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NUMERICAL ANALYSIS FOR STOCHASTIC DELAY INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract

This paper deals with a family of balanced methods which own the implicit iterative scheme in the diffusion term for the stochastic delay integro-differential equations. It is shown that the balanced implicit

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methods, which are fully implicit methods, give strong convergence rate of at least 1/2 and that the strong balanced methods can preserve the linear mean-square stability with the sufficiently small stepsize. Weak variants are also considered and their mean-square stability is analyzed. Some numerical experiments are given to demonstrate the conclusions and to show that the fully implicit methods are superior to those of the explicit methods in terms of mean-square stabilities.

1. Introduction

Stochastic delay integro-differential equations (SDIDEs) are often used to model some problems in a variety of application areas including population dynamics [1], engineering, physics [2], economy [3, 4] and so on. Unfortunately, stochastic delay differential equations rarely have explicit solutions. Thus, appropriate numerical methods are needed to apply in practice and to study their properties.

The numerical analysis of stochastic delay differential equations (SDDEs) is well studied, for instance, Baker and Buckwar [5, 6], Küchler and Platen [7], Mao and Sabanis [8], Liu et al. [9], Wang et al. [10], Cao and Zhang [11], Wu and Ding [12], Zhao et al. [13] and Zhang et al. [14, 15]. As for SDIDEs, there has been much less research of numerical schemes. Mao [17] discussed the stability of SDIDEs. Ding et al. [16] discussed the convergence and stability of the semi-implicit Euler method for SDIDEs. Tan and Wang [18] considered the convergence and mean-square stability of the split-step backward Euler (SSBE) method for SDIDEs. Li and Gan [19] investigated the mean-square exponential stability of stochastic theta methods for nonlinear SDIDEs.

However, it is already known that the majority of these discrete approximations for SDIDEs are not fully implicit methods, they are only implicit in the drift coefficient. These drift-implicit methods are well adapted for stiff systems with small stochastic noise intensity or additive noise. But in those cases in which the stochastic part plays an essential role in the dynamics, e.g., as it is with large multiplicative noise, the application of fully implicit methods also involving implicit stochastic terms is unavoidable. One of the most important fully implicit methods is the balanced implicit methods, which were firstly introduced by Milstein et al. [21] and used to solve stiff systems. Some recent papers which consider the balanced implicit methods for the stochastic differential equations include [20, 22-24].

Consider the following scalar linear stochastic delay integro-differential equation:

$$\begin{cases} dx(t) = \left(\alpha_1 x(t) + \alpha_2 x(t-\tau) + \alpha_3 \int_{t-\tau}^{t} x(s)ds\right)dt \\ \qquad\qquad + \left(\beta_1 x(t) + \beta_2 x(t-\tau) + \beta_3 \int_{t-\tau}^{t} x(s)ds\right)dW(t), \quad t > 0, \\ x(t) = \varphi(t), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in [-\tau, 0], \end{cases} \tag{1.1}$$

where $\alpha_i, \beta_i \in \mathbb{R}$ $(i = 1, 2, 3)$, τ is a positive fixed delay, $\varphi(t)$ is a $C([-\tau, 0]; \mathbb{R})$, $W(t)$ is a scalar Brownian motion, both defined on an appropriate complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right-continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets).

Lemma 1.1. *For any given* $T > 0$, *there exist positive numbers* η_1 *and* η_2 *such that the solution* $x(t)$ *of* (1.1) *satisfies*

$$\mathbb{E}(\sup_{-\tau\leq s\leq t} |x(s)|^2) \leq \eta_1[1 + \mathbb{E}\|\varphi\|^2], \tag{1.2}$$

for all $t \in [-\tau, T]$,

$$\mathbb{E}|x(t) - x(s)|^2 \leq \eta_2(t - s), \tag{1.3}$$

for any $0 \leq s < t \leq T$, $t - s < 1$.

For the proof of inequality (1.2), we refer to [25, Chapter 3, Theorem 5.1]. Further, inequality (1.3) can be obtained from (1.2).

Regarding numerical analysis of (1.1), Hu and Huang [26] investigated the mean-square stability of stochastic-methods for SDIDEs (1.1). However, to the best of our knowledge, there are no stability results of implicit methods for the system (1.1). In this paper, the balanced implicit methods are proposed for SDIDEs (1.1). Our aim is to investigate the strong mean-square convergence and mean-square stability of the balanced implicit methods of the system (1.1). The rest of the paper is organized as follows. In the subsequent section, Theorem 2.1 is established to showing the strong balanced implicit methods are convergent with strong order $1/2$. Section 3 and Section 4 deal with linear mean-square stability of the strong balanced implicit methods and the weak balanced implicit methods. In Section 5, some numerical experiments are given to demonstrate the conclusions. Finally, conclusion is made in Section 6.

2. Convergence of the Balanced Implicit Methods

Given a stepsize $h = \tau/m > 0$, a version of strong balanced methods for (1.1) is given by

$$\begin{cases} Y_{n+1} = Y_n + (\alpha_1 Y_n + \alpha_2 Y_{n-m} + \alpha_3 \widetilde{Y}_n)h \\ \qquad + (\beta_1 Y_n + \beta_2 Y_{n-m} + \beta_3 \widetilde{Y}_n)\Delta W_n + D_n(Y_n - Y_{n+1}), & n \geq 0, \\ Y_n = \varphi(nh), & n = -m, -m+1, \ldots, 0, \end{cases} \tag{2.1}$$

where Y_n is an approximation to $x(t_n)$ with $t_n = nh, \Delta W_n = W(t_{n+1}) - W(t_n)$. Here $\widetilde{Y}_n$ denotes

$$\widetilde{Y}_n = h \sum_{k=1}^{m} Y_{n-m+k}. \tag{2.2}$$

Here D_n is given by

$$\begin{aligned} D_n &= D_{0n}(Y_n, Y_{n-m}, \widetilde{Y}_n)h + D_{1n}(Y_n, Y_{n-m}, \widetilde{Y}_n)|\Delta W_n| \\ &= D_{0n}h + D_{1n}|\Delta W_n|, \end{aligned} \tag{2.3}$$

where $D_{0n} = D_{0n}(Y_n, Y_{n-m}, \widetilde{Y}_n)$, $D_{1n} = D_{1n}(Y_n, Y_{n-m}, \widetilde{Y}_n)$ are called *control functions*. In order to obtain our main results in this paper, we assume that D_{0n}, D_{1n} in equation (2.3) are constants, that is, $D_{0n} = D_0, D_{1n} = D_1$. In addition, D_0, D_1 which are uniformly bounded satisfy the following condition.

Assumption 1. For any real numbers $\alpha_0 \in [0, \overline{\alpha}]$, $\alpha_1 \geq 0$, where $\overline{\alpha} \geq h$ for all step sizes h considered, the constants D_0, D_1 satisfy $|1 + \alpha_0 D_0 + \alpha_1 D_1|^{-1} \leq H < \infty$.

Therefore, (2.1) can be rewritten as follows:

$$Y_{n+1} = Y_n + (1 + D_n)^{-1}[(\alpha_1 Y_n + \alpha_2 Y_{n-m} + \alpha_3 \widetilde{Y}_n)h$$
$$+ (\beta_1 Y_n + \beta_2 Y_{n-m} + \beta_3 \widetilde{Y}_n)\Delta W_n], \quad n \geq 0. \tag{2.4}$$

We denote by $x(t_{n+1})$ the value of the exact solution of (1.1) at the mesh point t_{n+1} and by Y_{n+1} the value of the approximation solution using (2.4). Furthermore, we denote by $\overline{Y}(t_{n+1})$ the value that is obtained when the exact solution values are inserted into the right-hand side of (2.4), that is,

$$\overline{Y}(t_{n+1}) = x(t_n) + (1 + D(t_n))^{-1}[(\alpha_1 x(t_n) + \alpha_2 x(t_{n-m}) + \alpha_3 \widetilde{x}(t_n))h$$
$$+ (\beta_1 x(t_n) + \beta_2 x(t_{n-m}) + \beta_3 \widetilde{x}(t_n))\Delta W_n]. \tag{2.5}$$

Here

$$D(t_n) = D_{0n}(x(t_n), x(t_{n-m}), \widetilde{x}(t_n))h + D_{1n}(x(t_n), x(t_{n-m}), \widetilde{x}(t_n))\Delta W_n$$
$$= D_{0n}(t_n)h + D_{1n}(t_n)\Delta W_n \tag{2.6}$$

and

$$\widetilde{x}(t_n) = h\sum_{k=1}^{m} x(t_{n-m+k}). \tag{2.7}$$

Based on the definition introduced in [27, 28], we give the following definitions.

Definition 2.1. The local error of method (2.1) is defined as follows:

$$\delta_{n+1} := x(t_{n+1}) - \overline{Y}(t_{n+1}), \quad n = 0, 1, \ldots, N-1.$$

The global error of method (2.1) is defined as follows:

$$\varepsilon_n := x(t_n) - Y_n, \quad n = 0, 1, \ldots, N.$$

Throughout this work, we use $C_1, C_2, \ldots$ to denote generic constants, independent of h.

Lemma 2.1. *There exist positive numbers* η_3 *and* η_4 *such that the numerical solution produced by the balanced method* (2.1) *to approximate the solution of equation* (1.1) *satisfies*

$$\max_{0 \leq n \leq N-1} |\mathbb{E}(\delta_{n+1})| \leq \eta_3 h^{\frac{3}{2}} \text{ as } h \to 0, \tag{2.8}$$

$$\max_{0 \leq n \leq N-1} (\mathbb{E}(\delta_{n+1})^2)^{\frac{1}{2}} \leq \eta_4 h \text{ as } h \to 0. \tag{2.9}$$

Proof. It follows from (2.5) and Definition 2.1, we get

$$\begin{aligned}
\delta_{n+1} = \int_{t_n}^{t_{n+1}} & \alpha_1(x(t) - x(t_n)) + \alpha_2(x(t-\tau) - x(t_{n-m})) \\
& + \alpha_3\left(\int_{t-\tau}^{t} x(s)ds - \widetilde{x}(t_n)\right)dt \\
& + \int_{t_n}^{t_{n+1}} \beta_1(x(t) - x(t_n)) + \beta_2(x(t-\tau) - x(t_{n-m})) \\
& + \beta_3\left(\int_{t-\tau}^{t} x(s)ds - \widetilde{x}(t_n)\right)dW(t) \\
& + \frac{D(t_n)}{1 + D(t_n)}[(\alpha_1 x(t_n) + \alpha_2 x(t_{n-m}) + \alpha_3 \widetilde{x}(t_n))h \\
& + (\beta_1 x(t_n) + \beta_2 x(t_{n-m}) + \beta_3 \widetilde{x}(t_n))\Delta W_n].
\end{aligned} \tag{2.10}$$

Thus, employing mathematical expectation and using the properties of the Itõ integral, we obtain

$$
\begin{aligned}
&|\mathbb{E}(\delta_{n+1})| \\
&\leq \int_{t_n}^{t_{n+1}} |\alpha_1| \cdot \mathbb{E}|x(t) - x(t_n)| dt + \int_{t_n}^{t_{n+1}} |\alpha_2| \cdot \mathbb{E}|x(t-\tau) - x(t_{n-m})| dt \\
&+ \int_{t_n}^{t_{n+1}} |\alpha_3| \cdot \mathbb{E}\left| \int_{t-\tau}^{t} x(s)ds - \widetilde{x}(t_n) \right| dt \\
&+ \left| \mathbb{E} \frac{D(t_n)}{1 + D(t_n)} (\alpha_1 x(t_n) + \alpha_2 x(t_{n-m}) + \alpha_3 \widetilde{x}(t_n)) h \right|. \qquad (2.11)
\end{aligned}
$$

For $\forall t \in [t_n, t_{n+1}]$, using (1.2), (1.3) and (2.7) yields

$$
\begin{aligned}
&\mathbb{E}\left| \int_{t-\tau}^{t} x(s)ds - \widetilde{x}(t_n) \right| \\
&\leq \sqrt{\eta_2} \tau h^{\frac{1}{2}} + 2\sqrt{\eta_1(1 + \mathbb{E}\|\varphi\|^2)} h \\
&= (\sqrt{\eta_2}\tau + 2\sqrt{\eta_1(1 + \mathbb{E}\|\varphi\|^2)}) h^{\frac{1}{2}}. \qquad (2.12)
\end{aligned}
$$

We notice the assumption that the D_0, D_1 are uniformly bounded, that is to say, there exists a positive constant B such that $|D_i| \leq B\,(i = 0, 1)$. Using Assumption 1, $\mathbb{E}|\Delta W_n| = \sqrt{\frac{2h}{\pi}}$, (1.2), (2.6) and (2.7) give

$$
\begin{aligned}
&\left| \mathbb{E} \frac{D(t_n)}{1 + D(t_n)} (\alpha_1 x(t_n) + \alpha_2 x(t_{n-m}) + \alpha_3 \widetilde{x}(t_n)) h \right| \\
&\leq H(Bh + B\sqrt{h}) h \cdot [(|\alpha_1| + |\alpha_2|)\sqrt{\eta_1(1 + \mathbb{E}\|\varphi\|^2)} + h^2 |\alpha_3| m \sqrt{\eta_1(1 + \mathbb{E}\|\varphi\|^2)}] \\
&\leq 2HB\sqrt{\eta_1(1 + \mathbb{E}\|\varphi\|^2)} (|\alpha_1| + |\alpha_2| + |\alpha_3|\tau) h^{\frac{3}{2}}. \qquad (2.13)
\end{aligned}
$$

Combining (1.3), (2.12), (2.13) with (2.11) yields

$$
\begin{aligned}
&|\, \mathbb{E}(\delta_{n+1})\,| \\
&\leq (|\,\alpha_1\,| + |\,\alpha_2\,|)\sqrt{\eta_2}\,h^{\frac{3}{2}} + |\,\alpha_3\,|(\sqrt{\eta_2}\,\tau + 2\sqrt{\eta_1(1+\mathbb{E}\|\,\varphi\,\|^2)})h^{\frac{3}{2}} \\
&\quad + 2HB\sqrt{\eta_1(1+\mathbb{E}\|\,\varphi\,\|^2)}\,(|\,\alpha_1\,| + |\,\alpha_2\,| + |\,\alpha_3\,|\tau)h^{\frac{3}{2}} \\
&= \eta_3 h^{\frac{3}{2}}.
\end{aligned}
$$

Here

$$
\begin{aligned}
\eta_3 &= (|\,\alpha_1\,| + |\,\alpha_2\,|)\sqrt{\eta_2} + |\,\alpha_3\,|(\sqrt{\eta_2}\,\tau + 2\sqrt{\eta_1(1+\mathbb{E}\|\,\varphi\,\|^2)}) \\
&\quad + 2HB\sqrt{\eta_1(1+\mathbb{E}\|\,\varphi\,\|^2)}\,(|\,\alpha_1\,| + |\,\alpha_2\,| + |\,\alpha_3\,|\tau).
\end{aligned}
$$

In the following, we will show $\max_{0\leq n\leq N-1}(\mathbb{E}(\delta_{n+1})^2)^{\frac{1}{2}} \leq \eta_4 h.$ Squaring and taking expectation on both sides of (2.10) and using the properties of the Itõ integral, we compute that

$$
\mathbb{E}(\delta_{n+1})^2 \leq 3h\int_{t_n}^{t_{n+1}} \mathbb{E}|\,\varsigma_1(t)\,|^2 dt + 3\int_{t_n}^{t_{n+1}} \mathbb{E}|\,\varsigma_2(t)\,|^2 dt + 3\mathbb{E}|\,\varsigma_3(t)\,|^2, \quad (2.14)
$$

where

$$
\varsigma_1(t) = \alpha_1(x(t) - x(t_n)) + \alpha_2(x(t-\tau) - x(t_{n-m})) + \alpha_3\left(\int_{t-\tau}^{t} x(s)ds - \widetilde{x}(t_n)\right),
$$

$$
\varsigma_2(t) = \beta_1(x(t) - x(t_n)) + \beta_2(x(t-\tau) - x(t_{n-m})) + \beta_3\left(\int_{t-\tau}^{t} x(s)ds - \widetilde{x}(t_n)\right),
$$

$$
\begin{aligned}
\varsigma_3(t) &= \frac{D(t_n)}{1+D(t_n)}[(\alpha_1 x(t_n) + \alpha_2 x(t_{n-m}) + \alpha_3\widetilde{x}(t_n))h \\
&\quad + (\beta_1 x(t_n) + \beta_2 x(t_{n-m}) + \beta_3\widetilde{x}(t_n))\Delta W_n].
\end{aligned}
$$

It follows from (1.3), we get

$$\mathbb{E}|\varsigma_1(t)|^2$$

$$= \mathbb{E}\left|\alpha_1(x(t)-x(t_n))+\alpha_2(x(t-\tau)-x(t_{n-m}))+\alpha_3\left(\int_{t-\tau}^{t} x(s)ds-\widetilde{x}(t_n)\right)\right|^2$$

$$\leq 3(\alpha_1^2+\alpha_2^2)\eta_2 h + 3\alpha_3^2\mathbb{E}\left|\int_{t-\tau}^{t} x(s)ds - \widetilde{x}(t_n)\right|^2. \tag{2.15}$$

Using the properties of the Itõ integral, the inequality $\left(\sum_{i=1}^{m} x_i\right)^2 \leq m\sum_{i=1}^{m} x_i^2$, (1.2), (1.3) and (2.7), we have

$$\mathbb{E}\left|\int_{t-\tau}^{t} x(s)ds - \widetilde{x}(t_n)\right|^2$$

$$= \mathbb{E}\left|\int_{t-\tau}^{t} x(s)ds - h\sum_{k=1}^{m} x(t_{n-m+k})\right|^2$$

$$\leq 3\tau m\eta_2 h^2 + 6\eta_1(1+\mathbb{E}\|\varphi\|^2)h^2$$

$$= 3\tau^2\eta_2 h + 6\eta_1(1+\mathbb{E}\|\varphi\|^2)h^2. \tag{2.16}$$

Combining (2.16) with (2.15) yields

$$\mathbb{E}|\varsigma_1(t)|^2 \leq 3(\alpha_1^2+\alpha_2^2)\eta_2 h + 3\alpha_3^2[3\tau^3\eta_2 h + 6\eta_1(1+\mathbb{E}\|\varphi\|^2)h^2]$$

$$= C_1 h. \tag{2.17}$$

By (1.3) and (2.16), we see that

$$\mathbb{E}|\varsigma_2(t)|^2$$

$$= \mathbb{E}\left|\beta_1(x(t)-x(t_n))+\beta_2(x(t-\tau)-x(t_{n-m}))+\beta_3\left(\int_{t-\tau}^{t} x(s)ds-\widetilde{x}(t_n)\right)\right|^2$$

$$\leq 3\beta_1^2\eta_2 h + 3\beta_2^2\eta_2 h + 3\beta_3^2[3\tau^2\eta_2 h + 6\eta_1(1 + \mathbb{E}\|\varphi\|^2)h^2]$$

$$= C_2 h. \tag{2.18}$$

Using Assumption 1, $|D_i| \leq B$, $E|\Delta W_n|^2 = h$, $E|\Delta W_n|^4 = 3h^2$, (1.2) and (2.7) yield

$$\begin{aligned}
&\mathbb{E}|\varsigma_3(t)|^2 \\
&= \mathbb{E}\left|\frac{D(t_n)}{1 + D(t_n)}[(\alpha_1 x(t_n) + \alpha_2 x(t_{n-m}) + \alpha_3 \widetilde{x}(t_n))h\right. \\
&\quad \left. + (\beta_1 x(t_n) + \beta_2 x(t_{n-m}) + \beta_3 \widetilde{x}(t_n))\Delta W_n]\right|^2 \\
&\leq 24H^2B^2h^3(\alpha_1^2 + \alpha_2^2)\eta_1(1 + \mathbb{E}\|\varphi\|^2) \\
&\quad + 48H^2B^2h^2(\beta_1^2 + \beta_2^2)\eta_1(1 + \mathbb{E}\|\varphi\|^2) \\
&\quad + 24H^2B^2(h\alpha_3^2 + 2\beta_3^2)\tau^2h^2\eta_1(1 + \mathbb{E}\|\varphi\|^2) \\
&= C_3h^2.
\end{aligned} \tag{2.19}$$

Substituting (2.17), (2.18), (2.19) into (2.14) yields

$$\begin{aligned}
&\mathbb{E}(\delta_{n+1})^2 \\
&\leq 3h\int_{t_n}^{t_{n+1}} C_1 h dt + 3\int_{t_n}^{t_{n+1}} C_2 h dt + 3C_3h^2 \\
&= 3C_1h^3 + 3C_2h^2 + 3C_3h^2 \\
&\leq (3C_1 + 3C_2 + 3C_3)h^2.
\end{aligned}$$

That is to say,

$$\max_{0\leq n\leq N-1}(\mathbb{E}(\delta_{n+1})^2)^{\frac{1}{2}} \leq \eta_4 h \text{ as } h \to 0.$$

where $\eta_4 = \sqrt{3C_1 + 3C_2 + 3C_3}$. The proof is completed.

Now we give the main theorem in this section.

Theorem 2.1. *The numerical solution produced by the balanced methods* (2.1) *converges to the exact solution of* (1.1) *on the mesh point in the mean-square sense with strong order* $\frac{1}{2}$, *i.e., there exists a positive constant* η_5 *such that*

$$\max_{1\le n\le N}(\mathbb{E}(\varepsilon_n)^2)^{\frac{1}{2}} \le \eta_5 h^{\frac{1}{2}}, \quad h \to 0.$$

Proof. By (2.1), we find that

$$\begin{aligned}
&\varepsilon_{n+1}\\
&= x(t_n) - Y_n + x(t_{n+1}) - \overline{Y}(t_{n+1}) - Y_{n+1} - x(t_n) + Y_n + \overline{Y}(t_{n+1})\\
&= \varepsilon_n + \delta_{n+1} + P_n, \qquad (2.20)
\end{aligned}$$

where $\varepsilon_n = x(t_n) - Y_n$, $\delta_{n+1} = x(t_{n+1}) - \overline{Y}(t_{n+1})$, $P_n = Y_n + \overline{Y}(t_{n+1}) - Y_{n+1} - x(t_n)$. It follows from equation (2.20), we have

$$\begin{aligned}
&\mathbb{E}(\varepsilon_{n+1}^2 \,|\, \mathcal{F}_{t_0})\\
&\le \mathbb{E}(\varepsilon_n^2 \,|\, \mathcal{F}_{t_0}) + 2\mathbb{E}(\delta_{n+1}^2 \,|\, \mathcal{F}_{t_0}) + 2\mathbb{E}(P_n^2 \,|\, \mathcal{F}_{t_0})\\
&\quad + 2|\, \mathbb{E}(\delta_{n+1}\varepsilon_n \,|\, \mathcal{F}_{t_0})\,| + 2|\, \mathbb{E}(\varepsilon_n P_n \,|\, \mathcal{F}_{t_0})\,|. \qquad (2.21)
\end{aligned}$$

We will now estimate the separate terms in (2.21) individually. Without loss of generality, we can assume $0 < h < 1$. From (2.9), we obtain

$$\mathbb{E}(\delta_{n+1}^2 \,|\, \mathcal{F}_{t_0}) \le \mathbb{E}[\mathbb{E}(\delta_{n+1}^2 \,|\, \mathcal{F}_{t_n})|\, \mathcal{F}_{t_0}] \le \eta_4^2 h^2. \qquad (2.22)$$

Using (2.4) and (2.5) gives

$$\begin{aligned}
P_n &= Y_n + \overline{Y}_{t_{n+1}} - Y_{n+1} - x(t_n)\\
&= [(1 + D(t_n))^{-1} - (1 + D_n)^{-1}][(\alpha_1 x(t_n) + \alpha_2 x(t_{n-m}) + \alpha_3 \widetilde{x}(t_n))h
\end{aligned}$$

$$+ (\beta_1 x(t_n) + \beta_2 x(t_{n-m}) + \beta_3 \widetilde{x}(t_n))\Delta W_n] + (1 + D_n)^{-1}$$

$$\cdot [\alpha_1 h(x(t_n) - Y_n) + \alpha_2 h(x(t_{n-m}) - Y_{n-m}) + \alpha_3 h(\widetilde{x}(t_n) - \widetilde{Y}_n)$$

$$+ (\beta_1(x(t_n) - Y_n) + \beta_2(x(t_{n-m}) - Y_{n-m}) + \beta_3(\widetilde{x}(t_n) - \widetilde{Y}_n))\Delta W_n]. \quad (2.23)$$

It follows equations (2.3) and (2.6), we get

$$[1 + D(t_n)]^{-1} - [1 + D_n]^{-1}$$

$$= [1 + D(t_n)]^{-1}[(D_{0n} - D_0(t_n))h + (D_{1n} - D_1(t_n))| \Delta W_n |] \cdot [1 + D_n]^{-1}. \quad (2.24)$$

Using Assumption 1, $\mathbb{E}| \Delta W_n | = \sqrt{\frac{2h}{\pi}}$, $| D_i | \le B$, (1.2), (2.2), (2.7) and (2.24) yield

$$| \mathbb{E}(P_n) |$$

$$\le 4H^2 B h^{\frac{3}{2}}[| \alpha_1 | + | \alpha_2 | + | \alpha_3 | \tau]\sqrt{\eta_1(1 + \mathbb{E}\| \varphi \|^2)}$$

$$+ H| \alpha_1 | h\mathbb{E}| \varepsilon_n | + H| \alpha_2 | h\mathbb{E}| \varepsilon_{n-m} | + H| \alpha_3 | h^2 \sum_{k=1}^{m} \mathbb{E}| \varepsilon_{n-m+k} |$$

$$+ C_4 h^{\frac{3}{2}} + C_5 h\mathbb{E}| \varepsilon_n | + C_6 h\mathbb{E}| \varepsilon_{n-m} | + C_7 h^2 \sum_{k=1}^{m} \mathbb{E}| \varepsilon_{n-m+k} |. \quad (2.25)$$

By (2.23),

$$\mathbb{E}P_n^2$$

$$\le 2\mathbb{E}[(1 + D(t_n))^{-1} - (1 + D_n)^{-1}]^2[(\alpha_1 x(t_n) + \alpha_2 x(t_{n-m}) + \alpha_3 \widetilde{x}(t_n))h$$

$$+ (\beta_1 x(t_n) + \beta_2 x(t_{n-m}) + \beta_3 \widetilde{x}(t_n))\Delta W_n]^2 + 2\mathbb{E}(1 + D_n)^{-2}$$

$$\cdot [\alpha_1 h(x(t_n) - Y_n) + \alpha_2 h(x(t_{n-m}) - Y_{n-m}) + \alpha_3 h(\widetilde{x}(t_n) - \widetilde{Y}_n)$$

$$+ (\beta_1(x(t_n) - Y_n) + \beta_2(x(t_{n-m}) - Y_{n-m}) + \beta_3(\widetilde{x}(t_n) - \widetilde{Y}_n))\Delta W_n]^2. \quad (2.26)$$

Using Assumption 1, $|D_i| \le B$, $\mathbb{E}|\Delta W_n|^2 = h$, $\mathbb{E}|\Delta W_n|^4 = 3h^2$, (1.2), (2.7) and (2.24) give

$$\mathbb{E}[(1 + D(t_n))^{-1} - (1 + D_n)^{-1}]^2[(\alpha_1 x(t_n) + \alpha_2 x(t_{n-m}) + \alpha_3 \tilde{x}(t_n))h$$

$$+ (\beta_1 x(t_n) + \beta_2 x(t_{n-m}) + \beta_3 \tilde{x}(t_n))\Delta W_n]^2$$

$$\le 12H^4B^2[2\alpha_1^2 + \beta_1^2 + 3\beta_1^2 + 2\alpha_2^2 + \beta_2^2 + 3\beta_2^2]\eta_1(1 + \mathbb{E}\|\varphi\|^2)h^2$$

$$+ 12H^4B^2[2\alpha_3^2 + \beta_3^2 + 3\beta_3^2]\tau^2\eta_1(1 + \mathbb{E}\|\varphi\|^2)h^2$$

$$= C_8h^2. \tag{2.27}$$

Using Assumption 1, (2.2) and (2.7), yield

$$\mathbb{E}(1 + D_n)^{-2} \cdot [\alpha_1 h(x(t_n) - Y_n) + \alpha_2 h(x(t_{n-m}) - Y_{n-m})$$

$$+ \alpha_3 h(x(t_n) - \tilde{Y}_n) + (\beta_1(x(t_n) - Y_n) + \beta_2(x(t_{n-m}) - Y_{n-m})$$

$$+ \beta_3(\tilde{x}(t_n) - \tilde{Y}_n))\Delta W_n]^2$$

$$\le H^2 6(\alpha_1^2 + \beta_2^3)h\mathbb{E}\varepsilon_n^2 + 6H^2(\alpha_2^2 + \beta_2^2)h\mathbb{E}\varepsilon_{n-m}^2$$

$$+ 6H^2(\alpha_3^2 + \beta_3^2)\tau h^2 \sum_{k=1}^{m} \mathbb{E}\varepsilon_{n-m+k}^2$$

$$= C_9 h\mathbb{E}\varepsilon_n^2 + C_{10}h\mathbb{E}\varepsilon_{n-m}^2 + C_{11}h^2 \sum_{k=1}^{m} \mathbb{E}\varepsilon_{n-m+k}^2. \tag{2.28}$$

Inserting (2.27), (2.28) into (2.26) yields

$$\mathbb{E}P_n^2 \le 2C_8h^2 + 2C_9h\mathbb{E}\varepsilon_n^2 + 2C_{10}h\mathbb{E}\varepsilon_{n-m}^2 + 2C_{11}h^2 \sum_{k=1}^{m} \mathbb{E}\varepsilon_{n-m+k}^2. \tag{2.29}$$

It is not difficult to find that

$$\mathbb{E}(P_n^2 \mid \mathcal{F}_{t_0}) \le 2C_8h^2 + 2C_9h\mathbb{E}(\varepsilon_n^2 \mid \mathcal{F}_{t_0}) + 2C_{10}h\mathbb{E}(\varepsilon_{n-m}^2 \mid \mathcal{F}_{t_0})$$

$$+ 2C_{11}h^2 \sum_{k=1}^{m} \mathbb{E}(\varepsilon_{n-m+k}^2 \mid \mathcal{F}_{t_0}). \tag{2.30}$$

It follows (2.8) that

$$
\begin{aligned}
&2|\,\mathbb{E}(\delta_{n+1}\varepsilon_n \,|\, \mathcal{F}_{t_0})\,| \\
&\leq 2[\mathbb{E}(\mathbb{E}(\delta_{n+1} \,|\, \mathcal{F}_{t_n}))^2 \,|\, \mathcal{F}_{t_0}]^{\frac{1}{2}}[\mathbb{E}(\varepsilon_n^2 \,|\, \mathcal{F}_{t_0})]^{\frac{1}{2}} \\
&\leq \eta_3^2 h^2 + h\mathbb{E}(\varepsilon_n^2 \,|\, \mathcal{F}_{t_0}).
\end{aligned}
\tag{2.31}
$$

By (2.25),

$$
\begin{aligned}
&2|\,\mathbb{E}(\varepsilon_n P_n \,|\, \mathcal{F}_{t_0})\,| \\
&\leq 2\mathbb{E}\left[\left(C_4 h^{\frac{3}{2}}\varepsilon_n + C_5 h|\,\varepsilon_n\,|^2 + C_6 h|\,\varepsilon_n\,||\,\varepsilon_{n-m}\,| + C_7 h^2 \sum_{k=1}^{m}|\,\varepsilon_{n-m+k}\,||\,\varepsilon_n\,|\right)\Big|\,\mathcal{F}_{t_0}\right] \\
&= C_4^2 h^2 + (1 + 2C_5 + C_6 + C_7\tau)h\mathbb{E}(\varepsilon_n^2 \,|\, \mathcal{F}_{t_0}) + C_6 h\mathbb{E}(\varepsilon_{n-m}^2 \,|\, \mathcal{F}_{t_0}) \\
&\quad + C_7 h^2 \sum_{k=1}^{m}\mathbb{E}(\varepsilon_{n-m+k}^2 \,|\, \mathcal{F}_{t_0}).
\end{aligned}
\tag{2.32}
$$

Combining (2.22), (2.30), (2.31), (2.32) with (2.21) yields

$$
\begin{aligned}
&\mathbb{E}(\varepsilon_{n-1}^2 \,|\, \mathcal{F}_{t_0}) \\
&\leq \mathbb{E}(\varepsilon_n^2 \,|\, \mathcal{F}_{t_0}) + 2\eta_4^2 h^2 + 4C_8 h^2 \\
&\quad + 4C_9 h\mathbb{E}(\varepsilon_n^2 \,|\, \mathcal{F}_{t_0}) + 4C_{10} h\mathbb{E}(\varepsilon_{n-m}^2 \,|\, \mathcal{F}_{t_0}) \\
&\quad + 4C_{11} h^2 \sum_{k=1}^{m}\mathbb{E}(\varepsilon_{n-m+k}^2 \,|\, \mathcal{F}_{t_0}) + \eta_3^2 h^2 + h\mathbb{E}(\varepsilon_n^2 \,|\, \mathcal{F}_{t_0}) \\
&\quad + C_4^2 h^2 + (1 + 2C_5 + C_6 + C_7\tau)h\mathbb{E}(\varepsilon_n^2 \,|\, \mathcal{F}_{t_0}) + C_6 h\mathbb{E}(\varepsilon_{n-m}^2 \,|\, \mathcal{F}_{t_0}) \\
&\quad + C_6 h^2 \sum_{k=1}^{m}\mathbb{E}(\varepsilon_{n-m+k}^2 \,|\, \mathcal{F}_{t_0}) \\
&= C_{12} h^2 + (1 + C_{13} h)\mathbb{E}(\varepsilon_n^2 \,|\, \mathcal{F}_{t_0}) + C_{14} h\mathbb{E}(\varepsilon_{n-m}^2 \,|\, \mathcal{F}_{t_0}) \\
&\quad + C_{15} h^2 \sum_{k=1}^{m}\mathbb{E}(\varepsilon_{n-m+k}^2 \,|\, \mathcal{F}_{t_0}).
\end{aligned}
\tag{2.33}
$$

Letting $G_n = \max_{0 \le i \le n} \mathbb{E}(\varepsilon_i^2 \,|\, \mathcal{F}_{t_0})(n = 0, 1, ...)$, it is not difficult to find $R_0 = 0$, hence (2.33) becomes

$$\begin{aligned} G_{n+1} &\le C_{12}h^2 + (1 + C_{13}h)R_n + C_{14}hR_n + C_{15}h^2 mG_n \\ &\le (1 + C_{16}h)^{n+1} G_0 + C_{12}h^2 \sum\nolimits_{j=0}^{n} (1 + C_{16}h)^j \\ &\le \frac{C_{12}}{C_{16}} (e^{C_{16}T} - 1)h. \end{aligned} \tag{2.34}$$

(2.34) leads to the estimate

$$\max_{0 \le n \le N} (\mathbb{E}(\varepsilon_n)^2)^{\frac{1}{2}} \le \eta_5 h^{\frac{1}{2}}, \quad h \to 0,$$

where $\eta_5 = \sqrt{\frac{C_{12}}{C_{16}} (e^{C_{16}T} - 1)}$.

Theorem 2.1 shows that the balanced methods have strong convergence rate of at least $1/2$. Having established the acceptable finite time convergence of the balanced methods, in the next section, we consider long-time stability.

3. Mean-square Stability of Strong Balanced Methods

We investigate the mean-square stability of the strong balanced methods in this section.

Since the system (1.1) has no explicit solution. The following lemma gives the sufficient condition on the stability for the analytic solution of the system (1.1).

Lemma 3.1 [26]. *Assume that* $\alpha_i, \beta_i, i = 1, 2, 3$ *satisfy*

$$\alpha_1 + |\alpha_2| + |\alpha_3|\tau + (|\beta_1| + |\beta_2| + |\beta_3|\tau)^2 < 0. \tag{3.1}$$

Then the solution of (1.1) *is asymptotically stable in the mean-square*, *that is*,

$$\lim_{n \to \infty} \mathbb{E}(x(t_n))^2 = 0.$$

Given parameters $\alpha_i, \beta_i, i = 1, 2, 3$ and stepsize h, we say the balanced implicit methods are *mean-square stable* if $\lim_{n\to\infty} \mathbb{E}(Y_n)^2 = 0$ for any Y_0. The following theorem will show the mean-square stability of the balanced implicit methods.

Theorem 3.1. *Under the condition* (3.1) *and Assumption* 1, *for the sufficiently small stepsize h*, *the balanced implicit methods* (2.1) *are mean-square stable for the linear system* (1.1).

Proof. Squaring and taking expectation on both sides of (2.4) yields

$$\begin{aligned}
&\mathbb{E}Y_{n+1}^2 \\
&= \mathbb{E}Y_n^2 + h^2\mathbb{E}(1 + D_n)^{-2}(\alpha_1 Y_n + \alpha_2 Y_{n-m} + \alpha_3 \widetilde{Y}_n)^2 \\
&\quad + 2h\mathbb{E}(1 + D_n)^{-1}\Delta W_n(\alpha_1 Y_n + \alpha_2 Y_{n-m} + \alpha_3 \widetilde{Y}_n)(\beta_1 Y_n + \beta_2 Y_{n-m} + \beta_3 \widetilde{Y}_n) \\
&\quad + \mathbb{E}(1 + D_n)^{-1}\Delta W_n^2(\beta_1 Y_n + \beta_2 Y_{n-m} + \beta_3 \widetilde{Y}_n)^2 \\
&\quad + 2h\mathbb{E}Y_n\mathbb{E}(1 + D_n)^{-1}(\alpha_1 Y_n + \alpha_2 Y_{n-m} + \alpha_3 \widetilde{Y}_n) \\
&\quad + 2\mathbb{E}Y_n(1 + D_n)^{-1}\Delta W_n(\beta_1 Y_n + \beta_2 Y_{n-m} + \beta_3 \widetilde{Y}_n).
\end{aligned} \tag{3.2}$$

Letting ξ be independent standard normal random variable, we know that

$$\begin{aligned}
&\mathbb{E}(\Delta W_n(1 + D_n)^{-1}) \\
&= \frac{\sqrt{h}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} x(1 + D_{0n}h + D_{1n}\sqrt{h}\,|\,x\,|)^{-1}dx = 0.
\end{aligned} \tag{3.3}$$

Similarly

$$\begin{aligned}
&\mathbb{E}(\Delta W_n(1 + D_n)^{-2}) \\
&= \frac{\sqrt{h}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} x(1 + D_{0n}h + D_{1n}\sqrt{h}\,|\,x\,|)^{-2}dx = 0.
\end{aligned} \tag{3.4}$$

By (3.3), (3.4), thus (3.2) becomes

$$\mathbb{E}Y_{n+1}^2 \leq [1 + h^2(3\alpha_1^2 + 3\alpha_2^2 + 3\alpha_3^2\tau^2)\mathbb{E}(1 + D_n)^{-2} + \mathbb{E}(1 + D_n)^{-2}\Delta W_n^2$$

$$\cdot (\beta_1^2 + \beta_2^2 + \beta_3^2\tau^2 + 2|\beta_1\beta_2| + 2|\beta_1\beta_3|\tau + 2|\beta_2\beta_3|\tau)$$

$$+ h(2\alpha_1 + 2|\alpha_2| + 2|\alpha_3|\tau)\mathbb{E}(1 + D_n)^{-1}] \cdot \max_{n-m\leq i\leq n} \mathbb{E}|Y_i|^2$$

$$= R(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, h, \Delta W_n)\max_{n-m\leq i\leq n} \mathbb{E}|Y_i|^2. \tag{3.5}$$

Here

$$R(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, h, \Delta W_n)$$

$$= 1 + h^2(3\alpha_1^2 + 3\alpha_2^2 + 3\alpha_3^2\tau^2)\mathbb{E}(1 + D_n)^{-2} + \mathbb{E}(1 + D_n)^{-2}\Delta W_n^2$$

$$\cdot (\beta_1^2 + \beta_2^2 + \beta_3^2\tau^2 + 2|\beta_1\beta_2| + 2|\beta_1\beta_3|\tau + 2|\beta_2\beta_3|\tau)$$

$$+ h(2\alpha_1 + 2|\alpha_2| + 2|\alpha_3|\tau)\mathbb{E}(1 + D_n)^{-1}. \tag{3.6}$$

From this, we see that $\lim_{n\to\infty} \mathbb{E}(Y_n)^2 = 0$ if and only if

$$R(\alpha, \beta, \eta, h, \Delta W_n) < 1. \tag{3.7}$$

It is not difficult to find that

$$\mathbb{E}(\Delta W_n^2(1 + D_n)^{-2})$$

$$= h - 2\mathbb{E}\left[\Delta W_n^2 \frac{D_n}{1 + D_n}\right] + \mathbb{E}\left[\Delta W_n^2 \frac{D_n^2}{(1 + D_n)^2}\right]. \tag{3.8}$$

Using Assumption 1 and properties of ΔW_n, D_n gives

$$\left|-2\mathbb{E}\left[\Delta W_n^2 \frac{D_n}{1 + D_n}\right]\right|$$

$$\leq 2H\mathbb{E}[\Delta W_n^2|D_{0n}h + D_{1n}|\Delta W_n||]$$

$$= O(h^{3/2}) \tag{3.9}$$

and

$$\left|\mathbb{E}\left[\Delta W_n^2 \frac{D_n^2}{(1+D_n)^2}\right]\right|$$

$$\leq H^2 \mathbb{E}[\Delta W_n^2 | D_{0n} h + D_{1n} | \Delta W_n ||^2] = O(h^2). \tag{3.10}$$

Inserting (3.9) and (3.10) into (3.8) yields

$$\mathbb{E}(\Delta W_n^2 (1+D_n)^{-2}) = h + o(h). \tag{3.11}$$

Similarly, we have

$$\mathbb{E}((1+D_n)^{-2}) = \mathbb{E}\left[1 - \frac{2D_n}{1+D_n} + \frac{D_n^2}{(1+D_n)^2}\right] = 1 + O(h^{1/2}) \tag{3.12}$$

and

$$\mathbb{E}((1+D_n)^{-1}) = \mathbb{E}\left[1 - \frac{D_n}{1+D_n}\right] = 1 + O(h^{1/2}). \tag{3.13}$$

Inserting (3.11), (3.12), (3.13) into (3.6) yields

$$\begin{aligned} &R(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, h, \Delta W_n) \\ &= 1 + (\beta_1^2 + \beta_2^2 + \beta_3^2\tau^2 + 2|\beta_1\beta_2| + 2|\beta_1\beta_3|\tau + 2|\beta_2\beta_3|\tau + 2\alpha_1 \\ &\quad + 2|\alpha_2| + 2|\alpha_3|\tau)h + o(h). \end{aligned}$$

For all sufficiently small stepsizes h, we obtain

$$\begin{aligned} &R(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, h, \Delta W_n) < 1 \\ &\Leftrightarrow \alpha_1 + |\alpha_2| + |\alpha_3|\tau + (|\beta_1| + |\beta_2| + |\beta_3|\tau)^2 < 0. \end{aligned}$$

The proof is completed.

4. Mean-square Stability of the Weak Balanced Implicit Methods

In this section, we will investigate the mean-square stabilities of the weak balanced implicit methods equipped with two-point random variables for the driving process.

Given a stepsize $h > 0$, the weak balanced implicit methods are defined by

$$\begin{cases} Y_{n+1} = Y_n + (1 + \hat{D}_n)^{-1}[(\alpha_1 Y_n + \alpha_2 Y_{n-m} + \alpha_3 \tilde{Y}_n)h \\ \qquad + (\beta_1 Y_n + \beta_2 Y_{n-m} + \beta_3 \tilde{Y}_n)\widehat{\Delta W_n}], & n \geq 0, \\ Y_n = \varphi(nh), & n = -m, -m+1, ..., 0, \end{cases} \tag{4.1}$$

where $\hat{D}_n = D_{0n}h + D_{1n}|\widehat{\Delta W_n}|, \mathbb{P}(\widehat{\Delta W_n} = \sqrt{h}) = \mathbb{P}(\widehat{\Delta W_n} = -\sqrt{h}) = 1/2.$

It is not difficult to find that

$$\mathbb{P}(\widehat{\Delta W_n}) = 0, \ \mathbb{E}(\widehat{\Delta W}_n^2) = h. \tag{4.2}$$

The following theorem will show that the weak balanced implicit methods (4.1) can preserve the mean-square stability of the system (1.1).

Theorem 4.1. *Under the condition* (3.1), *for the sufficiently small stepsize h*, *the weak balanced implicit methods* (4.1) *are mean-square stable for the linear system* (1.1).

Proof. Similarly to the proof of Theorem 3.1, we have

$$\begin{aligned} &\mathbb{E}Y_{n+1}^2 \\ &= \mathbb{E}Y_n^2 + h^2\mathbb{E}(1 - \hat{D}_n)^{-2}(\alpha_1 Y_n + \alpha_2 Y_{n-m} + \alpha_3 \tilde{Y}_n)^2 \\ &\quad + 2h\mathbb{E}(1 + \hat{D}_n)^{-1}\widehat{\Delta W_n}(\alpha_1 Y_n + \alpha_2 Y_{n-m} + \alpha_3 \tilde{Y}_n)(\beta_1 Y_n + \beta_2 Y_{n-m} + \beta_3 \tilde{Y}_n) \\ &\quad + \mathbb{E}(1 + \hat{D}_n)^{-2}\widehat{\Delta W}_n^2(\beta_1 Y_n + \beta_2 Y_{n-m} + \beta_3 \tilde{Y}_n)^2 \\ &\quad + 2h\mathbb{E}Y_n\mathbb{E}(1 + \hat{D}_n)^{-1}(\alpha_1 Y_n + \alpha_2 Y_{n-m} + \alpha_3 \tilde{Y}_n) \\ &\quad + 2\mathbb{E}Y_n(1 + \hat{D}_n)^{-1}\widehat{\Delta W_n}(\beta_1 Y_n + \beta_2 Y_{n-m} + \beta_3 \tilde{Y}_n). \end{aligned} \tag{4.3}$$

Since $\mathbb{P}(\widehat{\Delta W_n} = \sqrt{h}) = \mathbb{P}(\widehat{\Delta W_n} = -\sqrt{h}) = 1/2,$ we find that

$$\mathbb{E}(\widehat{\Delta W_n}(1 + \hat{D}_n)^{-1})$$

$$= \frac{1}{2}(\sqrt{h}(1 + D_{0n}h + D_{1n}\sqrt{h})^{-1}) + \frac{1}{2}(-\sqrt{h}(1 + D_{0n}h + D_{1n}\sqrt{h})^{-1}) = 0 \tag{4.4}$$

and similarly

$$\mathbb{E}(\widehat{\Delta W_n}(1 + \hat{D}_n)^{-2}) = 0. \tag{4.5}$$

By (4.4), (4.5), thus (4.3) becomes

$$\mathbb{E}Y_{n+1}^2$$

$$\leq [1 + h^2(3\alpha_1^2 + 3\alpha_2^2 + 3\alpha_3^2\tau^2)\mathbb{E}(1 + \hat{D}_n)^{-2} + \mathbb{E}(1 + \hat{D}_n)^{-2}\widehat{\Delta W}_n^2$$

$$\cdot (\beta_1^2 + \beta_2^2 + \beta_3^2\tau^2 + 2|\beta_1\beta_2| + 2|\beta_1\beta_3|\tau + 2|\beta_2\beta_3|\tau)$$

$$+ h(2\alpha_1 + 2|\alpha_2| + 2|\alpha_3|\tau)\mathbb{E}(1 + \hat{D}_n)^{-1}] \cdot \max_{n-m\leq i\leq n}\mathbb{E}|Y_i|^2$$

$$= R(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, h, \widehat{\Delta W_n})\max_{n-m\leq i\leq n}\mathbb{E}|Y_i|^2. \tag{4.6}$$

Here

$$R(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, h, \widehat{\Delta W_n})$$

$$= 1 + h^2(3\alpha_1^2 + 3\alpha_2^2 + 3\alpha_3^2\tau^2)\mathbb{E}(1 + \hat{D}_n)^{-2} + \mathbb{E}(1 + \hat{D}_n)^{-2}\widehat{\Delta W}_n^2$$

$$\cdot (\beta_1^2 + \beta_2^2 + \beta_3^2\tau^2 + 2|\beta_1\beta_2| + 2|\beta_1\beta_3|\tau + 2|\beta_2\beta_3|\tau)$$

$$+ h(2\alpha_1 + 2|\alpha_2| + 2|\alpha_3|\tau)\mathbb{E}(1 + \hat{D}_n)^{-1}. \tag{4.7}$$

We observe that

$$\mathbb{E}(\widehat{\Delta W}_n^2(1 + \hat{D}_n)^{-2})$$

$$= \mathbb{E}(\widehat{\Delta W}_n^2(1 + D_{0n}h + D_{1n}|\widehat{\Delta W_n}|)^{-2})$$

$$= h(1 + D_{0n}h + D_{1n}\sqrt{n})^{-2}.$$

There must exist $\hat{h}^* = \min\{1/(|D_0|^2 + |D_1|^2), 1\}$ such that for any $h \in (0, \hat{h}^*)$, $|D_{0n}h + D_{1n}\sqrt{n}| < 1$. By Taylor expansion,

$$(1 + D_{0n}h + D_{1n}\sqrt{h})^{-2} = 1 - 2D_{1n}\sqrt{h} + o(\sqrt{h}).$$

Hence, we obtain

$$\mathbb{E}(\widehat{\Delta W}_n^2(1 + \hat{D}_n)^{-2}) = h(1 - 2D_{1n}\sqrt{h} + o(\sqrt{h})) = h + o(h). \tag{4.8}$$

Similarly to (4.8), we find that

$$\mathbb{E}(1 + \hat{D}_n)^{-1} = \mathbb{E}(1 + \hat{D}_n)^{-2} = 1 + O(\sqrt{h}). \tag{4.9}$$

Substituting (4.8), (4.9) into (4.7) yields

$$\begin{aligned} &R(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, h, \widehat{\Delta W_n}) \\ &= 1 + (\beta_1^2 + \beta_2^2 + \beta_3^2\tau^2 + 2|\beta_1\beta_2| + 2|\beta_1\beta_3|\tau + 2|\beta_2\beta_3|\tau + 2\alpha_1 \\ &\quad + 2|\alpha_2| + 2|\alpha_3|\tau)h + o(h). \end{aligned}$$

For all sufficiently small stepsizes h, we find that

$$\begin{aligned} &R(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, h, \widehat{\Delta W_n}) < 1 \\ &\Leftrightarrow \alpha_1 + |\alpha_2| + |\alpha_3|\tau + (|\beta_1| + |\beta_2| + |\beta_3|\tau)^2 < 0. \end{aligned}$$

The proof is completed.

Theorem 3.1 and Theorem 4.1 show that strong balanced methods (2.1) and weak balanced methods (4.1) can well reproduce the mean-square stability of the system (1.1) for sufficiently small stepsize.

5. Numerical Experiments

In this section, several numerical examples are given to illustrate our theoretical results in the previous sections. Consider the scalar linear equation

$$\begin{cases} dx(t) = \left(\alpha_1 x(t) + \alpha_2 x(t-\tau) + \alpha_3 \int_{t-\tau}^{t} x(s)ds\right)dt \\ \qquad + \left(\beta_1 x(t) + \beta_2 x(t-\tau) + \beta_3 \int_{t-\tau}^{t} x(s)ds\right)dW(t), \quad t > 0, \\ x(t) = 1, \quad t \in [-\tau, 0]. \end{cases} \tag{5.1}$$

Denoting Y_n^i as the numerical approximation to $x^i(t_n)$ at step point t_n in ith simulation of all $K = 2000$ simulations. We use $\frac{1}{K}\sum_{i=1}^{K} |Y_n^i|^2$ to approximate $E|Y_n|^2$. For simply, we choose $D_{0j} = 1$; $D_{1j} = 1$ $(j = 0, 1, \ldots)$. All the graphs are drawn with the vertical axis scaled logarithmically.

We illustrate the mean-square stability of the strong balanced methods (2.1) via the two following examples:

Example 1. $\alpha_1 = -11; \alpha_2 = 6; \alpha_3 = 0.5, \beta_1 = 0.5, \beta_2 = 1, \beta_3 = 0.5, \tau = 1.$

Example 2. $\alpha_1 = -13; \alpha_2 = 4; \alpha_3 = 1, \beta_1 = 1, \beta_2 = 0.5, \beta_3 = 1, \tau = 1.$

To verify our result concerning mean-square stability for the balanced implicit methods, we illustrate it in Example 1 and Example 2. The values of the coefficients in Examples 1 and 2 satisfy the condition in Lemma 3.1, thus the system (1.1) is mean-square stable.

For the linear test equation (5.1), the balanced method obviously reduces to the drift-implicit method in the case $C_1 = 0$, $C_0 \neq 0$ and to the explicit method in the case $C_0 = C_1 = 0$. As $C_1 \neq 0$, due to the presence of balanced factor $C_1|\Delta W_n|$, the balanced method shows implicitness in the diffusion term, which has a potential to ensure good stability property. Thus, the effect of the balanced factor for the stability of the numerical method will be mainly analyzed in the following. More precisely, we investigate the variety of the stability through the numerical experiments with $C_0 = 0$ or $C_0 = 1$ and only varying $C_1 \in \{0, 1\}$. The parameter pair (C_0, C_1) is chosen as follows:

(1) $C_0 = 0,\ C_1 = 0;$

(2) $C_0 = 0,\ C_1 = 1;$

(3) $C_0 = 1,\ C_1 = 0;$

(4) $C_0 = 1,\ C_1 = 1.$

In Figures 1, 2, the black broken lines and the red star lines represent the solutions produced by the balanced method with $(C_0, C_1) = (0, 0)$ and the balanced method with $(C_0, C_1) = (0, 1)$, respectively. And the green solid lines and the blue solid lines represent the solutions produced by the balanced method with $(C_0, C_1) = (1, 0)$ and the balanced method with $(C_0, C_1) = (1, 1)$, respectively.

Applying the above four kinds of strong numerical methods to Example 1, we plot the numerical solutions of Example 1 in Figure 1. From Figure 1, one can easily observe that all the four numerical simulations are stable for small stepsize $h = \frac{1}{20}$. But when the stepsize h increases, different methods exhibit different behaviors. For example, the explicit method with $(C_0, C_1) = (0, 0)$ is not mean-square stable on $h = \frac{1}{8}$. However, with $C_0 = 0$ fixed and varying the parameter C_1, the numerical method with $(C_0, C_1) = (0, 1)$ is mean-square stable on $h = \frac{1}{8}$. For drift-implicit variant of the Euler-scheme, i.e., $C_0 = 1$, $C_1 = 0$, one can ensure good approximations in the first two pictures in Figure 1, where stepsizes $h = \frac{1}{7}, \frac{1}{6}$ were used. Unfortunately, such drift-implicit method becomes unstable as larger stepsizes $h = \frac{1}{7}, \frac{1}{6}$ were involved. Varying the parameter C_1 and leaving C_0 unchanged, we obtain the balanced method with $(C_0, C_1) = (1, 1)$, which successfully reproduces the mean-square stability of the test problem, even for large stepsizes $h = \frac{1}{7}, \frac{1}{6}$.

The above numerical results indicate that, with C_0 fixed the balanced method with $C_1 \neq 0$ ensures better stability behavior than the method with $C_1 = 0$. Below we try to explain such observation by the maximum allowable stepsizes of these numerical methods. In fact, we can obtain an estimate of the supremum of the stepsize h in (3.6) for the strong balanced methods (2.1). Noticing that ξ is a standard normal random variable and considering the convergence of the series in (3.7), we can compute (3.6) approximately as follows:

$$
\begin{aligned}
& R(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, h, \Delta W_n) \\
&= 1 + h^2(3\alpha_1^2 + 3\alpha_2^2 + 3\alpha_3^2\tau^2)\mathbb{E}(1 + D_n)^{-2} + \mathbb{E}(1 + D_n)^{-2}\Delta W_n^2 \\
&\quad \cdot (\beta_1^2 + \beta_2^2 + \beta_3^2\tau^2 + 2|\beta_1\beta_2| + 2|\beta_1\beta_3|\tau + 2|\beta_2\beta_3|\tau) \\
&\quad + h(2\alpha_1 + 2|\alpha_2| + 2|\alpha_3|\tau)\mathbb{E}(1 + D_n)^{-1} \\
&\approx 1 + h^2(3\alpha_1^2 + 3\alpha_2^2 + 3\alpha_3^2\tau^2) \cdot \frac{1}{\sqrt{2\pi}} \int_{-10}^{10} e^{-\frac{x^2}{2}} (1 + D_{0n}h + D_{1n}\sqrt{h}|x|)^{-2} dx \\
&\quad + (\beta_1^2 + \beta_2^2 + \beta_3^2\tau^2 + 2|\beta_1\beta_2| + 2|\beta_1\beta_3|\tau + 2|\beta_2\beta_3|\tau) \\
&\quad \cdot \frac{h}{\sqrt{2\pi}} \int_{-10}^{10} e^{-\frac{x^2}{2}} x^2 (1 + D_{0n}h + D_{1n}\sqrt{h}|x|)^{-2} dx \\
&\quad + h(2\alpha_1 + 2|\alpha_2| + 2|\alpha_3|\tau) \frac{1}{\sqrt{2\pi}} \int_{-10}^{10} e^{-\frac{x^2}{2}} (1 + D_{0n}h + D_{1n}\sqrt{h}|x|)^{-2} dx \\
&= 1 + f(h). \qquad (5.2)
\end{aligned}
$$

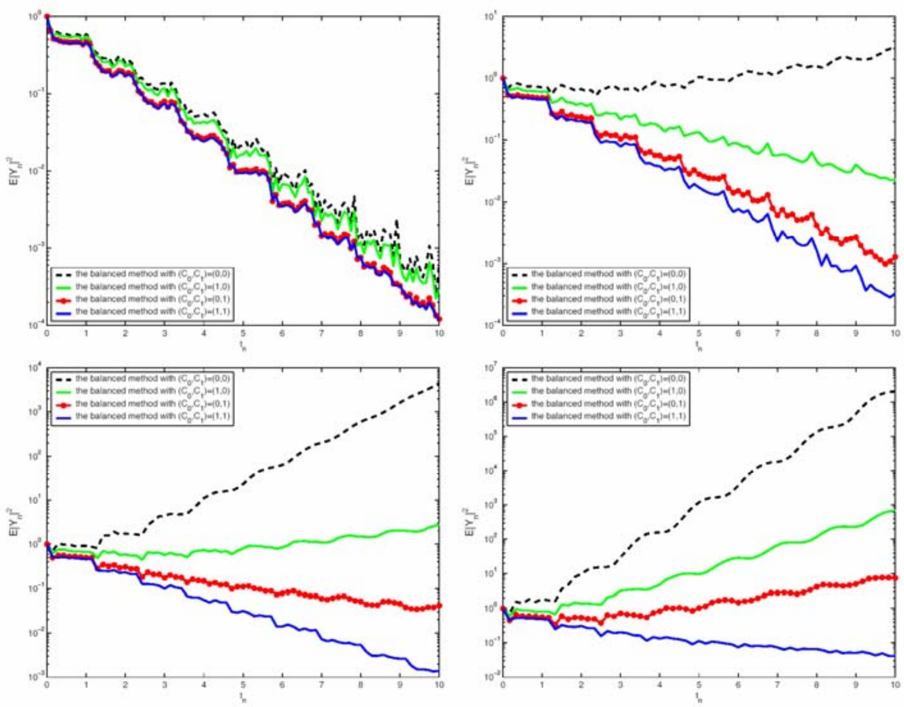

Figure 1. Stability behavior of strong balanced methods for Example 2: *upper left* : $h = \dfrac{1}{20}$; *upper right* : $h = \dfrac{1}{8}$; *lower left* : $h = \dfrac{1}{7}$; *lower right* : $h = \dfrac{1}{6}$.

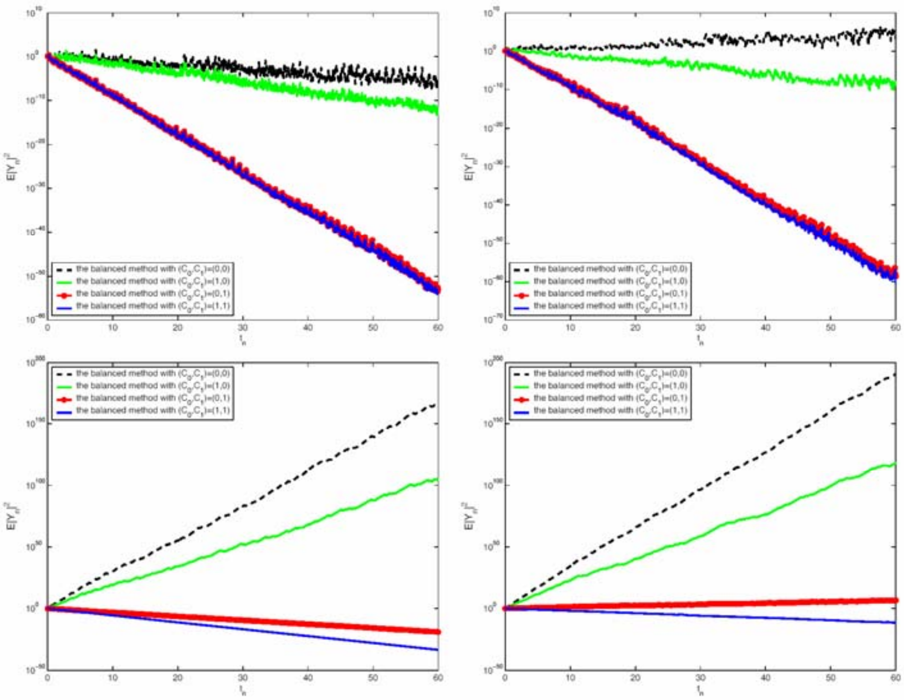

Figure 2. Stability behavior of weak balanced methods for Example 2: *upper left* : $h = \frac{1}{30}$; *upper right* : $h = \frac{1}{16}$; *lower left* : $h = \frac{1}{5}$; *lower right* : $h = \frac{1}{4}$.

Our aim is to find the supremum h_B such that $f(h) < 0$ for $h < h_B$. Applying the composite trapezoidal rule, we can approximate the three integrals of $f(h)$, where the integral interval $[-10, 10]$ is divided into 200 equal subintervals. It is obvious that $f(h)$ is a nonlinear function with respect to h. We use Newton-Raphson method to solve the nonlinear equation $f(h) = 0$ and obtain its zero root h such that $f(h) < 0$, $0 < h < h_B$.

Table 1. Upper bound h_B for stability of the four kinds of strong balanced methods for Example 1

	$C_0 = 0$	$C_0 = 1$
$C_1 = 0$	0.0106	0.0108
$C_1 = 1$	0.0135	0.0137

Consequently, the stepsize's supremum h_B can be derived for these four kinds of strong balanced methods and they are presented in Table 1 for Example 1. For Example 1, we have $h_B = 0.0106$ in the case $(C_0, C_1) = (0, 1)$, while $h_B = 0.0135$ for $(C_0, C_1) = (0, 1)$. Fixing $C_0 = 1$ and varying C_1 from $C_1 = 0$ to $C_1 = 1$, we get the corresponding stepsize's supremum h_B increasing from $h_B = 0.0108$ to $h_B = 0.0137$.

The numerical results in Figure 1 and the analysis of the stepsize's supremum show that, both the numerical results in Figure 1 and the analysis of the stepsize's supremum show that, to preserve stability the strong balanced methods with $C_1 \neq 0$ allow for larger range of the stepsize than the strong balanced methods with $C_1 = 0$.

Now let us begin stability tests for the weak numerical methods. Similarly to the strong numerical schemes, we can obtain an estimate of the maximum allowable stepsizes $\hat{h}_B$ in (4.7) for the weak numerical methods. Owing to the properties of $\hat{\xi}$ is a two-point random variable random variable, we can rewrite (4.7) as

$$R(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, h, \widehat{\Delta W_n})$$

$$= 1 + h^2(3\alpha_1^2 + 3\alpha_2^2 + 3\alpha_3^2\tau^2)\mathbb{E}(1 + \hat{D}_n)^{-2} + \mathbb{E}(1 + \hat{D}_n)^{-2}\widehat{\Delta W}_n^2$$

$$\cdot (\beta_1^2 + \beta_2^2 + \beta_3^2\tau^2 + 2|\beta_1\beta_2| + 2|\beta_1\beta_3|\tau + 2|\beta_2\beta_3|\tau)$$

$$+ h(2\alpha_1 + 2|\alpha_2| + 2|\alpha_3|\tau)\mathbb{E}(1 + \hat{D}_n)^{-1}$$

$$= 1 + \hat{f}(h). \tag{5.3}$$

Table 2. Upper bound $\hat{h}_B$ for stability of the four kinds of weak balanced methods for Example 2

	$C_0 = 0$	$C_0 = 1$
$C_1 = 0$	0.0337	0.0357
$C_1 = 1$	0.0456	0.0486

It is obvious that $\hat{f}(h)$ is a nonlinear function with respect to *h*. Using Newton-Raphson method, we solve the nonlinear equation $\hat{f}(h) = 0$ and obtain its zero root $\hat{h}_B$ such that $\hat{f}(h) < 0$ as $h < \hat{h}_B$. In this way, we can obtain the stepsize's supremum $\hat{h}_B$ of the weak balanced methods with four different parameter pairs, which are listed in Table 2. Similarly to the strong balanced methods, with $C_0 = 0$ fixed, the maximum allowable stepsize $\hat{h}_B$ increases from $\hat{h}_B = 0.0337$ to $\hat{h}_B = 0.0456$ as the method parameter C_1 varies from $C_1 = 0$ to $C_1 = 1$. Fixing $C_0 = 1$ and varying C_1 from $C_1 = 0$ to $C_1 = 1$, we get the corresponding stepsize's supremum $\hat{h}_B$ increasing from $\hat{h}_B = 0.0357$ to $\hat{h}_B = 0.0486$.

Applying the above numerical methods with four different parameter pairs to Example 2, we plot the numerical solutions of Example 2 in Figure 2. In Figure 2, we focus on the case when $C_0 = 0$ is fixed. For this case, $C_1 = 0$ is enough to guarantee stability on $h = \frac{1}{30}$, but fails to preserve stability on larger stepsize $h = \frac{1}{16}$. On the contrary, $C_1 = 1$ detects its good performances for both stepsizes. From Figure 2, one can observe a similar effect brought by increasing the parameter $C_1 = 0$ to $C_1 = 1$. Again, the weak balanced methods with $C_1 \neq 0$ possess better stability properties and have less restriction on the stepsize than the methods with $C_1 = 0$.

From the above, the balanced implicit methods admit better stable properties than the Euler-Maruyama method with the same stepsize. Overall, they are consistent with the established results.

6. Conclusion

In this work, we have examined the convergence and the mean-square stability of the balanced methods for the stochastic delay integro-differential equations. It is shown that the strong balanced implicit methods give strong convergence rate of at least $1/2$. The forgoing results show that both the strong balanced methods and weak balanced methods can reproduce the mean-square stability of the system with sufficiently small stepsize h. The theory result and the numerical experiment show that balanced methods which have the implicit diffusion term are indeed the superior schemes for relatively large stepsizes and admit better stability property than the balanced methods which have the explicit diffusion term, for example, the Euler scheme method.

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