is an edge of $\widetilde{G}$ and if $(x, xy), (y, xy) \in \text{Supp}(\Psi)$, then (x, xy) and (y, xy) are called *incidence pairs*.

Definition 2.13 [2]**.** Let $\widetilde{G} = (\sigma, \mu, \Psi)$ be a fuzzy incidence graph. Two vertices u and v joined by a path in a fuzzy incidence graph are said to be *connected*.

3. Main Results

In this section, we determine results concerning incidence cutvertices, incidence cutpairs, fuzzy incidence paths, fuzzy incidence tree for fuzzy incidence graphs.

Sequences of the type $u, (u, uv), v$ are not allowed as paths in an incidence graph. One needs $u, (u, uv), uv, (v, uv), v.$

In view of our definition of incidence walk, we can consider $(V \cup E, E^{(i)})$ as a bipartite graph.

An incidence graph $\widetilde{H} = (\tau, \nu, \Omega)$ is called a *partial fuzzy incidence subgraph* of the fuzzy incidence graph $\widetilde{G} = (\sigma, \mu, \Psi)$ if $\tau \subseteq \sigma, \nu \subseteq \mu,$ and $\Omega \subseteq \Psi.$ $\widetilde{H}$ is called a *fuzzy incidence subgraph* if $\tau = \sigma$ and $\nu = \mu.$ A partial fuzzy incidence subgraph $\widetilde{H}$ is said to *span* $\widetilde{G}$ if $\tau = \sigma$ and $\nu = \mu.$ In this case, $\widetilde{H}$ is called a *spanning fuzzy incidence subgraph* of $\widetilde{G}.$ A partial fuzzy incidence subgraph $\widetilde{H}$ of $\widetilde{G}$ is called *maximal* if $\Omega(x, xy) = \tau(x) \wedge \nu(xy) \wedge \Psi(x, xy).$

Let (σ, μ, Ψ) be a fuzzy incidence graph. Let $n \in \mathbb{N},$ the natural numbers. Let $u, v \in V.$ Define

$$\Psi^n(u, uv) = \vee\{\Psi(u_0, u_0v_1) \wedge \Psi(v_1, uv_1) \wedge \Psi(u_1, u_1v_2) \wedge \Psi(v_2, u_1v_2)$$
$$\wedge \cdots \wedge \Psi(u_{n-1}, u_{n-1}v_n) \wedge \Psi(v_n, u_{n-1}v_n)\Psi(v_n, u_0v_n) | u_0 = u,$$
$$v_n = v, u_i, v_i \in V, i = 1, ..., n-1\}.$$

Recall a (crisp) incidence graph with no cycles is called a *forest* and a connected forest is called a *tree*. We call a fuzzy incidence graph a *forest* if the graph consisting of its nonzero incidence pairs is a forest, and a *tree* if this graph is also connected. We call a fuzzy incidence graph a *fuzzy incidence forest* if it has partial fuzzy spanning incidence subgraph

$\widetilde{F} = (\sigma, \mu, \Omega)$ which is a forest, where for all pairs not in $\widetilde{F}$ (i.e., $\Omega(x, xy) = 0$), we have $\Psi(x, xy) < \Omega^{\infty}(x, xy)$. In other words, if (x, xy) is in $\widetilde{G}$, but not in $\widetilde{F}$, then there exists an incidence path in $\widetilde{F}$ between x and xy whose strength is greater than $\Psi(x, xy)$. A connected fuzzy incidence forest is called a *fuzzy incidence tree*.

Let $V = \{x, y, z\}$ and $E = \{xy, yz, xz\}$. Let σ, μ and Ψ be defined as follows:

$$\sigma(x) = 1, \sigma(y) = 1, \sigma(z) = 1,$$

$$\mu(xy) = 3/8, \mu(yz) = 3/4, \mu(xz) = 1/2,$$

$$\Psi(x, xy) = 3/8, \Psi(y, xy) = 3/8,$$

$$\Psi(y, yz) = 3/4, \Psi(z, yz) = 1/4,$$

$$\Psi(x, xz) = 1/2, \Psi(z, xz) = 1/2.$$

Then $\widetilde{F} = (\sigma, \mu, \Omega)$ is a fuzzy incidence tree, where $\Omega(z, yz) = 0$ and $\Omega = \Psi$ elsewhere.

Definition 3.1. Let $\widetilde{G} = (\sigma, \mu, \Psi)$ be the fuzzy incidence graph and $(u, uv) \in E^{(i)}$. Then (u, uv) is called an *incidence cutpair* if $\Psi'^{\infty}(x, xy) < \Psi^{\infty}(x, xy)$ for some $x, y \in V$, where $\Psi'(u, uv) = 0$ and $\Psi' = \Psi$ elsewhere.

Theorem 3.2. *Let $\widetilde{G}$ be a fuzzy incidence graph. Then the following conditions are equivalent*:

(1) $\Psi'^{\infty}(u, uv) < \Psi(u, uv)$.

(2) (u, uv) *is an incidence cutpair*.

(3) (u, vw) *is not the weakest incidence pair of any cycle*.

Proof. We prove the following three implications by contrapositive:

(1) $\Rightarrow$ (2) If (u, uv) is not an incidence cutpair, then $\Psi'^{\infty}(u, uv) = \Psi^{\infty}(u, uv) \geq \Psi(u, uv)$.

(2) $\Rightarrow$ (3) Suppose (u, uv) is the weakest pair in a cycle. Then any path involving (u, uv) can be converted into a path not involving (u, uv) but at least as strong by using the rest of the cycle as a path from u to uv. Thus, (u, uv) is not an incidence cutpair.

(3) $\Rightarrow$ (1) Suppose that $\Psi'^{\infty}(u, uv) \geq \Psi(u, uv)$. Then there is a path from u to v not involving (u, vu) that has strength $\geq \Psi(u, vu)$ and this path together with uv forms a cycle of which (u, uv) is the weakest pair.

Theorem 3.3. *Let* $\widetilde{G} = (\sigma, \mu, \Psi)$ *be the fuzzy incidence graph. If* $\exists t \in (0, 1]$ *such that* $(Supp(\sigma), Supp(\mu), \Psi^t)$ *is a tree, then* (σ, μ, Ψ) *is a fuzzy incidence tree. Conversely, if* (σ, μ, Ψ) *is a cycle and* (σ, μ, Ψ) *is a fuzzy incidence tree, then* $\exists t \in (0, 1]$ *such that* $(Supp(\sigma), Supp(\mu), \Psi^t)$ *is a tree.*

Proof. Suppose that t exists such that $(\text{Supp}(\sigma), \text{Supp}(\mu), \Psi^t)$ is a tree. Let $\Omega = \Psi$ on Ψ^t and $\Omega(x, xy) = 0$ if $(x, xy) \notin \Psi^t$. Since $(\text{Supp}(\sigma), \text{Supp}(\mu), \text{Supp}(\Omega))$ is a tree, (σ, μ, Ω) is a spanning fuzzy subgraph of (σ, μ, Ψ) such that (σ, μ, Ω) is a fuzzy tree. Suppose that $(u, uv) \in \text{Supp}(\Psi)\backslash\text{Supp}(\Omega)$. Then $(u, uv) \notin \Psi^t$ since $\Psi^t = \text{Supp}(\Omega)$. Thus, $\Psi(u, uv) < t \leq \Omega^{\infty}(u, uv)$. Hence (σ, μ, Ψ) is a fuzzy incidence tree.

Conversely, suppose (σ, μ, Ψ) is a cycle and (σ, μ, Ψ) is a fuzzy incidence tree. Then there exists a unique $(x, xy) \in \text{Supp}(\Psi)$ such that $\Psi(x, xy) = \wedge\{\Psi(u, uv) | (u, uv) \in \text{Supp}(\Psi)\}$. Let t be such that $\Psi(x, xy)$

$< t \leq \wedge\{\Psi(u, uvw \mid (u, uv) \in \text{Supp}(\Psi))\backslash\{(x, xy)\}\}$. Then $(\text{Supp}(\sigma), \text{Supp}(\mu),$ $\Psi^t)$ is a tree.

Theorem 3.4. *Let* $\widetilde{G} = (\sigma, \mu, \Psi)$ *be a fuzzy incidence graph and* $(u, uv) \in V \times E$. *If* (u, uv) *is an incidence cutpair, then* $\Psi(u, uv) = \Psi^{\infty}(u, uv)$.

Proof. Suppose $\Psi(u, uv) < \Psi^{\infty}(u, uv)$. Then there exists a strongest path from u to uv such that all pairs (x, xy) in the path are such that $\Psi(x, xy) > \Psi(u, uv)$. This path together with (u, uv) forms a cycle in which (u, uv) is the weakest pair and so (u, uv) is not an incidence cutpair.

Definition 3.5. Let $\widetilde{G} = (\sigma, \mu, \Psi)$ be a fuzzy incidence graph and $w \in V$. Then w is called an *incidence cutvertex* if there exist $u, v \in V$ such that $\Psi'^{\infty}(u, uv) < \Psi^{\infty}(u, uv)$ for $u \neq w \neq v$, where $\Psi'(w, zw) = 0, \forall z \in V$ and $\Psi' = \Psi$ elsewhere.

Theorem 3.6. *Let* $\widetilde{G} = (\sigma, \mu, \Psi)$ *be the fuzzy incidence graph. If w is a common vertex of at least two incidence cutpairs, then w is an incidence cutvertex.*

Proof. Let (w, u_1w) and (w, u_2w) be two incidence cutpairs. Then there exist u, v such that (w, u_1w) is on every strongest u-v path. If w is distinct from u and v, then w is an incidence cutvertex. Suppose $w = u$ or $w = v$. Then (w, u_1w) is on every strongest u-w path or (w, u_2w) is on every strongest w-v path. Suppose that w is not an incidence cutvertex. Then between every two vertices there exists at least one strongest path not containing w. In particular, there exists at least one strongest path P, joining u_1w and u_2w, not containing w. This path together with (w, u_1w) and (w, u_2w) forms a cycle.

We now consider two cases:

(1) Suppose that $u_1, (u_1, u_1w), u_1w, (w, u_1w), w, (w, wu_2), wu_2, (u_2, wu_2),$ u_2 is not a strongest path. Then clearly one of $(w, u_1w), (w, u_2w)$ or both become the weakest incidence pairs of the cycle which contradicts that (w, u_1w) and (w, u_2w) are two incidence cutpairs.

(2) Suppose that $u_1, (u_1, u_1w), u_1w, (w, u_1w), w, (w, wu_2), wu_2, (u_2, wu_2),$ u_2 is a strongest path joining u_1 to u_2. Then the strength of the strongest incidence path from u_1 to u_2 equals $\Psi(w, u_1w) \wedge \Psi(w, u_2w)$, the strength of *P*. Thus, the incidence pairs of *P* are at least as strong as $\Psi(w, u_1w)$ and $\Psi(w, u_2w)$ and so $(w, u_1w), (w, u_2w)$ or both are the weakest pairs of the cycle, a contradiction.

Theorem 3.7. *If* $\widetilde{G} = (\sigma, \mu, \Psi)$ *is a fuzzy incidence tree and*

$$\widetilde{G}^* = (Supp(\sigma), Supp(\mu), Supp(\Psi))$$

is not a tree, then there exists at least one incidence pair (u, uv) *for which* $\Psi(u, uv) < \Psi^{\infty}(u, uv)$.

Proof. Since $\widetilde{G}$ is a fuzzy incidence tree, there exists a fuzzy incidence spanning subgraph $\widetilde{F} = (\sigma, \mu, \Omega)$ which is a tree such that $\Psi(u, vw) < \Omega^{\infty}(u, vw)$ for all pairs (u, vw) not in $\widetilde{F}$. Also, $\Omega^{\infty}(u, uv) \leq \Psi^{\infty}(u, uv)$. Thus, $\Psi(u, uv) < \Psi^{\infty}(u, uv)$ for all (u, uv) not in $\widetilde{F}$ and by hypothesis there exists one incidence pair (u, uv) not in $\widetilde{F}$.

Definition 3.8. Let $\widetilde{G} = (\sigma, \mu, \Psi)$ be the fuzzy incidence graph. Then $\widetilde{G}$ is said to be *fuzzy incidence complete* if for all $(u, vw) \in V \times E$, $\Psi(u, vw) = \sigma(u) \wedge \mu(vw)$.

Note that if $\widetilde{G}$ is fuzzy incidence complete, then $\Psi(u, uv) = \sigma(u) \wedge \mu(uv) = \mu(uv) = \sigma(v) \wedge \mu(uv) = \Psi(v, uv)$.

Theorem 3.9. *If* $\widetilde{G} = (\sigma, \mu, \Psi)$ *is a fuzzy incidence tree, then* $\widetilde{G}$ *is not fuzzy incidence complete.*

Proof. Suppose $\widetilde{G}$ is fuzzy incidence complete. Then $\Psi(u, uv) = \Psi^{\infty}(u, uv)$ for all (u, uv). Since $\widetilde{G} = (\sigma, \mu, \Psi)$ is a fuzzy incidence tree, $\Psi(u, uv) < \Omega^{\infty}(u, uv)$ for all (u, uv) not in $\widetilde{F} = (\sigma, \mu, \Omega)$, a fuzzy incidence spanning subgraph of $\widetilde{G}$ which is a tree. Thus, $\Psi^{\infty}(u, uv) < \Omega(u, uv)$ which is impossible. Hence $\widetilde{G}$ is not fuzzy incidence complete.

Theorem 3.10. *Let* $\widetilde{G} = (\sigma, \mu, \Psi)$ *be a fuzzy incidence tree. Then the internal vertices of a fuzzy incidence spanning subgraph* $\widetilde{F}$ *which is a tree are the incidence cutvertices of* $\widetilde{G}$.

Proof. Let w be a vertex in $\widetilde{G}$ which is not an end vertex of $\widetilde{F}$. Then w is the common vertex of at least two incidence pairs in $\widetilde{F}$ which are incidence cutpairs of $\widetilde{G}$. Thus, by Theorem 3.6, w is an incidence cutvertex. Suppose w is an end vertex of $\widetilde{F}$. Then w is not an incidence cutvertex; else there would exist u, v distinct from w such that w is on every strongest u-v path and one such path certainly lies in $\widetilde{F}$. However, this is not possible since w is an end vertex of $\widetilde{F}$.

Corollary 3.11. *Let* $\widetilde{G} = (\sigma, \mu, \Psi)$ *be a fuzzy incidence tree. Then an incidence cutvertex is the common vertex of at least two incidence cutpairs.*

Theorem 3.12. *Let* $\widetilde{G} = (\sigma, \mu, \Psi)$ *be a fuzzy incidence graph. Then* $\widetilde{G}$ *is a fuzzy incidence tree if and only if the following conditions are equivalent for all* $u, v \in V$:

(1) (u, uv) *is an incidence cutpair.*

(2) $\Psi^{\infty}(u, uv) = \Psi(u, uv)$.

Proof. Suppose $\widetilde{G} = (\sigma, \mu, \Psi)$ is a fuzzy incidence tree. Suppose that (u, uv) is an incidence cutpair. Then $\Psi^{\infty}(u, uv) = \Psi(u, uv)$ by Theorem 3.4. Let (u, uv) be an incidence pair in $\widetilde{G}$ such that $\Psi^{\infty}(u, uv) = \Psi(u, uv)$. If $\widetilde{G}^{*} = (\text{Supp}(\sigma), \text{Supp}(\mu), \text{Supp}(\Psi))$ is a tree, then clearly (u, uv) is an incidence cutpair. If $\widetilde{G}^{*}$ is not a tree, then it follows from Theorem 3.7 that (u, uv) is in $\widetilde{F}$ and (u, uv) is an incidence cutpair.

Conversely, assume that (1) and (2) are equivalent. Let $\widetilde{T} = (\sigma, \mu, \Omega)$ be a maximal fuzzy incidence spanning tree for $\widetilde{G}$. If (u, uv) is in $\widetilde{T}$, $\Psi^{\infty}(u, uv) = \Psi(u, uv)$ and hence (u, uv) is an incidence cutpair. Now those are the only fuzzy incidence bridges in $\widetilde{G}$ for if possible let $(u', u'v')$ be an incidence cutpair of $\widetilde{G}$ which is not in $\widetilde{T}$. Consider a cycle C consisting of $(u', u'v')$ and the unique u'-$u'v'$ path in $\widetilde{T}$. Now incidence pairs of this u'-$u'v'$ path are incidence cutpairs and so they are not weakest pairs of C. Hence $(u', u'v')$ must be the weakest cutpair of C and thus cannot be an incidence cutpair.

Furthermore, for all incidence pairs $(u', u'v')$ not in $\widetilde{T}$, we have $\Psi(u', u'v') < \Omega^{\infty}(u', u'v')$; for if possible, let $\Psi(u', u'v') \geq \Omega^{\infty}(u', u'v')$. However, $\Psi(u', u'v') < \Psi^{\infty}(u', u'v')$, where strict inequality holds, since $(u', u'v')$ is not an incidence cutpair. Hence $\Omega^{\infty}(u', u'v') < \Psi^{\infty}(u', u'v')$ which is impossible, since $\Omega^{\infty}(u', u'v')$ is the strength of the unique u'-$u'v'$ path in $\widetilde{T}$ and $\Psi^{\infty}(u', u'v') = \Omega^{\infty}(u', u'v')$. Thus, $\widetilde{T}$ is the required spanning fuzzy incidence subgraph $\widetilde{F}$ which is a tree and hence $\widetilde{G}$ is a fuzzy incidence tree.

Acknowledgement

The author thanks the anonymous referees for their valuable suggestions which led to the improvement of the manuscript.

References

[1] T. Dinesh, Fuzzy incidence graphs, Ph.D. Thesis, Kannur University, Kerala, India, 2012.

[2] T. Dinesh, Fuzzy incidence graph - and introduction, Adv. Fuzzy Sets Syst. 21(1) (2016), 33-48.

[3] J. N. Mordeson and P. S. Nair, Fuzzy Graphs and Fuzzy Hypergraphs, Physica-Verlag, Vol. 46, 2000.

[4] A. Rosenfeld, Fuzzy graphs, L. A. Zadeh, K. S. Fu and M. Shimura, eds., Fuzzy Sets and their Applications, Academic Press, 1975, pp. 77-95.

[5] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965), 338-353.

[6] C. H. Smith, Tier rankings and the fight against human trafficking, Congress of the United States, Subcommittee on Africa, Global Health, Global Human Rights, and Int'l Organizations, 2013.

[7] D. Kar and D. Cartwright-Smith, Illicit Financial Flows from Developing Countries, Global Financial Integrity, A Program of the Center for International Policy, 2002-2006, pp. 1-67, www.gfip.org.